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Pavan Kumar Jha
Research Centre for Applied Science Technology (RECAST), T. U., Kirtipur, Nepal

Khageshwar Mandal
Department of Mathematics and Statistics, Padma Kanya Multiple Campus, T. U., Nepal

Corresponding Author:
Pavan Kumar Jha
Research Centre for Applied Science Technology (RECAST), T. U., Kirtipur, Nepal

Some fixed point results under generalized distance

Pavan Kumar Jha and Khageshwar Mandal

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Abstract
In this paper, the historical account of distances in metric space have introduced with generalize metric space which have applications to obtaining the solutions of various new important problems in nonlinear analysis.

Keywords: Fixed point theory, w-distance, Caristi's fixed point theorem, and Takahashi minimization theorem, etc.

Introduction

In 1996, *w-distance* introduced a notion of on a metric space by Kada *et al*. 1996 [4] and using this notion to improve the Caristi's fixed point theorem, and Ekeland variational principle. Using this notion of w-distance, Suzuki and Takahashi [9] have introduced notions of single-valued, Takahashi minimization theorem and multivalued weakly contractive i.e. *w*-contractive mappings and proved fixed point results for two or more mappings.

There are two useful lemma as follows

Lemma 1.1 [4]: Let (X,d) be a metric space and let p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$converging to 0.
Then, the following statements hold for every $x,y,z \in X$:

(a) If $p(x_n,y) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$for any $n \in N$, then $y = z$, in particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$;
(b) If $p(x_n,y_n) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for any $n \in N$, Then $y_n$converges to z;
(c) If $p(x_n,x_m) \leq \alpha_n$for any $n,m \in N$ With $m > n$, then $\{x_n\}$ is a Cauchy sequence;
(d) If $p(y,x_n) \leq \alpha_n$ for any $n \in N$, then x_n is a Cauchy sequence.

Lemma 1.2: Let (X,d) be a metric space and let p be a w-distance on X. Let K be a closed subset of X. Suppose that there exists $u \in X$
Such that $p(u,u) = 0$.
Then, $p(u,K) = 0$ if and only if $u \in K$,
Where $p(u,K) = inf_{y\in K}\, p(u,y)$.
In 1996, Kada, Suzuki and Takahashi improved Caristi's fixed point theorem as follows;

Theorem 1.1: [4] Let (X,d) be a complete metric space and p be a *w-distance* on X. Then, each Caristi mapping T on X with respect to p has a fixed point $x_n \in X$ and $p(x_0,x_0) = 0$.
In 1996, Suzuki and Takahashi improved the Banach Contraction Principle as follows;

Theorem 1.2: [5] Let (X,d) be a complete metric space and p be a *w-distance* on X. Then, each contraction mapping T on X with respect to p has a unique fixed point $x_0 \in X$ and $p(x_o,x_0) = 0$.

Theorem 1.3: Let (X, d) be a metric space. Then, X is complete if and only if every contraction mapping T on X with respect to p has a fixed point in X.
In 1998, Ume established the result as follows;

Theorem 1.4 [11]: Let (X, d) be a complete metric space and let p be a *w-distance* on X. Let $T : X \to X$ be a mapping such that for a fixed constant $h < 1$ and for all $x, y \in X, p(T(x), T(y)) \leq max\{p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx)\}$,
And

$inf\{p(x, u) + p(x, Tx) : x \in X\} > 0$, for every $u \in X$. With $u \neq f(u)$.

Then, T has a unique fixed point $x_0 \in X$ and $p(x_0, x_0) = 0$.
In 1998, Shioji, Suzuki and Takahashi proved the following lemma;

Lemma 1.3 [6]: Let (X, d) be a metric space with metric, let p be a *w-distance* on X, let T be a mapping of X into itself and let u be a point in X

Such that $\lim_{m,n\to\infty} p(T^m u, T^n u) = 0$. Then for every $x \in X$, $\lim_{k\to\infty} p(T^k u, x) = 0$. And

$\lim_{m,n\to\infty} p(x, T^k u) = 0$ exist. Moreover, let β and γ be functions from X to $[0, \infty)$ defined by

$\beta(x) = \lim_{k\to\infty} p(T^k u, x) = 0$. And $\gamma(x) = \lim_{k\to\infty} p(x, T^k u) = 0$.

Then the following hold:
(i) β is lower semicontinuous on X;
(ii) for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\beta(x) \leq \delta$ and $\beta(y) \leq \delta$ imply $d(x, y) \leq \varepsilon$. In particular, the set $\{x \in X : \beta(x) = 0\}$ consists of at most one point;
(iii) The functions q_0 and q_1 from $X \times X$ *to* $[0, \infty)$ defined by

$q_0(x, y) = \beta(x) + \beta(y)$ And $q_1(x, y) = \gamma(x) + \beta(y)$ are *w-distances* on X.

In 1998, Shioji, Suzuki and Takahashi proved the following theorem by using lemma 1.2;

Theorem 1.5 [6]: Let (X, d) be a metric space.

Then $WC_1(X) = WC_0(X) = WK_1(X) = WK_0(X) \subset WC_2(X) = WK_2(X)$.

In 2010, Iemoto, Takahashi and Yingtaweesittikul proved the following result by the help of above lemma;

Theorem 1.6 [3]: Let (X, d) be a metric space and let p be a *w-distance* on X such that $p(x, x) = 0$ for all $x \in X$.
Let T be a *p-contractively* non spreading mapping of X into itself.
Then T is in $WC_0(X)$.

In 2011, Hasegawa, Komiya and Takahashi proved the following theorem,

Theorem 1.7 [2]: Let (X, d) be a complete metric space and let T be a mapping of X into itself. Suppose that there exist a real number r with $0 \leq r < 1$ And an element $x \in X$ such that

$\{T^n x\}$ is bounded and $\mu_n d(T^n x, Ty) \leq r\mu_n d(T^n x, y), \forall y \in X$ For some mean μ on l^∞.

Then, the following results hold
(i) T has a unique fixed point u in X;
(ii) For every $z \in X$, the sequence $\{T^n z\}$ converges to u.

In 2013, Takahashi, Wong and Yao proved following fixed point a theorem by using lemma 1.2 with *w-distances* in complete metric spaces. Which is direct consequence of above theorem;

Theorem 1.8 [10]: Let (X, d) be a complete metric space, let $p \in w_0(X)$ and let $\{x_n\}$ be a sequence in X such that $\{p(x_n, x)\}$ is bounded for some $x \in X$. Let T be a mapping of X into itself. Suppose that there exist a real number $r \in [0,1)$ and a mean μ on ℓ^∞. Such that $\mu_n p(x_n, Ty) \leq r\mu_n p(x_n, y), \forall y \in X$. Then, the following hold:
(i) T has a unique fixed point u in X;
(ii) For every $z \in X$, the sequence $\{T^n z\}$ converges to u.

In 2013, the following theorems established by Takahashi, Wong and Yao;

Theorem 1.9 [10]: Let (X, d) be a complete metric space and let p be a *w-distance* on X. Let $T : X \to X$ be a *p-contractive* mapping, i.e., there exists a real number r with $0 \leq r < 1$ such that $p(Tx, Ty) \leq rp(x, y)$ For all $x, y \in X$.

Then, the following results hold

(i) T has a unique fixed point u in X;
(ii) For every $z \in X$, the sequence $\{T^n z\}$ converges to u.

Theorem 1.10: Let (X, d) be a complete metric space and let p be a*w-distance* on X. Let $T \in WK_1(X)$, i.e., there exists $\alpha \in [0,1/2)$ such that

$p(Tx, Ty) \leq \alpha\{p(Tx, x) + p(Ty, y)\}$ for all $x, y \in X$.

Then, the following hold

(i) T has a unique fixed point u in X;
(ii) For every $z \in X$, the sequence $\{T^n z\}$ converges to u.

Theorem 1.11: Let (X, d) be a complete metric space and let p be a*w-distance* on X such that $p(x, x) = 0$ for all $x \in X$. Let $T : X \to X$ be *p-contractively* nonspreading, i.e., there exists a real number γ with $0 \leq \gamma < 1/2$ such that $p(Tx, Ty) \leq \gamma\{p(Tx, y) + p(x, Ty)\}$ for all $x, y \in X$. Then, the following results hold:
(i) T has a unique fixed point u in X;
(ii) For every $z \in X$, the sequence $\{T^n z\}$ converges to u.

Lemma 1.4: Let (X, d) be a complete metric space and let p be a symmetric *w-distance* on X.
Let $T : X \to X$ be a *p-contractively* generalized hybrid mapping. Then T has a fixed point in X if and only if $\{p(T^n x, x)\}$is bounded for some $x \in X$. In this case, the following results hold:
(i) T has a unique fixed point u in X;
(ii) For every $z \in X$, the sequence $\{T^n z\}$ converges to u.

In general, a τ *-distance* p does not necessarily satisfy $p(x, x) = 0$. The metric d is a τ*-distance* on X. Each *w-distance* on a metric space X is also a τ*- distance* on X.Using the concept of τ *-distance*, Susuki improved the Banach Contraction Principle and Caristi's fixed point theorem as follows;

Theorem 1.12 [7]: Let (X, d) be a complete metric space and p be aτ *–distance* on X. Then, each contraction mapping T on X with respect to p has a unique fixed point $x_0 \in X$ and $p(x_0, x_0) = 0$.

Theorem 1.13 [7]: Let (X, d) be a complete metric space and p be aτ *-distance* on X. Then, each Caristi's mapping T on X with respect to p has a fixed point $x_0 \in X$ and $p(x_0, x_0) = 0$.
On the basis of this result, Suzuki obtained generalized Caristi's fixed point theorem as follows;

Theorem 1.14 [8]: Let (X, d) be a complete metric space, p be aτ *-distance* on X and
Let g $: X \to (0, \infty)$ be a function such that for some$r > 0$, $sup\{g(x) : x \in X, \psi(x) \leq \inf_{z \in X} \psi(z) + r\} < \infty$, where $\psi : X \to (0, \infty)$ is a lower semicontinuous function.
Let $T : X \to X$ be a map such that for each $x \in X$, $p(x, T(x)) \leq g(x)(\psi(x) - \psi(T(x)))$.
Then, there exists $x_0 \in X$ such that $T(x_0) = x_0$ *and* $p(x_0, x_0) = 0$.

Conclusion

Now a days, distances in metric space have been introduced and generalize metric space and its applications to obtaining the solutions of several new important problems in nonlinear analysis.

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