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R Subathira
Department of Statistics, Annamalai University, Annamalai Nagar, Tamil Nadu, India

N Vijayshankar
Assistant Professor, Department of Statistics, Annamalai University, Annamalai Nagar, Tamil Nadu, India

Corresponding Author:
R. Subathira
Department of Statistics, Annamalai University, Annamalai Nagar, Tamil Nadu, India

A stochastic approach using Garima distribution

R Subathira and N Vijayshankar

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Abstract
In this article we propose a power Garima distribution which includes Garima distribution as particular case. Its statistical properties including behavior of its probability density function for varying values of parameters, moments, hazard rate function and mean residual life function has been discussed. The estimation of the parameters of the distribution has been discussed using maximum likelihood estimation.

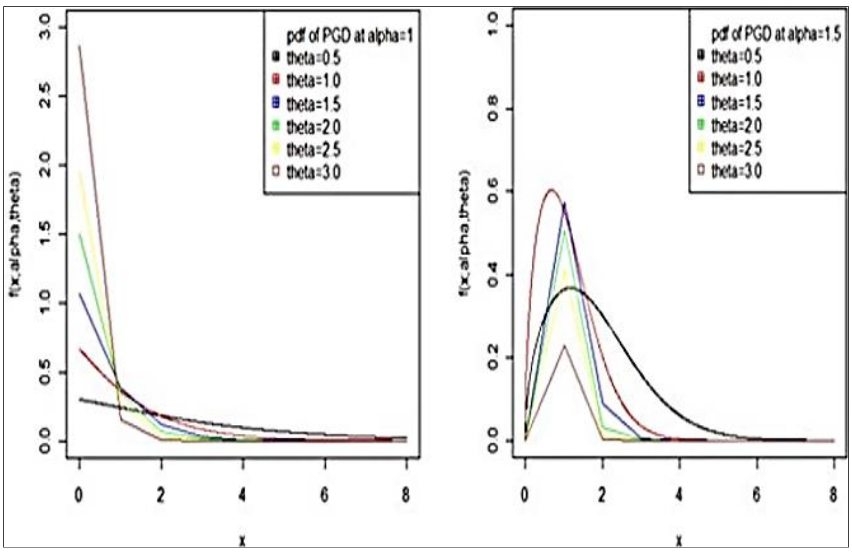
Keywords: Garima distribution, moments; hazard rate function; mean residual life function, maximum likelihood estimation

Introduction
In this research paper a continuous distribution named "Garima distribution" has been suggested for modelling data from real life time database. The important properties including its shape, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations have been discussed. The condition under which Garima distribution is over dispersed, uniform dispersed, and under-dispersed are presented along with other one parameter continuous distributions. The estimation of its parameter has been discussed using maximum likelihood estimation and method of moments. The application of the proposed distribution has been explained using a numerical example.

Power Garima Distribution (PGD)
The probability density function of power Garima distribution (PGD) is given by

$$f(x;\theta,\alpha)=\frac{\alpha\,\theta}{\theta+2}\left(1+\theta+\theta x^{\alpha}\right)x^{\alpha-1}e^{-\theta x^{\alpha}}\;;\; x>0,\theta>0,\;\alpha>0 \tag{7.1}$$

and the cumulative distribution function of power Garima distribution is given by



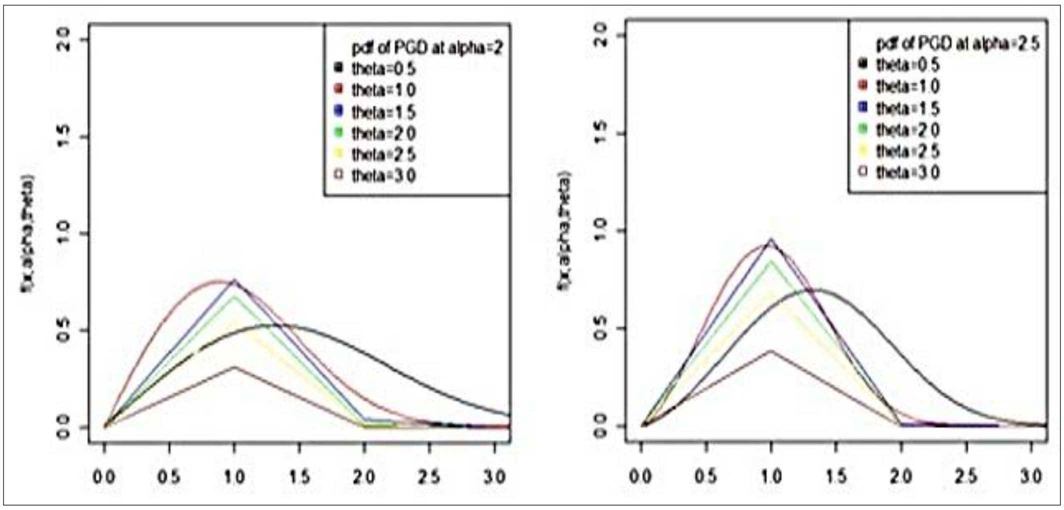


Fig 1: Graphs of pdf of PGD for varying values

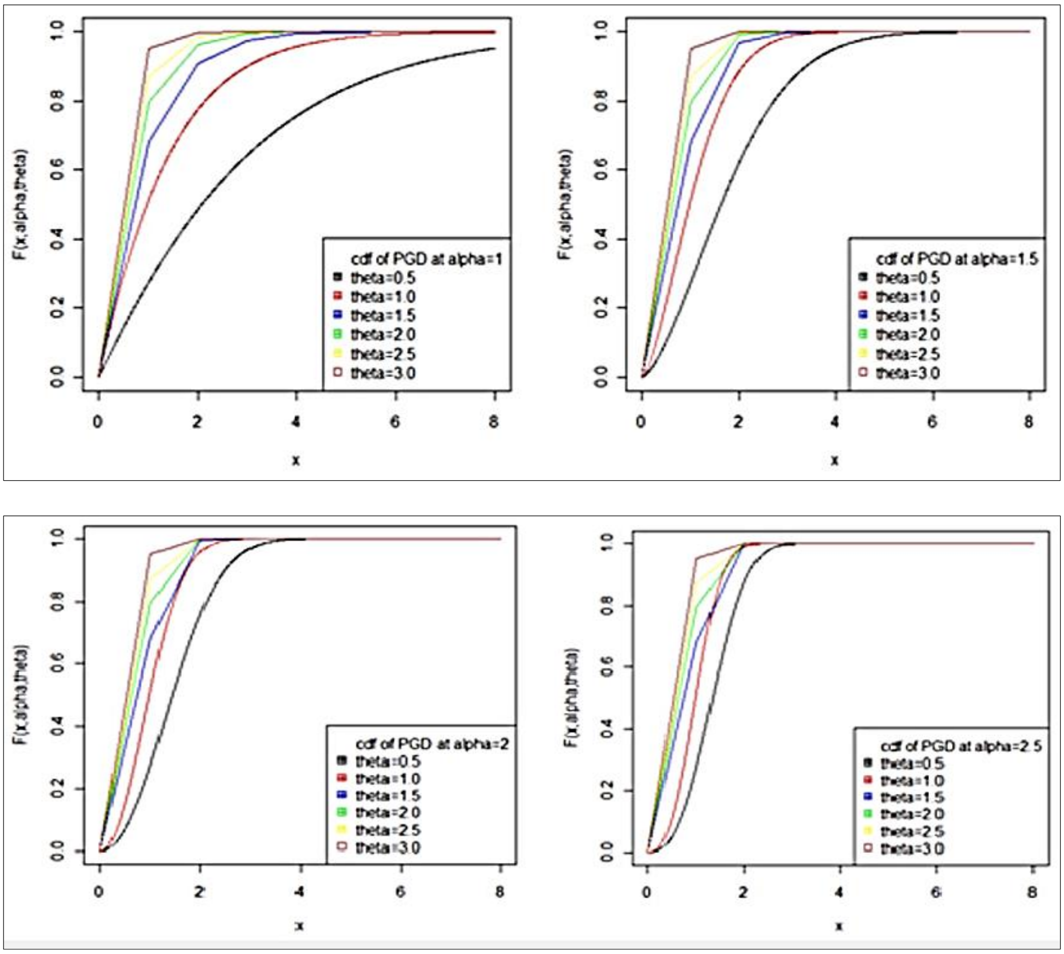


Fig 2: Graphs of the cdf of PGD for varying values of the

Reliability Measures

Survival function

The survival function is also known as reliability function. it can be computed as complement of the cumulative distribution function. The survival function or the reliability function of power Garima distribution is given by

$$S(x) = 1 - F_l(x;\theta,\alpha)$$

$$S(x) = 1 - \frac{1}{\Gamma\frac{(\alpha+1)}{\alpha} + \theta\Gamma\frac{(\alpha+1)}{\alpha} + \Gamma\frac{(2\alpha+1)}{\alpha}}\left[\gamma\left(\frac{(\alpha+1)}{\alpha},\theta x^{\alpha}\right) + \theta\gamma\left(\frac{(\alpha+1)}{\alpha},\theta x^{\alpha}\right) + \gamma\left(\frac{(2\alpha+1)}{\alpha},\theta x^{\alpha}\right)\right]$$

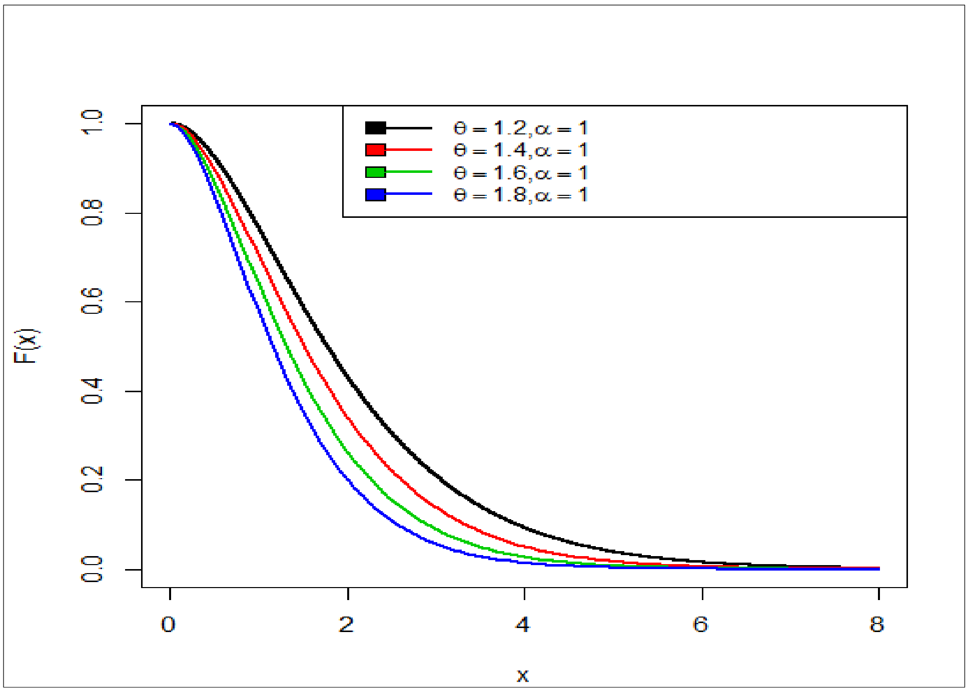
Hazard function

The hazard function is also known as hazard rate, instantaneous failure rate or force of mortality and is given by

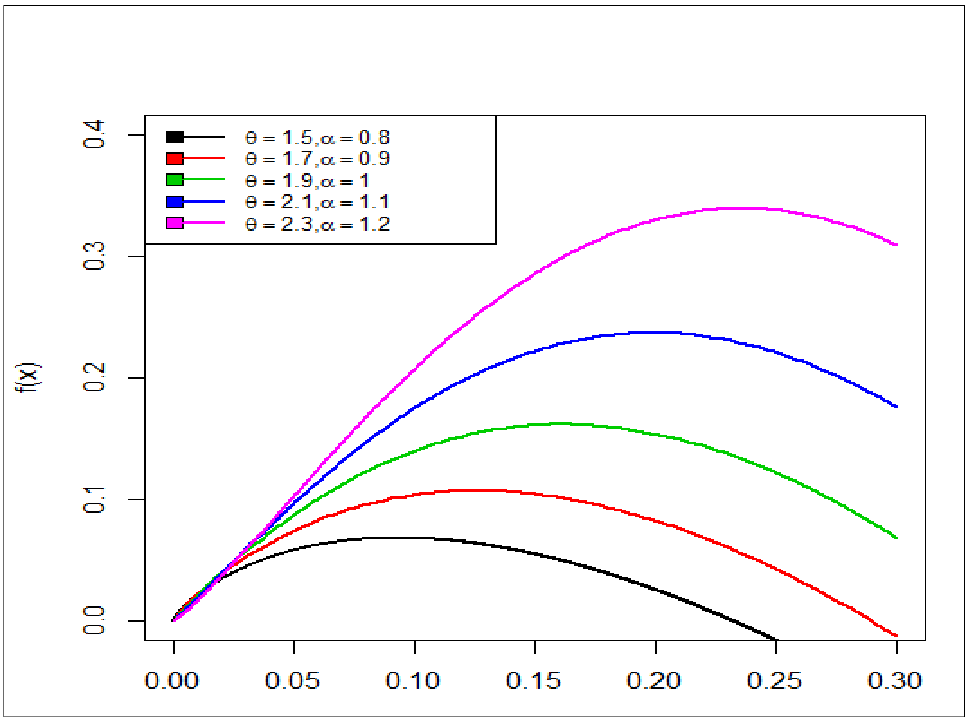
$$h(x) = \frac{f_l(x;\theta,\alpha)}{S(x)}$$

$$h(x) = \frac{x^{\alpha}\alpha\theta^{\frac{\alpha+1}{\alpha}}(1+\theta+\theta x^{\alpha})e^{-\theta x^{\alpha}}}{\left(\Gamma\frac{(\alpha+1)}{\alpha}+\theta\Gamma\frac{(\alpha+1)}{\alpha}+\Gamma\frac{(2\alpha+1)}{\alpha}\right)-\left(\gamma\left(\frac{(\alpha+1)}{\alpha},\theta x^{\alpha}\right)+\theta\gamma\left(\frac{(\alpha+1)}{\alpha},\theta x^{\alpha}\right)+\gamma\left(\frac{(2\alpha+1)}{\alpha},\theta x^{\alpha}\right)\right)}$$

Reliability function of Power Garima Distribution



Hazard function of power Garima Distribution



Structural Properties

Here, I discuss some statistical properties of power Garima distribution.

Moments

Let *X* represents the random variable of power Garima distribution with parameters θ and α, then the rth order moment *E(X r)* of power Garima distribution is obtained as

$$\mu_r' = E(X^r) = \int_0^\infty x^r f_l(x;\theta,\alpha)dx$$

$$= \int_0^\infty \frac{x^{\alpha+r}\alpha\theta^{\frac{\alpha+1}{\alpha}}(1+\theta+\theta x^\alpha)\,e^{-\theta x^\alpha}}{\Gamma\frac{(\alpha+1)}{\alpha}+\theta\Gamma\frac{(\alpha+1)}{\alpha}+\Gamma\frac{(2\alpha+1)}{\alpha}}dx$$

$$= \frac{\alpha\theta^{\frac{\alpha+1}{\alpha}}}{\Gamma\frac{(\alpha+1)}{\alpha}+\theta\Gamma\frac{(\alpha+1)}{\alpha}+\Gamma\frac{(2\alpha+1)}{\alpha}}\int_0^\infty x^{\alpha+r}(1+\theta+\theta x^\alpha)e^{-\theta x^\alpha}dx$$

$$= \frac{\alpha\theta^{\frac{\alpha+1}{\alpha}}}{\Gamma\frac{(\alpha+1)}{\alpha}+\theta\Gamma\frac{(\alpha+1)}{\alpha}+\Gamma\frac{(2\alpha+1)}{\alpha}}\left(\int_0^\infty x^{\alpha+r}e^{-\theta x^\alpha}dx+\theta\int_0^\infty x^{\alpha+r}e^{-\theta x^\alpha}dx+\theta\int_0^\infty x^{2\alpha+r}e^{-\theta x^\alpha}dx\right) \quad (7.3)$$

Put $x^\alpha = t \Rightarrow x = (t)^{\frac{1}{\alpha}}$

Also $\alpha x^{\alpha-1}dx = dt \Rightarrow dx = \frac{dt}{\alpha x^{\alpha-1}} = \frac{dt}{\alpha t^{\frac{\alpha-1}{\alpha}}}$

After simplification, equation (7.3) becomes

$$\mu_r' = E(X^r) = \frac{\Gamma\frac{(\alpha+r+1)}{\alpha}+\theta\Gamma\frac{(\alpha+r+1)}{\alpha}+\Gamma\frac{(2\alpha+r+1)}{\alpha}}{\alpha\theta^{\frac{r}{\alpha}}\left(\Gamma\frac{(\alpha+1)}{\alpha}+\theta\Gamma\frac{(\alpha+1)}{\alpha}+\Gamma\frac{(2\alpha+1)}{\alpha}\right)} \quad (7.4)$$

Substitute *r* = 1 and 2 in equation (7.4), we get mean and the second moment of power Garima distribution.

$$\mu_1' = E(X) = \frac{\Gamma\frac{(\alpha+2)}{\alpha}+\theta\Gamma\frac{(\alpha+2)}{\alpha}+\Gamma\frac{(2\alpha+2)}{\alpha}}{\alpha\theta^{\frac{1}{\alpha}}\left(\Gamma\frac{(\alpha+1)}{\alpha}+\theta\Gamma\frac{(\alpha+1)}{\alpha}+\Gamma\frac{(2\alpha+1)}{\alpha}\right)}$$

$$\mu_2' = E(X^2) = \frac{\Gamma\frac{(\alpha+3)}{\alpha}+\theta\Gamma\frac{(\alpha+3)}{\alpha}+\Gamma\frac{(2\alpha+3)}{\alpha}}{\alpha\theta^{\frac{2}{\alpha}}\left(\Gamma\frac{(\alpha+1)}{\alpha}+\theta\Gamma\frac{(\alpha+1)}{\alpha}+\Gamma\frac{(2\alpha+1)}{\alpha}\right)}$$

$$\text{Variance } (\mu_2) = \frac{\Gamma\frac{(\alpha+3)}{\alpha} + \theta\Gamma\frac{(\alpha+3)}{\alpha} + \Gamma\frac{(2\alpha+3)}{\alpha}}{\alpha\theta^{\frac{2}{\alpha}}\left(\Gamma\frac{(\alpha+1)}{\alpha} + \theta\Gamma\frac{(\alpha+1)}{\alpha} + \Gamma\frac{(2\alpha+1)}{\alpha}\right)} - \left(\frac{\Gamma\frac{(\alpha+2)}{\alpha} + \theta\Gamma\frac{(\alpha+2)}{\alpha} + \Gamma\frac{(2\alpha+2)}{\alpha}}{\alpha\theta^{\frac{1}{\alpha}}\left(\Gamma\frac{(\alpha+1)}{\alpha} + \theta\Gamma\frac{(\alpha+1)}{\alpha} + \Gamma\frac{(2\alpha+1)}{\alpha}\right)}\right)^2$$

Moment Generating Function

Moment generating function is the expected function of the random variable. we begin with the well-known definition of the moment generating function given by

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f_l(x;\theta,\alpha)dx$$

$$= \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \dots\right) f_l(x;\theta,\alpha)dx$$

$$= \int_0^\infty \sum_{j=0}^\infty \frac{t^j}{j!} x^j f_l(x;\theta,\alpha)dx$$

$$= \sum_{j=0}^\infty \frac{t^j}{j!} \mu_j'$$

$$= \sum_{j=0}^\infty \frac{t^j}{j!} \left(\frac{\Gamma\frac{(\alpha+j+1)}{\alpha} + \theta\Gamma\frac{(\alpha+j+1)}{\alpha} + \Gamma\frac{(2\alpha+j+1)}{\alpha}}{\alpha\theta^{\frac{j}{\alpha}}\left(\Gamma\frac{(\alpha+1)}{\alpha} + \theta\Gamma\frac{(\alpha+1)}{\alpha} + \Gamma\frac{(2\alpha+1)}{\alpha}\right)}\right)$$

$$M_X(t) = \frac{1}{\alpha\left(\Gamma\frac{(\alpha+1)}{\alpha} + \theta\Gamma\frac{(\alpha+1)}{\alpha} + \Gamma\frac{(2\alpha+1)}{\alpha}\right)} \sum_{j=0}^\infty \frac{t^j}{j!\theta^{\frac{j}{\alpha}}} \left(\Gamma\frac{(\alpha+j+1)}{\alpha} + \theta\Gamma\frac{(\alpha+j+1)}{\alpha} + \Gamma\frac{(2\alpha+j+1)}{\alpha}\right)$$

Characteristic function

Characteristic function is defined as the function of any real-valued random variable completely defines the probability distribution of a random variable. The characteristic function of power Garima distribution is given by

$$\varphi_X(t) = M_X(it)$$

$$M_X(it) = \frac{1}{\alpha\left(\Gamma\frac{(\alpha+1)}{\alpha} + \theta\Gamma\frac{(\alpha+1)}{\alpha} + \Gamma\frac{(2\alpha+1)}{\alpha}\right)} \sum_{j=0}^\infty \frac{it^j}{j!\theta^{\frac{j}{\alpha}}} \left(\Gamma\frac{(\alpha+j+1)}{\alpha} + \theta\Gamma\frac{(\alpha+j+1)}{\alpha} + \Gamma\frac{(2\alpha+j+1)}{\alpha}\right)$$

Maximum Likelihood Estimation and Fisher's Information matrix

In this section, we will discuss about the maximum likelihood estimation of power Garima distribution for estimating its parameters and also Fisher's Information matrix have been discussed. Let *x1, x2,…,xn* be a random sample of size *n* from the power Garima distribution, then the likelihood function can be written as

$$L(x) = \prod_{i=1}^n f_l(x;\theta,\alpha)$$

$$L(x)=\prod_{i=1}^{n}\left(\frac{x_i^{\alpha}\,\alpha\theta^{\frac{\alpha+1}{\alpha}}\left(1+\theta+\theta x_i^{\alpha}\right)e^{-\theta x_i^{\alpha}}}{\Gamma\frac{(\alpha+1)}{\alpha}+\theta\Gamma\frac{(\alpha+1)}{\alpha}+\Gamma\frac{(2\alpha+1)}{\alpha}}\right)$$

$$L(x)=\left(\frac{\alpha\theta^{\frac{\alpha+1}{\alpha}}}{\Gamma\frac{(\alpha+1)}{\alpha}+\theta\Gamma\frac{(\alpha+1)}{\alpha}+\Gamma\frac{(2\alpha+1)}{\alpha}}\right)^{n}\prod_{i=1}^{n}\left(x_i^{\alpha}\left(1+\theta+\theta x_i^{\alpha}\right)e^{-\theta x_i^{\alpha}}\right)$$

$$L(x)=\frac{\alpha\theta^{n\left(\frac{\alpha+1}{\alpha}\right)}}{\left(\Gamma\frac{(\alpha+1)}{\alpha}+\theta\Gamma\frac{(\alpha+1)}{\alpha}+\Gamma\frac{(2\alpha+1)}{\alpha}\right)^{n}}\prod_{i=1}^{n}\left(x_i^{\alpha}\left(1+\theta+\theta x_i^{\alpha}\right)e^{-\theta x_i^{\alpha}}\right)$$

The log likelihood function is given by

$$\log L=n\left(\frac{\alpha+1}{\alpha}\right)\log\alpha\theta-n\log\left(\Gamma\frac{(\alpha+1)}{\alpha}+\theta\Gamma\frac{(\alpha+1)}{\alpha}+\Gamma\frac{(2\alpha+1)}{\alpha}\right)+\alpha\sum_{i=1}^{n}\log x_i+\sum_{i=1}^{n}\log(1+\theta+\theta x_i^{\alpha})-\theta\sum_{i=1}^{n}x_i^{\alpha} \qquad (7.5)$$

The maximum likelihood estimates of θ and α can be obtained by differentiating equation (7.5) with respect to θ and α and must satisfy the following normal equations

$$\frac{\partial\log L}{\partial\theta}=\frac{n\alpha}{\alpha\theta}\left(\frac{\alpha+1}{\alpha}\right)-n\left(\frac{\Gamma\frac{(\alpha+1)}{\alpha}}{\Gamma\frac{(\alpha+1)}{\alpha}+\theta\Gamma\frac{(\alpha+1)}{\alpha}+\Gamma\frac{(2\alpha+1)}{\alpha}}\right)+\sum_{i=1}^{n}\left(\frac{\left(1+x_i^{\alpha}\right)}{\left(1+\theta+\theta x_i^{\alpha}\right)}\right)-\sum_{i=1}^{n}x_i^{\alpha}=0$$

$$\frac{\partial\log L}{\partial\alpha}=-\frac{n\theta}{\alpha^{3}\theta}-n\psi\left(\Gamma\frac{(\alpha+1)}{\alpha}+\theta\Gamma\frac{(\alpha+1)}{\alpha}+\Gamma\frac{(2\alpha+1)}{\alpha}\right)+\sum_{i=1}^{n}\log x_i+\sum_{i=1}^{n}\left(\frac{\theta x_i^{\alpha}\log x_i}{\left(1+\theta+\theta x_i^{\alpha}\right)}\right)-\theta\sum_{i=1}^{n}x_i^{\alpha}\log x_i=0$$

Where ψ (.) is the digamma function.
To obtain the confidence interval we use the asymptotic normality results. We have that if $\hat{\beta}=\left(\hat{\theta},\hat{\alpha}\right)$ denotes the MLE of $\beta=(\theta,\alpha)$, we can state the result as follows

$$\sqrt{n}(\hat{\beta}-\beta)\rightarrow N_2\left(0,I^{-1}(\beta)\right)$$

Where $I(\beta)$ is Fisher's information matrix is given by

$$I(\beta)=-\frac{1}{n}\begin{pmatrix}E\left(\frac{\partial^2\log L}{\partial\theta^2}\right) & E\left(\frac{\partial^2\log L}{\partial\theta\partial\alpha}\right)\\ E\left(\frac{\partial^2\log L}{\partial\alpha\partial\theta}\right) & E\left(\frac{\partial^2\log L}{\partial\alpha^2}\right)\end{pmatrix}$$

Where

$$E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) = -\frac{n\alpha^2}{(\alpha\theta)^2}\left(\frac{\alpha+1}{\alpha}\right) + n\left(\frac{\left(\Gamma\frac{(\alpha+1)}{\alpha}\right)^2}{\left(\Gamma\frac{(\alpha+1)}{\alpha} + \theta\Gamma\frac{(\alpha+1)}{\alpha} + \Gamma\frac{(2\alpha+1)}{\alpha}\right)^2}\right) - \sum_{i=1}^{n}\left(\frac{\left(1+x_i^{\alpha}\right)^2}{\left(1+\theta+\theta x_i^{\alpha}\right)^2}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) = \frac{3n\alpha^2\theta^2}{(\alpha^3\theta)^2} - n\psi'\left(\Gamma\frac{(\alpha+1)}{\alpha} + \theta\Gamma\frac{(\alpha+1)}{\alpha} + \Gamma\frac{(2\alpha+1)}{\alpha}\right)$$

$$+\sum_{i-1}^{n}\left(\frac{\left(1+\theta+\theta x_i^{\alpha}\right)\theta x_i^{\alpha}(\log x_i)^2 - \left(\theta x_i^{\alpha}\log x_i\right)^2}{\left(1+\theta+\theta x_i^{\alpha}\right)^2}\right) - \theta\sum_{i=1}^{n}\left(x_i^{\alpha}\log x_i\right)^2$$

$$E\left(\frac{\partial^2 \log L}{\partial\theta\,\partial\alpha}\right) = E\left(\frac{\partial^2 \log L}{\partial\alpha\,\partial\theta}\right) = -n\psi\left(\frac{\Gamma\frac{(\alpha+1)}{\alpha}}{\Gamma\frac{(\alpha+1)}{\alpha} + \theta\Gamma\frac{(\alpha+1)}{\alpha} + \Gamma\frac{(2\alpha+1)}{\alpha}}\right)$$

$$+\sum_{i=1}^{n}\left(\frac{\left(1+\theta+\theta x_i^{\alpha}\right)x_i^{\alpha}\log x_i - \theta x_i^{\alpha}\log x_i\left(1+x_i^{\alpha}\right)}{\left(1+\theta+\theta x_i^{\alpha}\right)^2}\right) - \sum_{i=1}^{n} x_i^{\alpha}\log x_i$$

where $\psi(.)'$ is the first order derivative of diagamma function. Since β being unknown, we estimate $I^{-1}(\beta)$ by $I^{-1}(\hat{\beta})$ and this can be used to obtain asymptotic confidence interval for θ and α.

Conclusion

In this paper, we have discussed a new generalization of power Garima distribution using stochastic approach. The different mathematical and statistical properties of the newly introduced distribution have been discussed. The maximum likelihood estimator is also used for estimating the parameters of the proposed distribution and also the fisher's information matrix have been discussed. Finally, the application of power Garima distribution fits better than the Garima, exponential and lindley distributions.

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