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Rajnish Kumar
Research Scholar, Department of Mathematics, J.P. University, Chapra, Bihar, India

Corresponding Author:
Rajnish Kumar
Research Scholar, Department of Mathematics, J.P. University, Chapra, Bihar, India

Recurrence relation for the polynomial set $A_n(x, y)$

Rajnish Kumar

Abstract
In this paper, an attempt has been made to express some recurrence relations for the generalized hypergeometric polynomial set $A_n(x, y)$ followed by important and interesting particular cases. Out of these particular results some of them stand for well known polynomials and some of thus are believed to be new. These recurrence relations are of at most important for mathematician, scientists and engineers.

Keywords: Recurrence relation, generalized hypergeometric polynomial, appell functions, generating function

1. Introduction

Singh, Jai Ram and Singh, D.N. [3] defined the polynomial set $A_n(x, y)$ by means of the generating function.

$$\left(1-\lambda_2 x^e y^{-e_4} t^{e_1}\right)^{\lambda_1} F\left[\begin{matrix}(A_p);(C_u)(E_r);\\(B_q);(D_v)(F_w);\end{matrix}\left(\lambda_3 x^{e_3} t\right),\lambda_4 y^{-e_2} t^{e_2}\right]$$

$$=\sum_{n=0}^{\infty} A_{n,e;e_1;e_2;e_3;e_4;(B_q);(D_v);(F_w)}^{\lambda_1;\lambda_2;\lambda_3;\lambda_4;(A_p);(C_u);(E_r)}(x,y)\,t^n \qquad \ldots (1.1)$$

Where λ1; λ2; λ3; λ4; are real and e, $e1$,$e2$,$e3$ and $e4$ are positive integers.
The left hand side of (1.1) contains generalized Appell function of two variables by Burchnall and Chaundy [1]. The polynomial set contains a number of parameters, for simplicity, the polynomial set denoted by $A_n(x, y)$, where n is the order of the polynomial set.
After little simplification (1.1) gives

$$A_n(x,y)=\sum_{\substack{h,s\geq 0\\ e_1h+e_2s\leq n}} \frac{\overline{\Delta}(h,s)}{(n-e_1h-e_2s)!} \qquad \ldots (1.2)$$

Where

$$\overline{\Delta}(h,s)=\frac{\left[(A_p)\right]_{n-e_1h-(e_2-1)s}\left[(C_u)\right]_{n-e_1h-e_2s}\left[(E_r)\right]_s(\lambda_1)_4\left(\lambda_2 x^e\right)^h\left(\lambda_3 x^{e_3}\right)^{n-e_1h-e_2s}\lambda_4^s}{\left[(B_q)\right]_{n-e_1h-(e_2-1)s}\left[(D_v)\right]_{n-e_1h-e_2s}\left[(F_w)\right]_s\, h!\quad s!\quad y^{e_1h+e_4s}}$$

2. Notations

A. (i) $(m) = 1, 2, 3, \ldots\ldots n-1, n$.
(ii) $(ap) = a1, a2, a3, \ldots\ldots ap$.
(iii) $(ap , i) = a1, a2, \ldots ai–1, ai+1 \ldots\ldots ap$.
(iv) $(ap, (s)) = as +1, as +2 \ldots\ldots ap$.

B. (i) $[(ap)] = a1, \; a2, \; \ldots\ldots ap.$

(ii) $$\left[\left(a_p\right)\right]_n = \left(a_1\right)_n \left(a_2\right)_n \left(a_3\right)_n \ldots\ldots \left(a_p\right)_n .$$

(iii) $$\left[\left(a_p\right)+m\left(p\right)\right] = \prod_{i=1}^{p}\left(a_i+m_i\right).$$

C. (i) $$\Delta\left(a,b\right) = \frac{b}{a}, \frac{b+1}{a}, \frac{b+2}{a}, \ldots\ldots \frac{b+a-1}{a}.$$

(ii) $$\Delta\left(m;a_p\right) = \left(\frac{a_i+r-1}{m}\right) \begin{matrix} r=1,m \\ i=1,p \end{matrix}$$

(iii) $\Delta(a; b \pm c \pm d) = \Delta(a; b + c + d)$ □$(a; b + c - d)$, $\Delta(a; b - c + d)$, $\Delta(a; b - c - d)$.

D. (i) $$\Delta_k\left[a\left(i\right);b\right] = \left(\frac{b}{a}\right)_k \left(\frac{b+1}{a}\right)_k \ldots\ldots \left(\frac{b+a-2}{a}\right)_k$$

(ii) $$\Delta_k\left[m;\left(a_p\right)\right] = \prod_{i=1}^{p}\prod_{r=1}^{m}\left(\frac{a;+r-1}{m}\right)_k$$

E. (i) $$\Gamma\left[\left(a_p\right)\right] = \prod_{i=1}^{p}\Gamma\left(a_i\right).$$

(ii) $$\Gamma\left[a+\frac{\left(m\right)}{m}\right] = \prod_{r=1}^{m}\Gamma\left(a+\frac{r}{m}\right).$$

(iii) $$\Gamma\left[\left(a;b\right)\right] = \prod_{r=1}^{a}\Gamma\left(\frac{b+r-1}{a}\right).$$

(vi) $$\Gamma\left[\Delta\left(m;\left(a_p\right)\right)\right] = \prod_{i=1}^{p}\prod_{r=1}^{m}\left(\frac{a_i+r-1}{m}\right).$$

F. (i) $$\Gamma_*\left(a\pm b\right) = \Gamma\left(a+b\right)\Gamma\left(a-b\right).$$

(ii) $$\Gamma_{**}\left(a+1\right) = \Gamma\left(a+b\right)\Gamma\left(a-b\right).$$

3. Recurrence Relations

Now, we shall derive few recurrence relations for the polynomials set $A_n(x, y)$.
From (1.2), we have

$$A_n\left(x^{g_1},y^{g_2}\right) = \sum_{h=0}^{\left[\frac{n}{e_1}\right]}\sum_{s=0}^{\left[\frac{n}{e_2}\right]} \frac{\left[\left(A_p\right)\right]_{n-e_1h-\left(e_2-1\right)s}\left[\left(C_u\right)\right]_{n-e_1h-e_2s}}{\left[\left(B_q\right)\right]_{n-e_1h-\left(e_2-1\right)s}\left[\left(D_v\right)\right]_{n-e_1h-e_2s}}$$

$$\times \frac{\left[\left(E_r\right)\right]_s \; \left(\lambda_1\right)_h \; \lambda_2^h \; \left(\lambda_3 x^{e_3}\right)^{n-e_1h-e_2s} \; \lambda_4^s}{\left[\left(F_w\right)\right]_s \; h! \; s! \; y^{e_1s+e_4h}\left(n-e_1h-e_2s\right)!} \qquad \ldots (3.1)$$

Differentiating (3.1) with respect to x, we arrive at

$$\frac{\partial}{\partial x}A_n\left(x^{g_1},y^{g_2}\right) = e_3g_1\sum_{h=0}^{\left[\frac{n-1}{e_1}\right]}\sum_{s=0}^{\left[\frac{n-1}{e_2}\right]} \frac{\left[\left(A_p\right)\right]_{n-e_1h-\left(e_2-1\right)s}}{\left[\left(B_q\right)\right]_{n-e_1h-\left(e_2-1\right)s}}$$

$$\times\frac{\left[(C_u)\right]_{n-e_1h-e_2s}\left[(E_r)\right]_s(\lambda_1)_h\lambda_2^h\ \lambda_4^s\ \lambda_3^{n-e_1h-e_2s}\left(x^{e_3g_1}\right)^{n-1-e_1h-e_2s}}{\left[(D_v)\right]_{n-e_1h-e_2s}\ \left[(F_w)\right]_s\ h!\quad s!\quad y^{e_4g_1h+e_2g_2s}(n-1-e_1h-e_2s)!} \qquad \ldots (3.2)$$

Now $A^{\lambda_1;\lambda_2;\lambda_3;\lambda_4;(A_p)+1,(C_u)+1,(E_r)}_{n-1,e;e_1;e_2;e_3;e_4;(B_q+1),(D_v)+1,;(F_w)}\left(x^{g_1},y^{g_2}\right)$

$$=\sum_{h=0}^{\left[\frac{n-1}{e_1}\right]}\sum_{s=0}^{\left[\frac{n-1}{e_2}\right]}\frac{\left[(A_p)+1\right]_{n-1-e_1h-(e_2-1)s}\left[(C_u)+1\right]_{n-1-e_1h-e_2s}}{\left[(B_q)+1\right]_{n-1-e_1h-(e_2-1)s}\left[(D_v)+1\right]_{n-1-e_1h-e_2s}}$$

$$\times\frac{\left[(E_r)\right]_s(\lambda_1)_h\lambda_2^h\ \lambda_4^s\ \lambda_3^{n-e_1h-e_2s-1}\left(x^{e_3g_1}\right)^{n-1-e_1h-e_2s}}{\left[(F_w)\right]_s\ h!\quad s!\quad \lambda_3y^{g_1e_4h+e_2g_2s}(n-1-e_1h-e_2s)!}$$

$$\therefore\ \frac{\left[(A_p)\right]\left[(C_u)\right]\lambda_3x^{e_3g_1-1}e_3g_1}{\left[(B_q)\right]\left[(D_v)\right]}A^{(A_p)+1,(C_u)+1}_{n-1,(B_q)+1,(D_v)+1}\left(x^{g_1},y^{g_2}\right)$$

$$=\sum_{h=0}^{\left[\frac{n-1}{e_1}\right]}\sum_{s=0}^{\left[\frac{n-1}{e_2}\right]}\frac{\left[(A_p)\right]_{n-e_1h-(e_2-1)s}\left[(C_u)\right]_{n-e_1h-e_2s}\left[(E_r)\right]_s}{\left[(B_q)\right]_{n-e_1h-(e_2-1)s}\left[(D_v)\right]_{n-e_1h-e_2s}\left[(F_w)\right]_s}$$

$$\times\frac{(\lambda_1)_h\lambda_2^h\ \lambda_4^s\ \lambda_3^{n-e_1h-e_2s-1}\left(x_3^{e_3g_1}\right)^{n-1-e_1h-e_2s}}{h!\quad s!\quad y^{e_4g_1h+e_2g_2s}(n-1-e_1h-e_2s)!} \qquad \ldots (3.3)$$

from (3.2) and (3.3), we arrive at

$$\frac{\partial}{\partial x}A_n\left(x^{g_1},y^{g_2}\right)=\frac{\left[(A_p)\right]\left[(C_u)\right]\lambda_3e_3g_1\ x^{e_3g_1-1}}{\left[(B_q)\right]\left[(D_v)\right]}A^{(A_p)+1,(C_u)+1}_{n-1,(B_q)+1,(D_v)+1}\left(x^{g_1},y^{g_2}\right) \qquad \ldots (3.4)$$

where $n \geq 1$

Corollary: On putting $g1 = 1 = g2$; we achieve

$$\frac{\partial}{\partial x}A_n(x,y)=\frac{\left[(A_p)\right]\left[(C_u)\right]\lambda_3e_3\ x^{e_3-1}}{\left[(B_q)\right]\left[(D_v)\right]}A^{(A_p)+1,(C_u)+1}_{n-1,(B_q)+1,(D_v)+1}(x,y) \qquad \ldots (3.5)$$

$n \geq 1$

The relation (3.4) can be thrown into the form

$$\left(x^{1-e_3g_1}\frac{\partial}{\partial x}\right)A_n(x,y)=\lambda_3e_3g_1\frac{\left[(A_p)\right]\left[(C_u)\right]}{\left[(B_q)\right]\left[(D_v)\right]}A^{(A_p)+1,(C_u)+1}_{n-1,(B_q)+1,(D_v)+1}\left(x^{g_1},\ y^{g_2}\right) \qquad \ldots (3.6)$$

Differentiating (3.6) successively *m*-times we achieve

$$\left(x^{1-e_3 g_1}\frac{\partial}{\partial x}\right)^m A_n\left(x^{g_1},y^{g_2}\right)=\frac{(\lambda_3 e_3 g_1)^m\left[(A_p)\right]_m\left[(C_u)\right]_m}{\left[(B_q)\right]_m\left[(D_v)\right]_m}$$

$$\times A^{(A_p)+m,(C_u)+m}_{n-m,(B_q)+m,(D_v)+m}\left(x^{g_1},y^{g_2}\right) \qquad \ldots (3.7)$$

where $n \geq m$

Corollary: On putting $g1 = 1 = g2$; we have

$$\left(x^{1-e_3}\frac{\partial}{\partial x}\right)^m A_n(x,y)$$

$$=\frac{(\lambda_3 e_3)^m\left[(A_p)\right]_m\left[(C_u)\right]_m}{\left[(B_q)\right]_m\left[(D_v)\right]_m} A^{(A_p)+m,(C_u)+m}_{n-m,(B_q)+m,(D_v)+m}(x,y) \qquad \ldots (3.8)$$

Where $n \geq m$ and m is a non-negative integer

Particular Cases of (3.5)

(i) On making the substitution $e = u = 0 = r$; $v = 1 = w = e3 = y$; $\lambda_3=\frac{1}{2}=\lambda_4$;

$B1 = 1 + \alpha 1$, $F1 = 1 + \beta$ and writting $\frac{x-1}{x+1}$ for x, we achieve

$$(x+1)\frac{d}{dx}P_n^{(\alpha,\beta)}(x)=nP_n^{(\alpha,\beta)}(x)+(n+\beta)P_{n-1}^{(\alpha+1,\beta)}(x)$$

Where $P_n^{(\alpha,\beta)}(x)$ are the Jacobi Polynomials [2].

(ii) If we put $u = 0 = r = e$; $v = 1 = w = e3 = y$; $\lambda_3=\frac{1}{2}=\lambda_4$; $D1 = 1 + \beta$; $w1 = 1 + \alpha$ and $\frac{x+1}{x-1}$ for x, we arrive at

$$(x-1)\frac{d}{dx}P_n^{(\alpha,\beta)}(x)=nP_n^{(\alpha,\beta)}(x)+(n+\alpha)P_{n-1}^{(\alpha,\beta+1)}(x)$$

Where $P_n^{(\alpha,\beta)}(x)$ are Jacobi Polynomials

(iii) For the values $u = 0 = r = e$; $v = 1 = w = e3 = y$; $\lambda_3=\frac{1}{2}=\lambda_4;\ D_1=\lambda+\frac{1}{2}=F_1$ and writting $\frac{x+1}{x-1}$ for x, we have

$$\frac{d}{dx}C_n^{(\lambda)}(x)=2\lambda C_{n-1}^{(\lambda+1)}(x)$$

Where $C_n^{\lambda}(x)$ are the Gegenbauer Polynomials

(iv) If we take $p = 0 = q = u = v = r = e$; $w = 1 = e3 = y = \square 3; \square e2 = q$; $F_1=\lambda+\frac{1}{2},\lambda_4=\frac{1}{4}$ and $\frac{x}{\sqrt{x^2-1}}$, for x, we get

$$(x^2-1)\frac{d}{dx}C_n^{(\lambda)}(x)=nxC_n^{(\lambda)}(x)-(2\lambda+n-1)C_{n-1}^{(\lambda)}(x)$$

Where $C_n^{\lambda}(x)$ are the Gegenbauer Polynomials

(v) On making the substitution $p = 0 = q = u = v = e = r = w$; $e3 = 1 = y$; $\lambda 3 = 2 = e2$; $\lambda 4 = -1$ we achieve

$$\frac{d}{dx} H_n(x) = 2nH_{n-1}(x)$$

where $Hn(x)$ are the Hermite Polynomials [2].

(vi) If we take $p = 0 = q = u = v = r = e$; $w = 1 = y = e3 = \square 3 \square = F1$; $\lambda_4 = \frac{1}{4}$; $e3 = 2$ and $\frac{x}{\sqrt{x^2-1}}$ for x, we get

$$\left(x^2-1\right)\frac{d}{dx} P_n(x) = n\left[xP_n(x) - P_{n-1}(x)\right]$$

where $Pn(x)$ are the Legendre Polynomials.

Particular Cases of (3.8)

(i) On making the substitution $e = u = 0 = r$; $v = 1 = w = e3 = y$; $\lambda_3 = \frac{1}{2} = \lambda_4$;

$B1 = 1 + \alpha 1$, $F1 = 1 + \beta$ and writting $\frac{x-1}{x+1}$ for x, we achieve

$$(x-1)\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = nP_n^{(\alpha,\beta)}(x) + (n+\alpha)P_{n-1}^{(\alpha,\beta+1)}(x)$$

where $P_n^{(\alpha,\beta)}(x)$ are the Jacobi Polynomials [2].

(ii) If we put $u = 0 = r = e$; $v = 1 = w = e3 = y$; $\lambda_3 = \frac{1}{2} = \lambda_4$; $D1 = 1 + \square$; $w1 = 1 + \square\square$ and $\frac{x+1}{x-1}$ for x, we arrive at

$$\frac{d^m}{dx^m}\left\{(1-x)^m P_n^{(\alpha,\beta)}\frac{x+1}{x-1}\right\} = \frac{(1+\beta)_m (1-x)^{n-m}}{(1+\beta)_{n-m}} P_{n-m}^{(\alpha+m,\beta)}\left(\frac{1+x}{1-x}\right)$$

where $P_n^{(\alpha,\beta)}(x)$ are Jacobi Polynomials.

(iii) For the values $u = 0 = r = e$; $v = 1 = w = e3 = y$; $\lambda_3 = \frac{1}{2} = \lambda_4$; $D_1 = \lambda + \frac{1}{2} = F_1$ and writting $\frac{x+1}{x-1}$ for x, we have

$$\frac{d^m}{dx^m}\left\{(x-1)^n C_n^{(\lambda)}\left(\frac{x+1}{x-1}\right)\right\} = \frac{(2\lambda)_n x^{n-m}}{(n-m)!} {}_2F_1\left[\begin{matrix} -n+m, \frac{1}{2}-\lambda-n; \\ \frac{1}{x} \\ \lambda+\frac{1}{2}; \end{matrix}\right]$$

Where $C_n^{\lambda}(x)$ are the Gegenbauer Polynomials

(iv) If we take $p = 0 = q = u = v = r = e$; $w = 1 = e3 = y = \lambda 3$; $e2 = q$; $F_1 = \lambda + \frac{1}{2}, \lambda_4 = \frac{1}{4}$ and $\frac{x}{\sqrt{x^2-1}}$, for x, we get

$$\frac{d^m}{dx^m}\left\{\left(x^2-1\right)^{\frac{n}{2}} C_n^{(\lambda)}\left(\frac{x}{\sqrt{x^2-1}}\right)\right\}=\frac{(2\lambda)_n\left(x^2-1\right)^{\frac{n-m}{2}}}{(2\lambda)_{n-m}} C_{n-m}^{(\lambda)}\left(\frac{x}{\sqrt{x^2-1}}\right)$$

where $C_n^{\lambda}(x)$ are the Gegenbauer Polynomials.

(v) On making the substitution $p = 0 = q = u = \mathrm{v} = e = r = w$; $e3 = 1 = y$; □3 = 2 = $e2$; $\lambda 4 = -1$ we achieve

$$\frac{d^m}{dx^m} H_n(x)=\frac{2^m n!}{(n-m)!} H_{n-m}(x)$$

where $Hn(x)$ are the Hermite Polynomials [2].

(vi) If we take $p = 0 = q = u = \mathrm{v} = r = e$; $w = 1 = y = e3 = \lambda 3 = F1$; $\lambda_4=\frac{1}{4}$; $e3 = 2$ and $\frac{x}{\sqrt{x^2-1}}$ for x, we get

$$\frac{d^m}{dx^m}\left\{\left(x^2-1\right)^{\frac{n}{2}} P_n\left(\frac{x}{\sqrt{x^2-1}}\right)\right\}=\frac{n!\left(x^2-1\right)^{\frac{n-m}{2}}}{(n-m)!} P_{n-m}\left(\frac{x}{\sqrt{x^2-1}}\right)$$

Where $Pn(x)$ are the Legendre Polynomials

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