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Some properties and applications of half-Cauchy exponential extension distribution

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Abstract
In this article, we have proposed a new distribution having three parameters using half Cauchy family of distribution called Half-Cauchy Exponential extension Distribution. The statistical properties as well as different characteristics like the hazard rate function (HRF), the probability density function (PDF), the cumulative distribution function (CDF), quantile function , skewness, and kurtosis of the proposed distribution are discussed. The parameters of the proposed distribution are estimated by using the least-square estimation (LSE), Cramer-Von-Mises (CVM) and maximum likelihood estimators (MLE) methods. All the computations are performed in R programming software. To assess the application of the proposed distribution, a real lifetime data set is analyzed and performed the goodness-of-fit. It is found that the Half -Cauchy exponential extension distribution performed better as compared to some existing distributions.

Keywords: Exponential extension distribution, half cauchy distribution, estimation, hazard function

1. Introduction

Probability distribution is vital part of probability analysis and applied statistics. The exponential distribution (ED) has a major application in modeling of survival and reliability data. Exponential distribution is particular case of the gamma distribution as well as geometric distribution, having memory less property. In addition it can be applied for the study of the Poisson point processes. During last few decades, the ED is extensively used as base line distribution to create more flexible families of probability distribution. Many researchers have done modifications and extensions of the ED to generate new distributions. Many of the researchers such as, Nadarajah and Kotz (2006) [18] has defined beta exponential. Generalized exponential distribution was proposed by [Gupta & Kundu (2007)] [11]. Similarly, Abouammoh and Alshingiti (2009) [2] presented the reliability based estimation of the generalized inverted ED. Beta generalized exponential (Barreto - Souza *et al*., 2010) [4], Exponential extension (EE) distribution by (Kumar, 2010) [12], Kumaraswamy exponential (Cordeiro and de Castro, 2011) [7], an extension of the ED by Nadarajah & Haghighi, 2011 [19]. Lemonte, A. J. (2013) [13] has defined a new exponential-type probability model. These models can have different types of failure rate function. These rate functions may be increasing, decreasing, and constant, bathtub-shaped and upside-down bathtub failure rate function. Gamma EE was presented by (Ristic and Balakrishnan, 2012). The transmuted EE distribution by (Merovci, 2013) [16]. Gomez *et al*. (2014) [10] have introduced a new generalization of the ED. The exponentiated exponential geometric (Louzada *et al*., 2014) [14] and Kumaraswamy transmuted ED (Afify *et al*., 2016) [3]. Mahdavi and Kundu (2017) have developed a new method for extension of the distribution by applying the ED.

Abdulkabir & Ipinyomi, (2020) [1] have introduced the Type II half-logistic exponentiated exponential distribution. Chaudhary and Kumar (2020) [5] has defined the extension of ED called the half logistic exponential extension distribution. Another extension of ED was presented by (Chaudhary *et al*. 2020) [6] named the truncated Cauchy power– exponential distribution.

In this section we have taken the half-Cauchy distribution which is the special case of the Cauchy distribution by breakdown of curve on origin such that it will consider only non-negative values. [Shaw (1995)] has used the half Cauchy distribution having a heavy-tailed, as an alternative to model spreading distances, since it can forecast more recurrent long-distance spreading events. In addition, the half-Cauchy distribution is also used by [Paradis *et al*. 2002] [21] to model ringing data on tits having two species in Ireland and Britain. Let us consider that a no- negative random variable X follows half-Cauchy distribution and its cumulative distribution function (CDF) is expressed as

$$R(x;\theta)=\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\theta}\right),\ \ x>0,\theta>0. \tag{1}$$

PDF corresponding to (3.1.1) is,

$$r(x;\theta)=\frac{2}{\pi}\left(\frac{\theta}{\theta^2+x^2}\right),\ \ x>0,\theta>0. \tag{2}$$

Here, we are interested to generate new model based on half-Cauchy family. The generating family of distribution was given by [Zografas & Balakrishnan, 2009] and CDF of family of distribution was obtained as

$$F(x)=\int_0^{-\ln[1-G(x)]} r(t)\ dt, \tag{3}$$

Here $G(x)$ is the CDF of any baseline distribution and $r(t)$ is the PDF of any distribution. The family of half-Cauchy with CDF is defined taking $r(t)$ which is the PDF of earlier defined half-Cauchy distribution defined in (2) as,

$$\begin{aligned}F(x)&=\int_0^{-\ln[1-G(x)]}\frac{2}{\pi}\frac{\theta}{\theta^2+t^2}\ dt \\ &=\frac{2}{\pi}\arctan\left(-\frac{1}{\theta}\ln\left[1-G(x)\right]\right);\ x>0,\theta>0\end{aligned} \tag{4}$$

The PDF corresponding to (4) can be expressed as,

$$f(x)=\frac{2}{\pi\theta}\frac{g(x)}{1-G(x)}\left[1+\left\{-\frac{1}{\theta}\log\left[1-G(x)\right]\right\}^2\right]^{-1};x>0,\theta>0 \tag{5}$$

The rest part of study is organized in different sections. In Section 2, model analysis where, the half Cauchy exponential extension distribution is defined and also we have presented the properties of the proposed model like survival, probability density, hazard rate, cumulative distribution, and cumulative hazard functions. Quantiles, measures of variability, measure of symmetry of the data based on quantiles along with measure of normality based on octiles. In Section 3, parameter estimation techniques are used. The application of the proposed model to real data set is presented in Section 4. In section 5 model comparisons is discussed and finally some concluding explanations are entered in Section 6.

2. Model Analysis

The Half Cauchy Exponential Extension (HCEE) distribution:

The extension of the exponential distribution was defined by Nadarajah & Haghighi (2011) [19]. We named it as exponential extension distribution. The CDF of exponential extension distribution is defined as

$$F(x;\beta,\lambda)=1-e^{\{1-(1+\lambda x)^{\beta}\}};\quad \beta,\lambda>0,\ x>0 \tag{6}$$

The PDF corresponding to (6) is

$$f(x;\beta,\lambda)=\beta\lambda(1+\lambda x)^{\beta-1}e^{\{1-(1+\lambda x)^{\beta}\}};\quad \beta,\lambda>0,\ x>0 \tag{7}$$

Substituting (6) and (7) in (4) and (5) we obtained CDF of half-Cauchy exponential extension distribution, which is defined as,

$$F(x)=\frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}\left\{1-(1+\lambda x)^{\beta}\right\}\right\},\ \ x>0,\ \beta,\lambda,\theta>0. \tag{8}$$

And the PDF of half-Cauchy exponential extension can be expressed as

$$f(x)=\frac{2}{\pi}\frac{\beta\lambda}{\theta}(1+\lambda x)^{\beta-1}\left[1+\left[-\frac{1}{\theta}\left\{1-(1+\lambda x)^{\beta}\right\}\right]^{2}\right]^{-1};\ x>0,\ \beta,\lambda,\theta>0. \tag{9}$$

Survival function

The survival function of HCEE is,

$$S(x)=1-\frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}\left\{1-(1+\lambda x)^{\beta}\right\}\right\},\ \ x>0,\ \beta,\lambda,\theta>0. \tag{10}$$

Hazard Rate Function

Hazard rate function of HCEE model with parameters (β,λ,θ) is,

$$h(x)=\frac{f(x)}{1-F(x)}=\frac{2}{\pi}\frac{\beta\lambda}{\theta}(1+\lambda x)^{\beta-1}\left[1-\frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}\left\{1-(1+\lambda x)^{\beta}\right\}\right\}\right]^{-1}$$

$$\left[1+\left[-\frac{1}{\theta}\left\{1-(1+\lambda x)^{\beta}\right\}\right]^{2}\right]^{-1};\ 0<x<\infty \tag{11}$$

Reverse hazard function of HCEE

The reverse hazard function of HCEE is as

$$h_{rev}(x)=\frac{f(x)}{F(x)}$$

$$=\frac{\beta\lambda}{\theta}(1+\lambda x)^{\beta-1}\left[\arctan\left\{-\frac{1}{\theta}\left\{1-(1+\lambda x)^{\beta}\right\}\right\}\right]^{-1}\left[1+\left[-\frac{1}{\theta}\left\{1-(1+\lambda x)^{\beta}\right\}\right]^{2}\right]^{-1} \tag{12}$$

Cumulative Hazard Function:

The chf of the HCEE (β,λ,θ) is defined as

$$H(x)=\int_{-\infty}^{x}h(y)dy\ =-\log[1-F(x)]$$

$$=-\log\left[1-\frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}\left\{1-(1+\lambda x)^{\beta}\right\}\right\}\right]. \tag{13}$$

The various shapes of pdf and hazard rate function of HCEE (β,λ,θ) with various possible parameter values are displayed in Figure 1.

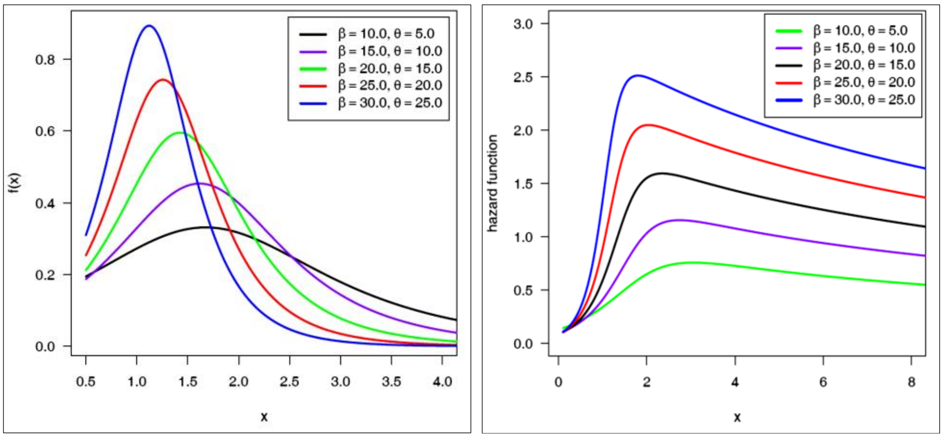

Fig 1: Graphs of PDF (left side) and hazard function (right side) for fixed λ, and various values of β and θ.

Quantile function

Let *X* is a non-negative random variable with CDF $F_X(x)$ then quantile function can be defined as,

$$Q_X(p) = F_X^{-1}(p)$$

$$Q_x(p) = \frac{1}{\lambda}\left[\left\{1+\theta\tan\left(\frac{\pi p}{2}\right)\right\}^{1/\beta} - 1\right];\quad 0 < p < 1 \tag{14}$$

The random deviate generation for the HCEE(β, λ, θ) is,

$$x = \frac{1}{\lambda}\left[\left\{1+\theta\tan\left(\frac{\pi u}{2}\right)\right\}^{1/\beta} - 1\right];\quad 0 < u < 1 \tag{15}$$

Skewness and Kurtosis

The Quantile coefficient of measure of symmetry is,

$$S_B = \frac{Q(0.75) - 2Q(0.5) + Q(0.25)}{Q(0.75) - Q(0.25)}$$

and

Expression for testing normality by using octiles of the data is

$$K - Moors = \frac{Q(0.875) - Q(0.625) - Q(0.125) + Q(0.375)}{Q(3/4) - Q(1/4)}$$

3. Parameter Estimation

3.1. Maximum Likelihood Estimation

In this subsection, we have presented MLE method of the HCEE model to estimate the parameters. Let $\underline{x} = (x_1, \ldots, x_n)$ be sample with n item from HCEE(β, λ, θ). Log likelihood function of HCEE is

$$\ell = n\ln(2/\pi) + n\ln\left(\frac{\beta\lambda}{\theta}\right) + (\beta-1)\sum_{i=1}^{n}\ln B(x_i) - \sum_{i=1}^{n}\ln\left[1+\left[-\frac{1}{\theta}\left\{1-B(x_i)^{\beta}\right\}\right]^2\right] \tag{16}$$

First order derivative of (16) with respect to β, λ and θ are

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\ln B(x_i) + \frac{2}{\theta^2}\sum_{i=1}^{n}B(x_i)^{\beta}\ln B(x_i)\left\{1-B(x_i)^{\beta}\right\}\left[1+\left[-\frac{1}{\theta}\left\{1-B(x_i)^{\beta}\right\}\right]^2\right]^{-1}$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} + (\beta-1)\sum_{i=1}^{n}\frac{x_i}{B(x_i)} + \frac{\beta}{\theta^2}\sum_{i=1}^{n}x_i B(x_i)^{\beta-1}\left\{1-B(x_i)^{\beta}\right\}\left[1+\left[-\frac{1}{\theta}\left\{1-B(x_i)^{\beta}\right\}\right]^2\right]^{-1}$$

$$\frac{\partial\ell}{\partial\theta} = -\frac{n}{\theta} - \frac{2}{\theta}\sum_{i=1}^{n}\left[-\frac{1}{\theta}\left\{1-B(x_i)^{\beta}\right\}\right]^2\left[1+\left[-\frac{1}{\theta}\left\{1-B(x_i)^{\beta}\right\}\right]^2\right].$$

Where $B(x_i) = 1 + \lambda x_i$

Equating $\frac{\partial\ell}{\partial\beta} = \frac{\partial\ell}{\partial\lambda} = \frac{\partial\ell}{\partial\theta} = 0$ and solving simultaneously for the β, λ and θ we can obtain the MLE of the HCEE(β, λ, θ). Normally, it is not possible to solve above non-linear equations, so with the aid of suitable programming language, we can solve them easily. Let $\underline{\Theta} = (\hat{\beta}, \hat{\lambda}, \hat{\theta})$ is MLE of parameters $\underline{\Theta} = (\beta, \lambda, \theta)$ then resulting asymptotic normality will be $\left(\underline{\Theta} - \underline{\Theta}\right) \rightarrow N_3\left[0, \left(I\left(\underline{\Theta}\right)\right)^{-1}\right]$ and corresponding Fisher's information matrix is,

$$I(\underline{\Theta}) = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial \beta^2}\right) & E\left(\frac{\partial^2 l}{\partial \beta \partial \lambda}\right) & E\left(\frac{\partial^2 l}{\partial \beta \partial \theta}\right) \\ E\left(\frac{\partial^2 l}{\partial \beta \partial \lambda}\right) & E\left(\frac{\partial^2 l}{\partial \lambda^2}\right) & E\left(\frac{\partial^2 l}{\partial \lambda \partial \theta}\right) \\ E\left(\frac{\partial^2 l}{\partial \beta \partial \theta}\right) & E\left(\frac{\partial^2 l}{\partial \lambda \partial \theta}\right) & E\left(\frac{\partial^2 l}{\partial \theta^2}\right) \end{pmatrix}$$

Since $\underline{\Theta}$ is unknown so it is worthless that the MLE has an asymptotic variance $\left(I\left(\underline{\Theta}\right)\right)^{-1}$. Using the estimated value of parameters one can approximate the asymptotic variance and observed fisher information matrix $O(\underline{\Theta})$ can be used as an estimate of $I\left(\underline{\Theta}\right)$ in form of hessian matrix H as,

$$O(\underline{\Theta}) = -\begin{pmatrix} \frac{\partial^2 l}{\partial \hat{\beta}^2} & \frac{\partial^2 l}{\partial \hat{\beta} \partial \hat{\lambda}} & \frac{\partial^2 l}{\partial \hat{\beta} \partial \hat{\theta}} \\ \frac{\partial^2 l}{\partial \hat{\beta} \partial \hat{\lambda}} & \frac{\partial^2 l}{\partial \hat{\lambda}^2} & \frac{\partial^2 l}{\partial \hat{\theta} \partial \hat{\lambda}} \\ \frac{\partial^2 l}{\partial \hat{\beta} \partial \hat{\theta}} & \frac{\partial^2 l}{\partial \hat{\theta} \partial \hat{\lambda}} & \frac{\partial^2 l}{\partial \hat{\theta}^2} \end{pmatrix}_{\left|(\hat{\beta}, \hat{\lambda}, \hat{\theta})\right.} = -H\left(\underline{\Theta}\right)_{\left|\left(\underline{\Theta} = \underline{\Theta}\right)\right.}$$

By using Newton-Raphson for optimization of the log- likelihood function we can get variance and covariance matrix as,

$$\left[-H\left(\underline{\Theta}\right)_{\left|\left(\underline{\Theta} = \underline{\Theta}\right)\right.}\right]^{-1} = \begin{pmatrix} \mathrm{Var}(\hat{\beta}) & \mathrm{Cov}(\hat{\beta}, \hat{\lambda}) & \mathrm{Cov}(\hat{\beta}, \hat{\theta}) \\ \mathrm{Cov}(\hat{\beta}, \hat{\lambda}) & \mathrm{Var}(\hat{\lambda}) & \mathrm{Cov}(\hat{\lambda}, \hat{\theta}) \\ \mathrm{Cov}(\hat{\beta}, \hat{\theta}) & \mathrm{Cov}(\hat{\lambda}, \hat{\theta}) & \mathrm{Var}(\hat{\theta}) \end{pmatrix} \tag{17}$$

Now, using asymptotic normality of MLEs, approximate *100(1-γ) %* C.I for β, λ and θ of HCEE (β, λ, θ) taking $Z_{\gamma/2}$ as the upper percentile is,

$$\hat{\beta} \pm Z_{b/2}\sqrt{\mathrm{Var}(\hat{\beta})},, \ \hat{\lambda} \pm Z_{b/2}\sqrt{\mathrm{Var}(\hat{\lambda})} \text{ and } \hat{\theta} \pm Z_{b/2}\sqrt{\mathrm{Var}(\hat{\theta})}.$$

3.2. Least-Square Estimation

Use of least-square estimation for estimating the unknown parameters β, λ and θ of HCEE distribution minimizes the function

$$B\left(X; \beta, \lambda, \theta\right) = \sum_{i=1}^{n}\left[F(X_i) - \frac{i}{n+1}\right]^2 \tag{18}$$

with respect to β, λ and θ.

Let us consider $\{X_1, X_2, \ldots, X_n\}$ be a random sample with n units from ordered random variable $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ having CDF as $F(X_i)$ the LSE of β, λ and θ can be calculated by minimizing the relation (3.3.4) with respect to β, λ and θ.

$$B\left(X; \beta, \lambda, \theta\right) = \sum_{i=1}^{n}\left[\frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}\left\{1 - (1 + \lambda x_i)^{\beta}\right\}\right\} - \frac{i}{n+1}\right]^2 \tag{19}$$

Differentiating the relation (3.3.4) with respect to β, λ and θ as,

$$\frac{\partial B}{\partial \beta}=\frac{2}{\theta}\sum_{i=1}^{n}\left[\left\{\frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}\left\{1-(1+\lambda x_i)^{\beta}\right\}\right\}-\frac{i}{n+1}\right\}(1+\lambda x_i)^{\beta}\log(1+\lambda x_i)\right]$$

$$\left[1+\left\{\frac{1}{\theta}\left\{1-(1+\lambda x_i)^{\beta}\right\}\right\}^2\right]^{-1}.$$

$$\frac{\partial B}{\partial \lambda}=\frac{2\beta}{\theta}\sum_{i=1}^{n}\left[\left\{\frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}\left\{1-(1+\lambda x_i)^{\beta}\right\}\right\}-\frac{i}{n+1}\right\}x_i(1+\lambda x_i)^{\beta-1}\right]$$

$$\left[1+\left\{\frac{1}{\theta}\left\{1-(1+\lambda x_i)^{\beta}\right\}\right\}^2\right]^{-1}$$

$$\frac{\partial B}{\partial \lambda}=\frac{2\beta}{\theta}\sum_{i=1}^{n}\left[\left\{\frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}\left\{1-(1+\lambda x_i)^{\beta}\right\}\right\}-\frac{i}{n+1}\right\}x_i(1+\lambda x_i)^{\beta-1}\right]$$

$$\left[1+\left\{\frac{1}{\theta}\left\{1-(1+\lambda x_i)^{\beta}\right\}\right\}^2\right]^{-1}$$

We can minimize following relation to get weighted least square estimators as

$$B(X;\beta,\lambda,\theta)=\sum_{i=1}^{n}w_i\left[F(X_{(i)})-\frac{i}{n+1}\right]$$

where w_i are weight and is given as

$$w_i=\frac{1}{Var(X_{(i)})}=\frac{(n+1)^2(n+2)}{i(n-i+1)}$$

By minimizing relation (20) we can estimate unknown parameters β, λ and θ using weighted least square method with respect to β, λ and θ.

$$B(X;\beta,\lambda,\theta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}\left\{1-(1+\lambda x_i)^{\beta}\right\}\right\}-\frac{i}{n+1}\right]^2 \tag{20}$$

3.3. Cramer-Von-Mises estimation

By minimizing relation (21) we can get estimated values of unknown parameters β, λ and θ using Cramer-Von-Mises technique of estimation

$$C(X;\beta,\lambda,\theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_{i:n}\mid\beta,\lambda,\theta)-\frac{2i-1}{2n}\right]^2$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}\left\{1-(1+\lambda x_i)^{\beta}\right\}\right\}-\frac{2i-1}{2n}\right]^2 \tag{21}$$

Differentiation of (21) with respect to β, λ and θ will give following results,

$$\frac{\partial C}{\partial \beta}=\frac{2}{\theta}\sum_{i=1}^{n}\left\{\frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}\left\{1-(1+\lambda x_i)^{\beta}\right\}\right\}-\frac{2i-1}{2n}\right\}$$

$$(1+\lambda x_i)^{\beta}\log(1+\lambda x_i)\left[1+\left\{\frac{1}{\theta}\left\{1-(1+\lambda x_i)^{\beta}\right\}\right\}^{2}\right]^{-1}$$

$$\frac{\partial C}{\partial \lambda}=\frac{2\beta}{\theta}\sum_{i=1}^{n}\left\{\frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}\left\{1-(1+\lambda x_i)^{\beta}\right\}\right\}-\frac{2i-1}{2n}\right\}x_i(1+\lambda x_i)^{\beta-1}$$

$$\left[1+\left\{\frac{1}{\theta}\left\{1-(1+\lambda x_i)^{\beta}\right\}\right\}^{2}\right]^{-1}$$

$$\frac{\partial C}{\partial \theta}=\frac{-2\beta}{\theta^2}\sum_{i=1}^{n}\left\{\frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}\left\{1-(1+\lambda x_i)^{\beta}\right\}\right\}-\frac{i}{n+1}\right\}\left\{1-(1+\lambda x_i)^{\beta}\right\}$$

$$\left[1+\left\{\frac{1}{\theta}\left\{1-(1+\lambda x_i)^{\beta}\right\}\right\}^{2}\right]^{-1}$$

After solving above non-linear equations $\frac{\partial C}{\partial \beta}=0, \frac{\partial C}{\partial \lambda}=0$ and $\frac{\partial C}{\partial \theta}=0$ we can get the CVM estimators of the parameters.

4. Application to Real Dataset

In this part of the study, applicability and validity of HCEE model taking a real data set is explained. The data is the breaking stress of carbon fibers having 50 mm length (GPa). The data was used by (Nichols & Padgett, 2006) [20]. The data is used previously in many researchers. The dataset is given as 0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82,2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90, 2.56 Here we have plotted the graph of profile log-likelihood function for the parameters β, λ and θ in Figure 2 and the profile graphs show that ML estimates can be uniquely calculated.

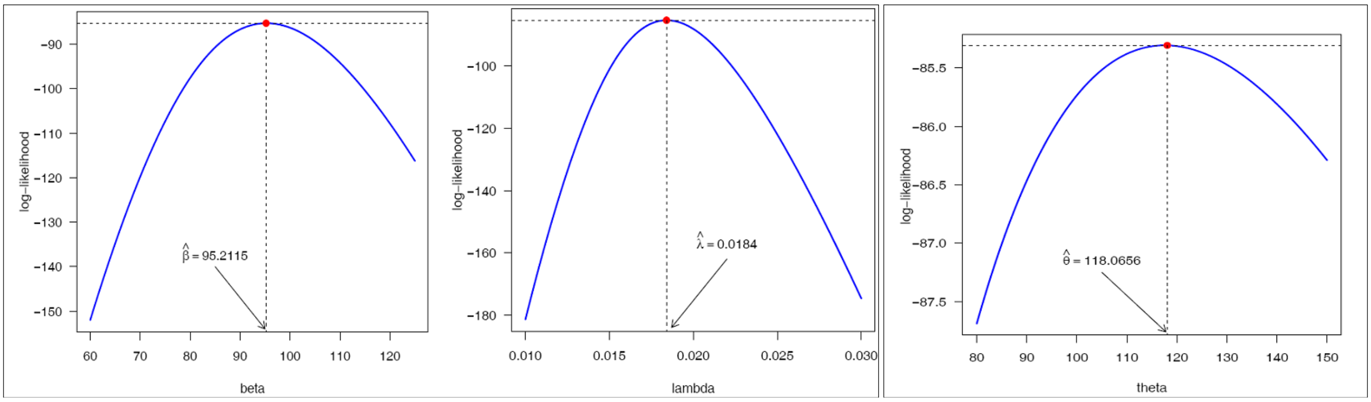

Fig 2: Profile plot of log-likelihood function of the parameters β, λ and θ.

Using R software (R Core Team, 2020) having optim () function, calculations of MLEs of HCEE model is done by maximizing the likelihood function (3.3.1) [Mailund, 2017] [14]. We got the value of Log-Likelihood as l = -85.3059. Estimated parameters value with their standard errors (SE) for β, λ and θ and 95% asymptotic confidence interval (ACI) in Table 1.

Table 1: MLE and SE for β, λ and θ and 95% ACI

Parameter	MLE	SE	95%ACI
beta	95.2125	5.5210	(84.3888, 106.0312)
lambda	0.01840	0.0013	(0.0159, 0.0209000)
theta	118.065	3.8100	(110.6324, 125.567)

We have also plotted the P-P plot (left) and Q-Q plot (right) for testing the normality of model

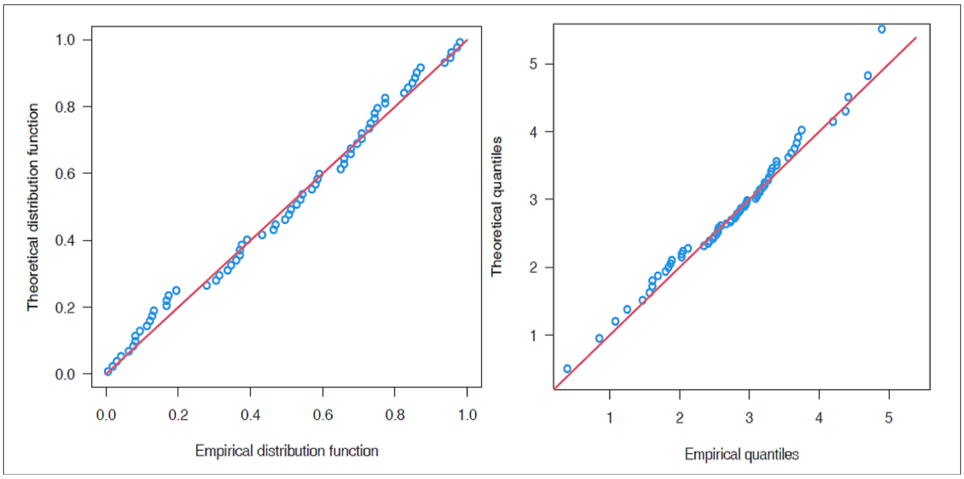

Fig 3: The P-P plot in left panel and Q-Q plot in right panel of the HCEE distribution.

Using above estimated parameter values by all three methods we have calculated log-likelihood and AIC criterion and are presented in Table 2.

Table 2: Estimated parameters, log-likelihood, and AIC of HCEE

Method	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	LL	AIC
MLE	95.2115	0.0184	118.0656	-85.3062	176.6123
LSE	87.6167	0.0194	101.8119	-85.3523	176.7046
CVE	92.0534	0.0190	115.6711	-85.3100	176.6201

For testing the goodness of fit of the model, the KS, W and A^2 statistic along with respective p-value of MLE, LSE and CVE estimates are presented in Table 3

Table 3: The KS, W and A^2 statistics and p-value

Method	*KS(p-value)*	*W(p-value)*	*A^2(p-value)*
MLE	0.0691(0.9110)	0.0488(0.8853)	0.3530(0.8934)
LSE	0.0675(0.9247)	0.0499(0.8783)	0.3515(0.8947)
CVE	0.0670(0.9281)	0.0488(0.8848)	0.3447(0.9009)

Figure 4 exhibits the Q-Q plot and Histogram and the density function of fitted distributions of estimation methods using MLE, LSE and CVM of the proposed model HCEE distribution.

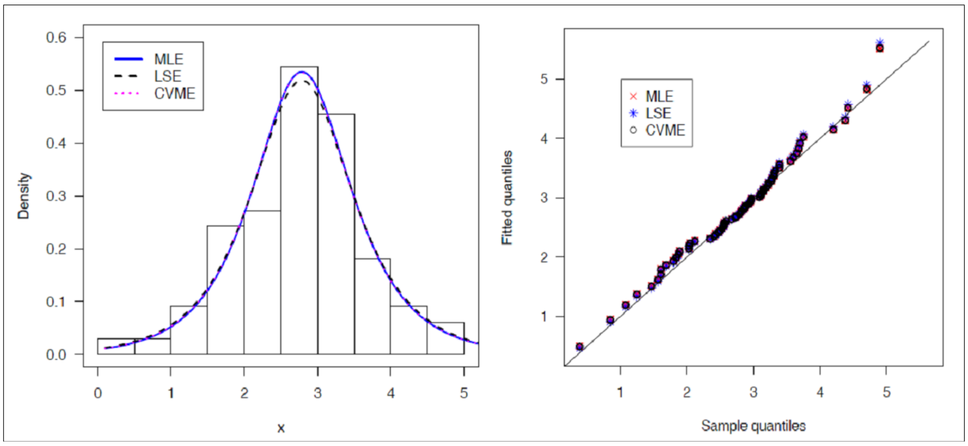

Fig 4: The Q-Q plot in right panel and Histogram & the density function of fitted distributions in left panel of MLE, LSE and CVM of HCEE distribution.

5. Model Comparison

Here we have compared the applicability of the model HCEE with other models used by researchers using the same data set. Here we have chosen five distributions to compare the potentiality of the HCEE. Five distributions considered are;

Exponentiated Exponential Poisson (EEP), Generalized Exponential Extension, (GEE), Power Cauchy (PC) distribution, Generalized gompertz distribution**,** and Exponentiated Weibull Distribution (EW). We also displayed the histogram of the dataset, graph for goodness-of-fit of HCEE model along with models taken in considerations in left panel of Figure 5. In right panel of the graph, empirical CDF and fitted CDF are shown. From plotting it is observed that HCEE distribution fits the dataset better than other models taken in consideration.

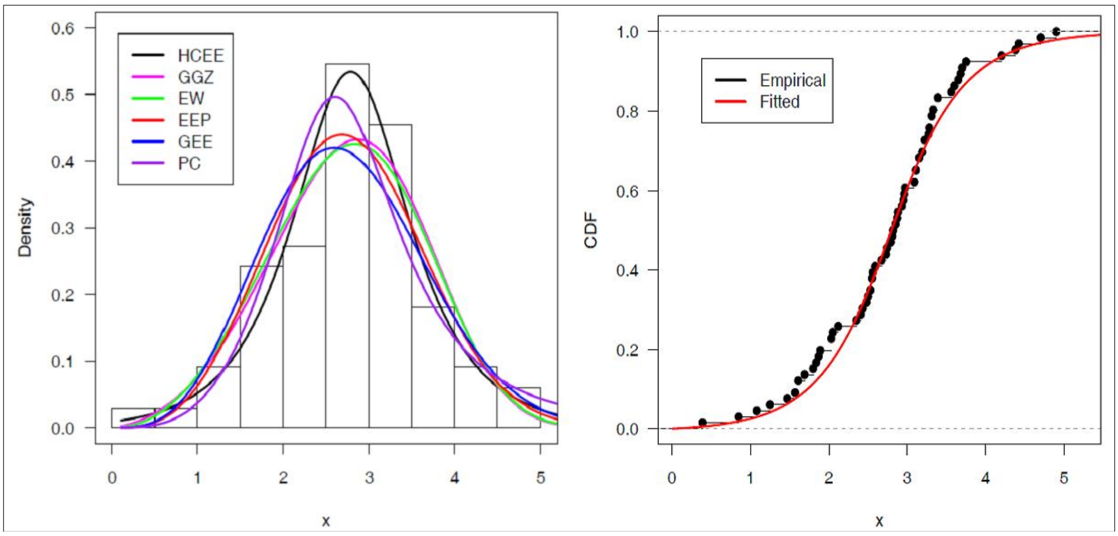

Fig 5: The Histogram and the density function plot of fitted distributions in left panel & Empirical cdf versus estimated cdf plot in right panel of HCEE distribution.

We have illustrated the Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC), and Hannan-Quinn information criterion (HQIC) for the evaluation of the applicability of HCEE distribution. These values are presented in Table 4. Validation criteria obtained clearly show that model HCEE has fewer values than the value of other considered models. This suggests that proposed model fit data better than other models taken in considerations.

Table 4: Log-likelihood (LL), AIC, BIC, CAIC and HQIC

Model	LL	AIC	BIC	CAIC	HQIC
HCEE	-85.3062	176.6123	183.1813	176.9994	179.2080
GGZ	-85.6858	177.3716	183.9406	177.7587	179.9673
EW	-85.9447	177.8894	184.4584	178.2765	180.4851
EEP	-86.6899	179.3798	185.9488	179.7669	181.9755
GEE	-87.2704	180.5408	187.1098	180.9279	183.1365
PC	-90.5126	185.0252	189.4045	185.2157	186.7557

Testing of goodness of fit of fitted model and comparing it with other model is essential part of statistical modeling. For comparing the goodness-of-fit of the HCEE distribution with competing distributions, we have calculated the values of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) test statistics. These test statistic values are displayed in Table 5. It is found that the HCEE distribution has lesser values of the test statistics and higher *p*-value in every methods of goodness of fit. This concludes that the HCEE distribution fits real data set more consistently compared to other models taken in consideration.

Table 5: The KS, AD, CVM statistics and their corresponding p-value

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
HCEE	0.0691(0.9110)	0.0488(0.8853)	0.3530(0.8934)
GGZ	0.0833(0.7498)	0.0715(0.7443)	0.4457(0.8020)
EW	0.0809(0.7805)	0.0813(0.6861)	0.4846(0.7620)
EEP	0.0895(0.6662)	0.1014(0.5796)	0.5657(0.6804)
GEE	0.1096(0.4065)	0.1530(0.3812)	0.7816(0.4940)
PC	0.0963(0.5731)	0.1246(0.4782)	1.0733(0.3207)

Models for comparison

Following are the PDF of the models taken in consideration for comparisons of the proposed models

(i). Exponentiated Exponential Poisson (EEP)

The probability density function of EEP (Ristić & Nadarajah, 2014) [22] is

$$f(x)=\frac{\alpha\beta\lambda}{\left(1-e^{-\lambda}\right)}e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1}\exp\left\{-\lambda\left(1-e^{-\beta x}\right)^{\alpha}\right\}$$

(ii). Generalized Exponential Extension (GEE)

The pdf of GEE was introduced by (Lemonte, 2013) [13]. Distribution has upside down bathtub shaped hazard rate function having three parameters as

$$f_{GEE}(x)=\alpha\beta\lambda\,(1+\lambda x)^{\alpha-1}\,exp\left\{1-(1+\lambda x)^{\alpha}\right\}\left[1-exp\left\{1-(1+\lambda x)^{\alpha}\right\}\right]^{\beta-1};x>0$$

(iii). Power Cauchy distribution

Power Cauchy (PC) distribution was given by (Rooks *et al*., 2010) .PDF of the model is

$$f_{PC}(x;\alpha,\lambda)=\frac{2\alpha}{\pi x}\left(\frac{x}{\lambda}\right)^{\alpha}\left\{1+\left(\frac{x}{\lambda}\right)^{2\alpha}\right\}^{-1};x>0,\alpha>0,\lambda>0$$

(iv). Generalized gompertz distribution

The probability density function of Gompertz distribution (El-Gohary *et al*., 2013) [9] having three parameters as α, λ and θ is

$$f_{GGZ}(x)=\theta\lambda e^{\alpha x}e^{-\frac{\lambda}{\alpha}\left(e^{\alpha x}-1\right)}\left[1-\exp\left(-\frac{\lambda}{\alpha}\left(e^{\alpha x}-1\right)\right)\right]^{\theta-1};\ \lambda,\theta>0,\alpha\geq 0,x\geq 0$$

(v). Exponentiated Weibull (EW)Distribution

A new family of distributions, namely the Exponentiated Exponential Distribution was introduced by Gupta *et al*. (1998) [26]. EW is a generalization of the Exponentiated Exponential Family as well as the Weibull Family. EW distribution also has a very nice physical interpretation.

$$f_{EW}(x)=\alpha\beta\lambda x^{\beta-1}\exp\left(-\alpha x^{\beta}\right)\left(1-\exp\left(-\alpha x^{\beta}\right)\right)^{\lambda-1}$$

We have also plotted the graph of the empirical CDF and the estimated fitted CDF to illustrate the goodness of fit of the HCEE model in figure 6.

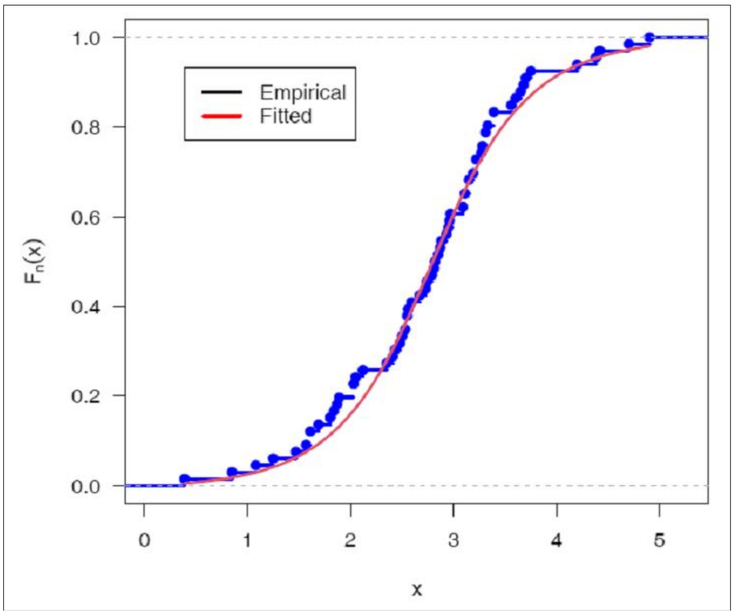

Fig 6: The plot of empirical CDF and estimated CDF of HCEE model.

6. Conclusion

Here, a new distribution named half Cauchy exponential extension distribution is presented. A detailed analysis of some basic statistical characteristics of new model is done. We have derived some important expression for the hazard rate, survival, the quantile function etc. We have also calculated summary of the data as mean, median skewness and kurtosis etc and are displayed. For estimating the unknown parameters, three important and more effective methods of estimation as MLE, LSE and CVME are used. Here we found that the CVM method is relatively better than MLE and LSE methods. PDF curve of the proposed model HCEE has different shaped density curve for different values of parameters showing that curve is positively skewed having increasing-decreasing hazard rate curve. Here we have also considered a real dataset and compared the applicability and suitability of the proposed model with some other models introduced by the earlier researchers using same dataset and same methods of parameter estimation. It is found that the proposed model fits data more adequately than the other models taken in considerations. We also considered different methods of model validation techniques to test the validity showing that proposed model has lesser values of information criteria than competing models. So we can conclude that model HCEE will play significant role in data analysis and new model formulations.

7. References

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