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A canonical representation of maximal, indecomposable semi groups of constant-rank nonnegative r-potent matrices in Mn(R)

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Abstract
An r-potent matrix in $Mn(\mathbb{R})$ is an $n \times n$ matrix satisfying $Er = E$. A multiplicative semigroup $\mathcal{S}$ in $Mn(\mathbb{R})$ is said to be decomposable if there exists a special kind of common invariant subspace called standard invariant subspace for each $A \in \mathcal{S}$. A semi-group $\mathcal{S}$ of non-negative r-potent matrices in $Mn(\mathbb{R})$ is known to be decomposable if rank $(S) > r - 1$ for all S in $\mathcal{S}$. Further, a semigroup $\mathcal{S}$ in $Mn(\mathbb{R})$ of nonnegative matrices will be called a full semigroup if $\mathcal{S}$ has no common zero row and no common zero column. We have studied the structure of maximal semi-groups of non-negative r-potent matrices in $Mn(\mathbb{R})$ under the special condition of fullness. The objectives of this paper are twofold: (1) To find conditions under which semigroups of nonnegative r-potent matrices can be expressed as a direct sum of maximal rank-one indecomposable semigroups of r-potent matrices; and (2) To obtain a canonical representation of maximal indecomposable rank-one semigroups of r-potent matrices which in the light of the above result gives a complete characterization of such semigroups having constant rank.

Keywords: r-potent, maximal, fullness, indecomposable, canonical, geometric characterization

Introduction

A nonnegative semigroup of r-potent matrices with constant rank m, where $m > r - 1$, is known (see Theorem 3.5 (i) in [9]) to have the diagonal form under a special condition of fullness:

$$\begin{pmatrix} S_1 & & & & \\ & S_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & S_m \end{pmatrix}$$

Where each diagonal block $\mathcal{S}_i = \{S_i : S \in \mathcal{S}\}$ is an indecomposable semigroup of constant-rank r-potent matrices. Further, in Theorem 4.1(i) [9], it was proved that under the property of fullness, a semigroup $\mathcal{S}$ of nonnegative r-potent matrices with constant rank is maximal if and only if it can be expressed as a direct sum of maximal constant rank indecomposable semigroups of nonnegative r-potent matrices.

In this paper, we extend the results presented in [9]. In Section 2 of this paper, we establish the maximality of a direct sum of indecomposable semigroups of nonnegative rank-one r-potent matrices in $Mn(\mathbb{R})$. We further give an overview of the relevant salient results. In Section 3, we provide a geometric characterization and a canonical representation of maximal indecomposable semigroups of nonnegative rank-one r-potent matrices in $(\mathbb{R})$. Inspired by Theorem 5.1 of [1] where a complete structure of maximal indecomposable semigroups of 2-potent matrices (or idempotents) of rank one was obtained, we establish in this paper a generalization of this result to any such semigroup of r-potent matrices $Mn(\mathbb{R})$.

Maximality of a Direct Sum of Indecomposable Semigroups of Nonnegative Rank-One r-Potent Matrices in $M(\mathbb{R})$

Before proving our main theorem (Theorem 2.5), we state two lemmas which are instrumental in proving it. The proofs of these two lemmas are given in A. Marwaha's paper [2] (see Lemmas 4.7 and 4.8) and therefore are only stated here. Moreover, the lemmas hold true for general nonnegative semigroups and are thus applicable to semigroups of idempotents or more generally, to semigroups of r-potent matrices.

Prior to stating the lemmas, we give some preliminary definitions and terminology which are being used in the proof of Theorem 2.5.

Definition 2.1 Let $\mathcal{S}$ be a semigroup of matrices in $(\mathbb{R})$. We denote by $\mathcal{L}at'\mathcal{S}$, the lattice of all standard subspaces which are invariant under every member of $\mathcal{S}$.

Definition 2.2 A semigroup $\mathcal{S}$ in $Mn(\mathbb{R})$ of nonnegative matrices will be called a full semigroup if $\mathcal{S}$ has no common zero row and no common zero column.

Lemma 2.3 Let $\mathcal{S}$ be an indecomposable, nonnegative semigroup in $Mn(\mathbb{R})$ and e_i be any basis vector. Then $\mathcal{V}\{\mathcal{S}e_i\}$ contains a positive vector.

(By $\mathcal{V}\{\mathcal{S}e_i\}$, we mean the linear span of the vectors $\mathcal{S}e_i$).

Lemma 2.4 Let $\mathcal{S}$ be a direct sum of m nonnegative, indecomposable semigroups $\mathcal{S}_1, \cdots\cdots, \mathcal{S}_m$, so that each member of $\mathcal{S}$ has block diagonal representation.

$$\begin{pmatrix} S_1 & & & & \\ & S_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & S_m \end{pmatrix}$$

Where

$S_i \in \mathcal{S}_i, i = 1,2, \cdots\cdots, m$ with respect to a fixed decomposition $M_1 \oplus \cdots\cdots \oplus M_m$ of $\mathbb{R}^n$ into standard subspaces. Then every $M \in \mathcal{L}at'\mathcal{S}$ is of the form $M = \oplus^m{}_{i=1}\, \varepsilon_i M_i$, where ε_i is either 0 or 1.

Theorem 2.5 A direct sum of m maximal, indecomposable nonnegative rank-one semigroups of r-potent matrices is a maximal semigroup of constant rank m.

Proof For $m = 1$, the result is trivially true. Therefore, let $m > 1$. Suppose $\mathcal{S}_1, \mathcal{S}_2, \cdots\cdots, \mathcal{S}_m$ are m maximal indecomposable, nonnegative rank-one semigroups of r-potent matrices and consider their direct sum and denote it by $\mathcal{S}$.

Then every member S of $\mathcal{S}$ is of the form

$$\begin{pmatrix} S_1 & & & & \\ & S_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & S_m \end{pmatrix}$$

where $S_i \in \mathcal{S}_i, i = 1,2, \cdots\cdots, m$

If $\mathcal{S}$ is not maximal, then let $\mathcal{S} \subseteq \mathcal{S}'$, where $\mathcal{S}'$ is a semigroup of r-potents with constant rank m. Clearly $\mathcal{S}$ is full because of its structure and therefore $\mathcal{S}'$ is full too. This gives that $\mathcal{S}'$ is a direct sum of m rank-one indecomposable, nonnegative semigroups of r-potent matrices. Let us denote them by $\mathcal{S}_1', \mathcal{S}_2', \cdots\cdots, \mathcal{S}_m'$ respectively.

Clearly, $\mathcal{L}at'\mathcal{S}' \subseteq \mathcal{L}at'\mathcal{S}$. By Lemma 2.4, the cardinality of $\mathcal{L}at'\mathcal{S}$ is the same as that of $\mathcal{L}at'\mathcal{S}'$ which is 2^m. Therefore, we must have $\mathcal{L}at'\mathcal{S} = \mathcal{L}at'\mathcal{S}'$. After permutation of basis, if required, we obtain $\mathcal{S}_i \subseteq \mathcal{S}_i'$. But maximality of $\mathcal{S}_i$ gives that $\mathcal{S}_i = \mathcal{S}_i'$ for each $i = 1,2, \cdots\cdots, m$. Hence $\mathcal{S}$ is maximal.

The following theorem has been proved in [9] (see Theorem 3.5(i)). We are stating the theorem for the sake of completeness.

Theorem 2.6 Let $\mathcal{S}$ be a semigroup of nonnegative r-potent matrices in $Mn(\mathrm{R})$ with constant rank $m > r - 1$. If $\mathcal{S}$ is full, then there exists a permutation matrix P such that for any $S \in \mathcal{S}$, $P{-}1SP$ has the block diagonal form

$$\begin{pmatrix} S_1 & & & & \\ & S_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & S_m \end{pmatrix},$$

Where each $\mathcal{S}i = \{S : S \in \mathcal{S}\}$ is an indecomposable semigroup of constant-rank nonnegative r-potent matrices.

It is clear that Theorem 2.5 proves the converse of Theorem 2.6 in case the semigroups are maximal satisfying the property of fullness and therefore we get the following characterization of maximal nonnegative semigroups of r-potent matrices of constant rank.

Theorem 2.7 Let $\mathcal{S}$ be a nonnegative semigroup of r-potent matrices in $Mn(\mathrm{R})$ of constant rank m. If $\mathcal{S}$ is full, then $\mathcal{S}$ is maximal if and only if

$$\mathcal{S} = \left\{ \begin{pmatrix} S_1 & & & & \\ & S_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & S_m \end{pmatrix} : S_i \in \mathcal{S}_i\,, i = 1,2, \dots, m \right\},$$

Where $\mathcal{S}_i$ is a maximal constant-rank indecomposable semigroup of nonnegative r-potent matrices.

The result in the next section (Theorem 3.1) gives a geometrical characterization of maximal semigroups of rank one nonnegative r-potent matrices of order n x n, which in view of Theorem 2.5 shall result in giving a complete geometrical characterization of maximal semigroups of nonnegative r-potent matrices of constant rank. The observations following Theorem 3.1 give a canonical representation of the geometrical form of such semigroups.

A Canonical Representation of Maximal Indecomposable Semigroups of Nonnegative Rank-One r-Potent Matrices in $M_n(\mathbb{R})$

We begin by first finding the general form of all maximal indecomposable nonnegative semigroups of rank-one matrices in $M_n(\mathbb{R})$.

A nonzero, nonnegative, rank-one matrix in $M_n(\mathbb{R})$ is of the form xy^*, where x, y are nonzero, nonnegative vectors in $\mathbb{R}^n$. Let us denote $E = xy^*$, where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \text{ belong to } \mathbb{R}^n.$$

Then

$$E = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & & \vdots \\ x_ny_1 & x_ny_n & \cdots & x_ny_n \end{bmatrix}$$

Further, if E is an r-potent, then $E^r = E$ and it can be easily verified that each of n^2 equations corresponding to the $(i,j)^{th}$ entries of E^r and E respectively reduce to the equation

$$(x_1y_1 + x_2y_2 + \cdots\cdots + x_ny_n)^r = 1 \Leftrightarrow x_1y_1 + x_2y_2 \cdots\cdots + x_ny_n = 1$$

Which is equivalent to saying that $y^*x = 1$.
Thus, if $\mathcal{S}$ is a semigroup of nonnegative rank-one r-potent matrices, then we can find sets $\mathcal{X}$ and $\mathcal{Y}$ in the nonnegative cone of $\mathbb{R}^n$,viz., $\mathbb{R}^n{}_+ = \{x \in \mathbb{R}^n : x \geq 0\}$, so that $\mathcal{S} \subseteq \mathcal{X}\mathcal{Y}^*$ where

$$\mathcal{X}\mathcal{Y}^* = \{xy^*, x \in \mathcal{X}, y \in \mathcal{Y}\}$$

Such that $(x^*y)^r = x^*y$ and $y^*x = 1$ for all $x \in \mathcal{X}$ and for all $y \in \mathcal{Y}$.
Further, if $\mathcal{S}$ is maximal, then we must have $\mathcal{S} = \mathcal{X}\mathcal{Y}^*$ for some $\mathcal{X}, \mathcal{Y}$ in $\mathbb{R}^n{}_+$.
Our aim is to find the general form of $\mathcal{X}$ and $\mathcal{Y}$ for a maximal, nonnegative semigroup of rank-one r-potent matrices in $M_n(\mathbb{R})$.
Observe that if $tx_1 + (1-t)x_2$ for $0 \leq t \leq 1$ is any convex combination of arbitrary vectors x, x_2 in $\mathcal{X}$, then

$$y^*(tx_1 + (1-t)x_2) = t(y^*x_1) + (1-t)(y^*x_2)$$

$$= t \cdot 1 + (1-t) \cdot 1 = 1$$

and this holds good for all $y \in \mathcal{Y}$.
This shows that for a maximal $\mathcal{X}\mathcal{Y}^*$, $\mathcal{X}$ must contain all convex combinations of its members. Thus, with no loss of generality, we can assume that $\mathcal{X}$ has a positive vector, say $a = (a_i), a_i > 0$ for all i.
Now, any $x \in \mathcal{X}$ satisfies the system of equations $y^*x = 1$ for all $y \in \mathcal{Y}$. In particular, $x = a$ is a solution of the system. Thus the solution set of the system can be expressed as

$$\mathcal{X} = \{a + \mathcal{V}\} \cap \mathbb{R}^n{}_+ \ (1)$$

Where
$\mathcal{V} = \{v \in \mathbb{R}^n{}_+ : \ y^*v = 0 \ \forall \ y \in \mathcal{Y}\}$.

Now
$\mathcal{Y} = \{y \in \mathbb{R}^n{}_+ : y^*x = 1\}$

As $a \in \mathcal{X}$, $y^*a = 1$ for all $y \in \mathcal{Y}$. Also $\mathcal{Y} \subseteq \mathcal{V}^\perp$. Thus
$\mathcal{Y} \subseteq \{y \in \mathbb{R}^n{}_+ : y^*a = 1\} \cap \mathcal{V}^\perp$.
The maximality of $\mathcal{X}\mathcal{Y}^*$ implies that $\mathcal{Y}$ is exactly equal to this set. In other words,

$$\mathcal{Y} = \{y \in \mathbb{R}^n{}_+ : y^*a = 1\} \cap \mathcal{V}^\perp \ (2)$$

Next we show that if $\mathcal{X}$ and $\mathcal{Y}$ are given as in (1) and (2) for some positive a and some set $\mathcal{V}$, then $\mathcal{S} = \mathcal{X}\mathcal{Y}^*$, a semigroup of nonnegative rank-one r-potent matrices in $M_n(\mathbb{R})$ is maximal.
Suppose $\mathcal{S}$ is contained in a semigroup $\mathcal{S}_0$ of rank-one matrices where

$$\mathcal{S}_0 \subseteq \{zt^* : \ z, t \ \in \ \mathbb{R}^n{}_+\}$$

Where $\mathcal{S}_0$is a semigroup of nonnegative rank-one r-potent matrices in $M_n(\mathbb{R})$.

Let $s = zt^* \in \mathcal{S}_0$. Since $\mathcal{S}_0$ is a semigroup, $xy^* \cdot zt^* \in \mathcal{S}_0$ for all $xy^* \in \mathcal{S}$. Therefore

$$1 = tr(xy^* \cdot zt^*) = y^*z \cdot tr(xt^*) = y^*z \cdot t^*x$$

With no loss of generality, we can assume that $y^*z = 1$ and $t^*x = 1$. Now
$t^*x = 1$ for all $x \in \mathcal{X}$
In particular, for $x = a$ and $x = a + v$,
$t^*a = 1$ and $t^*(a + v) = 1 \ \forall \ v \in \ \mathcal{V}$
$\Rightarrow \ t^*a = 1$ and $t^*v = 0 \ \forall \ v \in \mathcal{V}$
$\Rightarrow \ t \in \ \mathcal{Y}$

We also have $y^*z = 1 \ \forall \ y \in \mathcal{Y}$
But $y^*a = 1 \ \forall \ y \in \mathcal{Y}$
Thus $y^*z = 1 = \ y^*a \ \forall \ y \in \mathcal{Y}$
$\Rightarrow \ y^*(z - a) = 0 \ \forall \ y \in \mathcal{Y}$
$\Rightarrow \ (z - a) \in \mathcal{Y}^\perp = \mathcal{V}$
$\Rightarrow z \in \{a + \mathcal{V}\} \cap \ \mathbb{R}^n{}_+ = \ \mathcal{X}$

Thus $zt^* \in \ \mathcal{S}$ which implies that $\mathcal{S}$ is maximal.
Next, we would like to see what type of $\mathcal{V}$ will give rise to maximal indecomposobale semigroups. We observe that if $v = (v_i) \in \ \mathcal{V}$, where $v_i \geq 0 \ or \ \leq 0$, then since $\sum_i v_i y_i = 0 \ \forall \ y = (y_i) \in \ \mathcal{Y}, y_i \geq 0$, a nonzero component of v, say v_i will render the i^{th} component y_i of each $y \in \ \mathcal{Y}$ zero, thus yielding a decomposable band $\mathcal{X}\mathcal{Y}^*$.
This shows that every vector of $\mathcal{V}$ must necessarily be a 'mixed' vector, *i.e.* a vector having both negative and positive entries (possibly some zeros too).
The above exposition can be summarized as follows:.

Theorem 3.1 A maximal, nonnegative indecomposable semigroup of rank-one r-potent matrices in $M_n(\mathbb{R})$ is of the form $\mathcal{X}\mathcal{Y}^*$, where $\mathcal{X} = \{a + \mathcal{V}\} \cap \mathbb{R}^n{}_+$, for some positive vector a in $\mathbb{R}^n{}_+$ and $\mathcal{Y} = \{y \ \in \ \mathbb{R}^n{}_+ : y^*a = 1\} \ \cap \ \mathcal{V}^\perp$, where $\mathcal{V}$ is a proper subspace of $\mathbb{R}^n{}_+$ containing only mixed vectors. In other words, $\mathcal{V}$ is a proper subspace of $\mathbb{R}^n{}_+$ intersecting the nonnegative cone of $\mathbb{R}^n{}_+$ in only the zero vector.
Observe that the selection of a positive vector in $\mathcal{X}$ gives rise to a particular maximal indecomposable semigroup of the said kind. Therefore, we would like to give a classification of all such maximal semigroups up to similarity. We begin with some observations.

Observation 1: Our first observation is that instead of $\mathcal{X}\mathcal{Y}^*$, we can without any loss of generality, consider $\mathcal{X}_1\mathcal{Y}_1{}^*$, where $\mathcal{X}_1 = \{a' + \ \mathcal{V}\} \ \cap \ \mathbb{R}^n{}_+$ with $a' \in a + \ \mathcal{V}$ and a' may not be positive, and $\mathcal{Y}_1 = \{y \ \in \mathbb{R}^n{}_+ : y^*a' = 1\} \ \cap \mathcal{V}^\perp$. This is not hard to see since $a' \in a + \ \mathcal{V}$ implies that $a' + \ \mathcal{V} = a + \ \mathcal{V}$ and so $\mathcal{X} = \mathcal{X}_1$. Also if $y \in \mathcal{Y}_1$, then $y^*a' = 1$, $y \in \ \mathcal{V}^\perp \cap \mathbb{R}^n{}_+$. But $a' = a + \ v_o$ for some $v_o \in \mathcal{V}$.
Therefore $y^*a + y^*v_o = 1$. As $y^*v_o = 0$, we obtain $y^*a = 1$. This gives $y \in \mathcal{Y}$ and so $\mathcal{Y}_1 \subseteq \mathcal{Y}$. Now let $y \in \mathcal{Y}$. Then $y \in \ \mathbb{R}^n{}_+ \cap \mathcal{V}^\perp$ and $y^*a = 1 \ \Rightarrow \ y^*(a' + v_{oo}) = 1$ for some $v_{oo} \in \mathcal{V}$
$\Rightarrow \ y^*a' = 1$ as $y^*v_{oo} = 0$
$\Rightarrow \ \mathcal{Y} \subseteq \mathcal{Y}_1$
Therefore $\mathcal{X}\mathcal{Y}^* = \mathcal{X}_1\mathcal{Y}_1{}^*$.

This shows that if $\mathcal{X}$ has a positive element a, we can pick up any nonnegative a' such that $a' \in a + \ \mathcal{V}$ and work with $\mathcal{X}_1 = \{a' + \ \mathcal{V}\} \ \cap \ \mathbb{R}^n{}_+$ instead of $\mathcal{X} = \{a + \ \mathcal{V}\} \ \cap \ \mathbb{R}^n{}_+$.

Observation 2: Suppose we are given an indecomposable nonnegative semigroup of rank-one r-potent matrices in $M_n(\mathbb{R})$ as described above. Then we obtain the canonical representation of $\mathcal{X}$ in the following manner:
We begin by selecting the unique vector a' in $\mathcal{V}^{\perp}$ which is closest to the vector a. This is possible because $\mathcal{V}^{\perp}$ is a closed convex set. Clearly $a' - a \in \mathcal{V}$ so that $a' + \mathcal{V} = a + \mathcal{V}$. Furthermore, we can normalize a' and consider to be of norm 1, with no loss of generality.
By Observation 1, we can consider $\mathcal{X} = \{a' + \mathcal{V}\} \cap \mathbb{R}^n{}_+$ and this would be our canonical representation of $\mathcal{X}$.

Observation 3: Now let $\mathcal{X}_1\mathcal{Y}_1{}^*$ and $\mathcal{X}_2\mathcal{Y}_2{}^*$ be two maximal indecomposable semigroups of the said type given in their canonical forms:

$\mathcal{X}_1 = \{a_1 + \mathcal{V}_1\} \cap \mathbb{R}^n{}_+$, $\mathcal{X}_2 = \{a_2 + \mathcal{V}_2\} \cap \mathbb{R}^n{}_+$
$\mathcal{Y}_1 = \{y \in \mathbb{R}^n{}_+ : y^* a_1 = 1\} \cap \mathcal{V}_1{}^{\perp}$
$\mathcal{Y}_2 = \{y \in \mathbb{R}^n{}_+ : y^* a_2 = 1\} \cap \mathcal{V}_2{}^{\perp}$

Where a_1, a_2 are unique vectors of norm 1 in $\mathcal{V}_1{}^{\perp}$ and $\mathcal{V}_2{}^{\perp}$ respectively.
Now suppose $\mathcal{X}_1\mathcal{Y}_1{}^* = \mathcal{X}_2\mathcal{Y}_2{}^*$. Fix any Y_1 in $\mathcal{Y}_1$ and let y_i be any nonzero component in $Y_1 (y_i > 0)$. Then $y_i\mathcal{X}_1$ is a positive scalar multiple of $\mathcal{X}_2$ which implies that $\mathcal{X}_1 \subseteq \mathbb{R}^+\mathcal{X}_2$. Similarly, we get that $\mathcal{X}_2 \subseteq \mathbb{R}^+\mathcal{X}_1$. By the same reasoning for $\mathcal{Y}_1$ and $\mathcal{Y}_2$, we obtain $\mathcal{Y}_1 \subseteq \mathbb{R}^+\mathcal{Y}_2$ and $\mathcal{Y}_2 \subseteq \mathbb{R}^+\mathcal{Y}_1$.
Consider $\mathcal{X}_1 \subseteq \mathbb{R}^+\mathcal{X}_2$. Since $a_1 \in \mathcal{X}_1$, there exists $r \in \mathbb{R}^+$ such that $a_1 = r(a_2 + v_2)$ for some $v_2 \in \mathcal{V}_2$

$\Rightarrow a_1 - ra_2 \in \mathcal{V}_2$
$\Rightarrow a_1 \in ra_2 + \mathcal{V}_2$ or $\frac{1}{r}a_1 \in a_2 + \mathcal{V}_2$

This implies that a_2 is at a shortest distance from $\frac{1}{r}a_1$ but a_2 and $\frac{1}{r}a_1$ are both in $\mathcal{V}_2{}^{\perp}$. Therefore $\frac{1}{r}a_1 = a_2$. Also as $\|a_1\| = \|a_2\| = 1$, we get $r = 1$, *i.e.* $a_1 = a_2$.
Next consider $\mathcal{Y}_1 \subseteq \mathbb{R}^+\mathcal{Y}_2$. For $y_1 \in \mathcal{Y}_1$, there exists $c > 0$ such that $y_1 = cy_2$ for some $y_2 \in \mathcal{Y}_2$. Also $y_1{}^* a_1 = 1$. Therefore $cy_2{}^*.a_1 = 1$. But $a_1 = a_2$. Thus $cy_2{}^*.a_2 = 1$ which gives $c = 1$ as $y_2{}^* a_2 = 1$. This implies that $y_1 = y_2 \in \mathcal{Y}_2$ and so $\mathcal{Y}_1 \subseteq \mathcal{Y}_2$. Similarly, we can show that $\mathcal{Y}_2 \subseteq \mathcal{Y}_1$ and hence $\mathcal{Y}_1 = \mathcal{Y}_2$. This further gives that $\mathcal{V}_1 = \mathcal{V}_2$ as $\mathcal{V}_1 = \mathcal{Y}_1{}^{\perp}$ and $\mathcal{V}_2 = \mathcal{Y}_2{}^{\perp}$. Hence if $\mathcal{X}_1\mathcal{Y}_1{}^* = \mathcal{X}_2\mathcal{Y}_2{}^*$, we get that $\mathcal{X}_1 = \mathcal{X}_2$, $\mathcal{Y}_1 = \mathcal{Y}_2$ and $a_1 = a_2$.
We now give some definitions that will be used to obtain the final canonical form up to similarity.

Definition 3.2 By an admissible similarity transformation, we mean an operator (matrix) permuting the basis or a diagonal operator (matrix) with positive entries or a combination of both.

Definition 3.3 Two positive vectors $a = (a_i)$ and $b = (b_i)$ in $\mathbb{R}^{\boldsymbol{n}}$ are said to be equivalent if and only if there exists an admissible similarity L such that $La = b$.
It is evident that admissible similarity defines an equivalence relation and this decomposes the positive vectors in $\mathbb{R}^{\boldsymbol{n}}$ into equivalence classes.
Let a_1 and a_2 be two positive vectors in the same equivalence class, giving rise to the maximal indecomposable nonnegative rank-one semigroups of r-potent matrices in $M_n(\mathbb{R})$, *viz.* $\mathcal{X}_1\mathcal{Y}_1{}^*$ and $\mathcal{X}_2\mathcal{Y}_2{}^*$ respectively, where

$\mathcal{X}_1 = \{a_1 + \mathcal{V}_1\} \cap \mathbb{R}^n{}_+$
$\mathcal{Y}_1 = \{y \in \mathbb{R}^n{}_+ : y^* a_1 = 1\} \cap \mathcal{V}_1{}^{\perp}$

And

$\mathcal{X}_2 = \{a_2 + \mathcal{V}_2\} \cap \mathbb{R}^n{}_+$
$\mathcal{Y}_2 = \{y \in \mathbb{R}^n{}_+ : y^* a_2 = 1\} \cap \mathcal{V}_2{}^{\perp}$

Then the admissible similarity, say L with $La_1 = a_2$ is such that

$$L(\mathcal{X}_1\mathcal{Y}_1{}^*)L^{-1} = \mathcal{X}_2\mathcal{Y}_2{}^* \qquad (3)$$

Write $\mathcal{X}_1{}' = L\mathcal{X}_1 = \{La_1 + L\mathcal{V}_1\} \cap \mathbb{R}^n{}_+ = \{a_1{}' + \mathcal{V}_1{}'\} \cap \mathbb{R}^n{}_+$, where $a_1{}' = La_1$, $\mathcal{V}_1{}' = L\mathcal{V}_1$ and $\mathcal{Y}_1{}' = (L^{-1})^*\mathcal{Y}_1$, then it is easy to verify that

$\mathcal{Y}_1{}' = \{y \in \mathbb{R}^n{}_+ : y^*.La_1 = 1\} \cap (L\mathcal{V}_1)^{\perp}$.

Then (3) can be written as $\mathcal{X}_1{}'\mathcal{Y}_1{}'^* = \mathcal{X}_2\mathcal{Y}_2{}^*$.
Hence, we can conclude by saying that if $La_1 + L\,\mathcal{V}_1$ is the canonical representation of $\mathcal{X}_1{}'$, in other words, La_1 is the unique vector of norm 1 in $(L\mathcal{V}_1)^{\perp}$ closest to every vector in $La_1 + L\,\mathcal{V}_1$, then by the case considered above, we shall obtain $La_1 = a_2, L\mathcal{X}_1 = \mathcal{X}_2$ and $L\mathcal{Y}_1 = \mathcal{Y}_2$. Otherwise, express $\mathcal{X}_1{}'$ in its canonical form with vector β, say and obtain

$\beta = a_2$, $L\mathcal{X}_1 = \mathcal{X}_2$, $L\mathcal{Y}_1 = \mathcal{Y}_2$ and $L\,\mathcal{V}_1 = \mathcal{V}_2$.

Conclusion

It will be interesting and worthwhile to study the geometric characterization of maximal constant rank semigroups of nonnegative r-potent operators in infinite dimensions. For this, we shall consider $\mathcal{X}$ to be a separable, locally compact Hausdorff space and μ a Borel measure on $\mathcal{X}$ such that $\mu(X) < \infty$ and let $\mathcal{L}^2(\mathcal{X})$ denote the Hilbert space of (equivalent classes) complex-valued measurable and square integrable functions on $\mathcal{X}$. Conditions leading to decomposability of single nonnegative r-potent operator defined on $\mathcal{L}^2(\mathcal{X})$ and semigroups of nonnegative r-potents operations in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ have been studied in [6]. As a particular case, A. Marwaha in her paper [7] has independently proved results of decomposability of semigroups of nonnegative idempotents, called 'bands' (an idempotent is an r-potent with $r = 2$) in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ and also studied the structure in entirety of maximal constant-rank bands in view of their decomposability under the condition of fullness.
In continuation of the results proved in this paper, it will be noteworthy to study the conditions which yield a geometric characterization for maximal semigroups of nonnegative r-potent operators of constant rank on the infinite-dimensional Hilbert Space $\mathcal{L}^2(\mathcal{X})$.

References

1. Marwaha A. Decomposability and structure of non-negative bands in (R), Linear Algebra and its Application. 1999;291:63-82.
2. Thukral R.S, Marwaha A. Decomposability of non-negative r-potent matrices, International Journal of Pure and Applied Mathematics. 2014;94(5):705-724.
3. Rajavi H, Rosenthal P. Simultaneous Triangularization, Springer-Verlag, Canada; c2000.
4. Marcus M, Minc H. A survey of matrix theory and matrix inequalities, Allyn and Bacon; c1964.

5. McCloskey JP. Characterizations of r-potent matrices, Mathematical Proceedings of the Cambridge Philosophical Society, 1984;96:213-222.
6. Thukral RS, Marwaha A. Decomposability of r-potent operators on L2(X), International Journal of Pure and Applied Mathematics. 2015;100(1):29-52.
7. Marwaha A. Decomposability and structure of nonnegative bands in infinite dimensions, Journal of Operator Theory. 2002;47(1):37-61.
8. Marwaha A. A geometric characterization of nonnegative bands, Canadian Mathematical Bulletin, Cambridge University Press. 2004;47(2):257-263.
9. Marwaha A, Thukral RS. Structure of decomposable semigroups of nonnegative $\boldsymbol{r}$-Potent matrices in Mn(R), International Journal of Statistics and Applied Mathematics, Maths. 2022;7(4): 215-225.