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Finite double integral representation for the classical polynomial Set $J_n\{(x_m), y\}$ of several variables

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Abstract

In the present paper, an attempt has been made to express a Finite Double Integral Representation for the Classical polynomial set $J_n\{(x_m), y\}$ of Several Variables. Many interesting new results may be obtained as particular cases out of these particular results some of them stand for well known and some of them are believed to be new. These are of at most important for mathematician scientists, engineers and physical sciences because these occur in the solution of differential equation Integral equation etc.

AMS Subject Classification: Special function-33C

Keywords: Hypergeometric function, generating function, integral representation

Introduction

We have derived generalized hypergeometric polynomial set $J_n\{(x_m), y\}$ of several variables by means generating relation

$$\begin{aligned}
 & (\xi + \lambda y^{-e} t^e)^{-\sigma} F \left[\begin{matrix} \lambda_1; (a_g); \\ (b_h); \end{matrix} \mu_1 y^{-e_1} t^{e_1} \right] \\
 & \times F \left[\begin{matrix} (A_r); (C_p); (\alpha_{u_k}) \\ (B_s); (D_q); (\beta_{v_k}) \end{matrix} \mu x_1^d t, \mu_2 x_2^{e_2} y^{-e_2} t^{e_2} \dots \dots \mu_k x_k^{e_k} t^{e_k} \right] \\
 & = \sum_{n=0}^{\infty} J_{n, e, e_1; e_2; e_3; \dots; e_m; (b_k); (B_s); (D_q); (\beta_{v_k})}^{\xi; \lambda; \lambda_1; \sigma; d; \mu; \mu_1; \mu_2 \dots \dots \mu_m; (a_g); (A_r); (C_p); (\alpha_{u_k})} \{(x_m), y\} \dots (1.1)
 \end{aligned}$$

Where $\xi, \lambda, \lambda_1, \sigma, \delta; \mu, \mu_1, \mu_2 \dots \dots \mu_m$ are real and $e, e_1, e_2, \dots \dots e_m$ are natural number.

The left hand side of (1.2) are the product of generalized hypergeometric function and Lauricella function in the notation of Burchnall and Chaundy [1]. The polynomial set contain number of parameters. For simplicity we shall denote

$$J_{n, e, e_1; e_2; e_3; \dots; e_m; (b_k); (B_s); (D_q); (\beta_{v_k})}^{\xi; \lambda; \lambda_1; \sigma; d; \mu; \mu_1; \mu_2 \dots \dots \mu_m; (a_g); (A_r); (C_p); (\alpha_{u_k})} \{(x_m), y\} = J_n\{(x_m), y\}$$

Where n denotes the order of the generalized polynomial set. After little simplification (1.1) give

$$\begin{aligned}
 J_n\{(x_m), y\} &= \xi \sigma \sum_{m=0}^n \sum_{m_1=0}^{[n-e]} \sum_{m_2=0}^{[n-em-e_1 m_1]} \dots \dots \sum_{m_k=0}^{[n-em-e_1 m_1 - e_2 m_2 \dots \dots e_{k-1} m_{k-1}]} \\
 & \times \frac{[(A_r)]_{n-em-e_1 m_1 - (e_2-1)m_2 - \dots - (e_k-1)m_k} [(C_p)]_{n-ek-e_1 k_1 - e_2 k_2 - \dots - e_k m_k}}{[(B_s)]_{n-em-e_1 m_1 - (e_2-1)m_2 - \dots - (e_k-1)m_k} [(D_q)]_{n-ek-e_1 k_1 - e_2 k_2 - \dots - e_k m_k}}
 \end{aligned}$$

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$$\begin{aligned} & \times \frac{[(\alpha_{u_k})]_{m_k} [(a_g)]_{m_1} (\sigma)_{m(-\lambda)}^{m(-\lambda_1)_{m_1}} \mu^{m_1} (\mu_2 x_2^{e_2})^{m_2}}{[(\beta_{v_k})]_{m_k} [(b_h)]_{m_1} \xi^m m! m_1! y^{em+e_1 m_1+e_2 m_2} m_2!} \\ & \times \frac{(\mu x_1^{e_k})^{m_k} (\lambda x_1^e)^{n, em-e_1 m_1-e_2 m_2-\dots-e_k m_k}}{m_k! (n, em-e_1 m_1-e_2 m_2-\dots-e_k m_k)!} \end{aligned} \quad \dots (1.2)$$

2. Notations

- I.** (i) $(n) = 1, 2, \dots, n-1, n$.
- (ii) $(a_p) = a_1, a_2, a_3, \dots, a_p$.
- (iii) $(a_p; i) = a_1, a_2, a_3, \dots, a_{i-1}, a_{i+1}, \dots, a_p$.

- II.** (i) $[(a_p)] = a_1, a_2, a_3, \dots, a_p$.
- (ii) $[(a_p)]_n = \prod_{i=1}^p (a_i)_n = (a_1)_n (a_2)_n (a_3)_n \dots (a_p)_n$

- III.** (i) $\Delta(a; b) = \frac{b}{a}, \frac{b+1}{a} + \dots + \frac{b+a-1}{a}$.
- (ii) $\Delta(a(1); b) = \frac{b}{a}, \frac{b+1}{a}, \frac{b+2}{a}, \dots + \frac{b+a-2}{a}$
- (iv) $\Delta(a; b \pm c \pm d) = \Delta(a; b + c + d), \Delta(a; b + c - d),$
 $\Delta(a; b - c + d), \Delta(a; b - c - d),$

- IV.** (i) $\Delta_k[a; b] = \prod_{r=1}^a \left(\frac{b+r-1}{a}\right)_k = \left(\frac{b}{a}\right)_k \left(\frac{b+1}{a}\right)_k \dots \left(\frac{b+a-1}{a}\right)_k$
- (ii) $\Delta_k[a(1); b] = \left(\frac{b}{a}\right)_k \left(\frac{b+1}{a}\right)_k \dots \left(\frac{b+a-2}{a}\right)_k$
- (iii) $\Delta_k[m; (a_p)] = \prod_{i=1}^b \prod_{r=1}^m \left(\frac{a_i+r-1}{m}\right)_k$

- V.** (i) $\Gamma[(a_p)] = \prod_{i=1}^p (a_i)$
- (ii) $\Gamma[(a_p); (s)] = \prod_{i=s+1}^p (a_i)$
- (iii) $\Gamma[(a; b)] = \prod_{r=1}^a \Gamma\left(\frac{b+r-1}{a}\right)$
- (iv) $\Gamma[\Delta(m); (a_p)] = \prod_{i=1}^p \prod_{r=1}^m \Gamma\left(\frac{(a_i)+r-1}{m}\right)$

- VI.** (i) $\Gamma(a \pm b) = \Gamma(a + b)\Gamma(a - b)$
- (ii) $\Gamma_{* *}(a + b) = \Gamma(a + b)\Gamma(a + b)$

$$M = \frac{[(A_r)]_n [(C_p)]_n (\lambda x_1^d)^n}{[(B_s)]_n [(D_q)]_n n!}$$

3. Theorem

For $e_2 > 1, \dots, e_k > 1$, we have

$$\begin{aligned} J_n\{(x_m), y\} &= M \frac{\Gamma(\mu+\nu)}{\Gamma(\mu)\Gamma(\nu)} \int_0^1 \int_0^1 \frac{(1-\xi)^{\mu-1} (1-\eta)^{\nu-1}}{(1-\xi\eta)^{\mu+\nu-1}} \\ & \times F_{s+q+h+u:1:1: \alpha_{u_1}, \alpha_{u_2}, \dots, \alpha_{u_k}}^{r+p+g+v+1:1:1: \beta_{v_1}, \beta_{v_2}, \dots, \beta_{v_k}} \left[\frac{[(-n): e, e_1, e_2; \dots, e_k]}{\dots}, \right. \\ & [(1 - (B_s) - n): e, e_1, e_2 - 1 \dots e_k - 1], [(1 - (D_q) - n): e, e_1, e_2; \dots, e_k] \\ & [(1 - (A_r) - n): e, h_1, e_2 - 1 \dots e_k - 1], [(1 - (C_p) - n): e, e_1, e_2; \dots, e_k] \\ & \left. \left[\frac{[(\alpha_{u_1}): 1], [(\alpha_{u_2}): 1] \dots [(\alpha_{u_k}): 1] [(a_g): 1] [\sigma: 1] [\lambda_1: 1] \left[\frac{\mu+\nu}{2}: 1\right] \left[\frac{\mu+\nu+1}{2}: 1\right]}{[(\beta_{v_1}): 1], [(\beta_{v_2}): 1] \dots [(\beta_{v_k}): 1], [(b_h): 1] [(\mu): 1] [(\mu): 1]} \right. \right. \\ & \left. \frac{-\lambda(-1)^{e(r+s+p+q+g+h+u+v+1)}}{\xi^e (\mu x_1^d y)^e}, \frac{\mu_1(-1)^{e_1(r+s+p+q+g+h+u+v+1)}}{(\mu x_1^d y)^{e_1}}, \right. \\ & \left. \frac{\mu_2 x_2^{e_2} (-1)^{e_2(r+s+p+q+g+h+u+v+1)+r+s}}{(\mu x_1^d y)^{e_2}}, \dots, \frac{\mu_k x_k^{e_k} (-1)^{e_k(r+s+p+q+g+h+u+v+1)+r+s}}{(\mu x_1^d y)^{e_k}} \right] d\xi d\eta \end{aligned} \quad \dots (3.1)$$

Proof: we have

$$\begin{aligned}
 I_2 &= \int_0^1 \int_0^1 \frac{(1-\xi)^{\mu-1} \eta^\mu (1-\eta)^{\nu-1}}{(1-\xi\eta)^{\mu+\nu-1}} \sum_{m=0}^{\lfloor n \rfloor} \sum_{m_1=0}^{\lfloor \frac{n-em}{e_1} \rfloor} \sum_{m_2=0}^{\lfloor \frac{n-em-e_1 m_1}{e_2} \rfloor} \\
 &\times \sum_{m_k=0}^{\lfloor \frac{n-em-e_1 m_1 \dots - e_{k-1} m_{k-1}}{e_k} \rfloor} \frac{[(A_r)]_{n-em-e_1 m_1-(e_2-1)m_2-\dots-(e_k-1)m_k}}{[(B_s)]_{n-em-e_1 m_1-(e_2-1)m_2-\dots-(e_k-1)m_k}} \\
 &\times \frac{[(C_p)]_{n-em-e_1 m_1-e_2 m_2-\dots-e_k m_k} [(\alpha_{u_k})]_{m_k} [(a_g)]_{m_1}}{[(D_q)]_{n-em-e_1 m_1-e_2 m_2-\dots-e_k m_k} [(\beta_{v_k})]_{m_k} [(b_h)]_{m_1}} \\
 &\times \frac{(\sigma)_m (-\lambda)_m (\lambda_1)_{m_1} \mu_1^{m_1} (\mu_2 x_2^{e_2})^{m_2} \dots (\mu_k x_k^{e_k})^{m_k}}{\xi^m m! m_1! y^{em+e_1 m_1+e_2 m_2} m_2! \dots (m_k)!} \\
 &\times \frac{(\mu x_1^d)^{n-em-e_1 m_1-e_2 m_2-\dots-e_k m_k} \binom{\mu+\nu}{2}_{m_1} \binom{\mu+\nu+1}{2}_{m_1}}{(n-em-e_1 m_1-e_2 m_2-\dots-e_k m_k)! (\mu)_{m_1} (\nu)_{m_1}} \\
 &= \sum_{m=0}^{\lfloor n \rfloor} \sum_{m_1=0}^{\lfloor \frac{n-em}{e_1} \rfloor} \sum_{m_2=0}^{\lfloor \frac{n-em-e_1 m_1}{e_2} \rfloor} \dots \sum_{m_k=0}^{\lfloor \frac{n-em-e_1 m_1 \dots - e_{k-1} m_{k-1}}{e_k} \rfloor} \\
 &\times \frac{[(A_r)]_{n-em-e_1 m_1-(e_2-1)m_2-\dots-(e_k-1)m_k} [(C_p)]_{n-em-e_1 m_1-e_2 m_2-\dots-e_k m_k}}{[(B_s)]_{n-em-e_1 m_1-(e_2-1)m_2-\dots-(e_k-1)m_k} [(D_q)]_{n-em-e_1 m_1-e_2 m_2-\dots-e_k m_k}} \\
 &\times \frac{[(\alpha_{u_k})]_{m_k} [(a_g)]_{m_1} (\sigma)_m (-\lambda)_m (\lambda_1)_{m_1} \mu_1^{m_1} (\mu_2 x_2^{e_2})^{m_2}}{[(\beta_{v_k})]_{m_k} [(b_h)]_{m_1} \xi^m m! m_1! y^{em+e_1 m_1+e_2 m_2} m_2!} \\
 &\times \frac{(\mu_k x_k^{e_k})^{m_k} (\mu x_1^d)^{n-em-e_1 m_1-e_2 m_2-\dots-e_k m_k} \binom{\mu+\nu}{2}_{m_1} \binom{\mu+\nu+1}{2}_{m_1}}{m_k! (n-em-e_1 m_1-e_2 m_2-\dots-e_k m_k)! (\mu)_{m_1} (\nu)_{m_1}} \\
 &\times \int_0^1 \int_0^1 \frac{(1-\xi)^{\mu+m_1-1} \eta^{\mu+m_1} (1-\eta)^{\nu+m_1-1}}{(1-\xi\eta)^{\mu+\nu-1+2m_1}} d\xi dn. \\
 &= M \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \sum_{m, m_1, m_2, \dots, m_k}^{\infty} \frac{[1-(B_s)-n]_{em+e_1 m_1+(e_2-1)m_2+\dots+(e_k-1)m_k}}{[1-(A_r)-n]_{em+e_1 m_1+(e_2-1)m_2+\dots+(e_k-1)m_k}} \\
 &\times \frac{[1-(D_q)-n]_{em+e_1 m_1+e_2 m_2+\dots+e_k m_k} [(\alpha_{u_k})]_{m_k} [(a_g)]_{m_1}}{[1-(C_p)-n]_{em+e_1 m_1+e_2 m_2+\dots+e_k m_k} [(\beta_{v_k})]_{m_k} [(b_h)]_{m_1}} \\
 &\times \frac{(\sigma)_m (\lambda_1)_{m_1} (-\lambda)^m (-1)^{e(r+s+p+q+g+h+u+v+1)m}}{m! \xi^m (\mu x_1^d y)^{em}} \\
 &\times \frac{\mu_1 (-1)^{e_1(r+s+p+q+g+h+u+v+1)m_1} (\mu_2 x_2^{e_2})^{m_2} (-1)^{e_2(r+s+p+q+g+h+u+v+1)r+s} m_2}{m_1! (\mu x_1^d y)^{e_1 m_1} m_2! (\mu x_1^d y)^{e_2 m_2}} \\
 &\times \frac{(\mu_k x_k^{e_k})^{m_k} (-1)^{e_k(r+s+p+q+g+h+u+v+1)r+s} m_2}{m_k! (\lambda x_1^d)^{e_k m_k}} \\
 &\times (-n)_{em+e_1 m_1+e_2 m_2+\dots+e_k m_k} \dots (3.2)
 \end{aligned}$$

The single terminating factor $(-n)_{em+e_1 m_1+e_2 m_2+\dots+e_k m_k}$ makes all summation in (3.2) runs upto \square . Then we finally achieve.

$$I_2 = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} J_n \{ (x_m), y \}.$$

Hence, the theorem
We have from [2].

$$\int_0^1 \int_0^1 \frac{(1-\xi)^{\mu-1} \eta^\mu (1-\eta)^{\nu-1}}{(1-\xi\eta)^{\mu+\nu-1}} d\xi dn = \beta(\mu, \nu)$$

Particular Cases of (3.1)

(i) On making the substitution $p = 0 = q = r = s = g = u_k = v_k; m = m_1 = h = 0 = \lambda_1 = -\lambda = \mu_1 = y = \mu; b_1 = 1 + \alpha$, and $x_1 = \frac{1}{y}$, in (3.1) we get

$$L_n^{(\alpha)}(y) = \frac{\Gamma(\mu+\nu)(1+\alpha)_n}{\Gamma(\mu)\Gamma(\nu)y^n n!} \int_0^1 \int_0^1 \frac{(1-\xi)^{\mu-1} \eta^\mu (1-\eta)^{\nu-1}}{(1-\xi\eta)^{\mu+\nu-1}} \times F \left[\begin{matrix} -n, \frac{\mu+\nu}{2}, \frac{\mu+\nu-1}{2}; \\ y \\ 1 + \alpha, \mu, \nu; \end{matrix} \right] d\xi d\eta$$

Where $L_n^{(\alpha)}(y)$ are the laguerre polynomials

(ii) On taking $r = 0 = s = p = g = u_k = v_k; q = h = y = d = e = e_1 = v_1 = \square_1 = \square\square = \square_1 = -\square; \mu = \frac{1}{2}; = D_1 = \frac{1}{2} = b_1$; and putting $x_1 = \frac{x-1}{x+1}$, we arrive at

$$T_n(x) = \frac{\Gamma(\mu+\nu)}{n! \Gamma(\mu)\Gamma(\nu)} \left(\frac{x-1}{2}\right)^n \int_0^1 \int_0^1 \frac{(1-\xi)^{\mu-1} \eta^\mu (1-\eta)^{\nu-1}}{(1-\xi\eta)^{\mu+\nu-1}} \times F \left[\begin{matrix} -n, \frac{1}{2} - n, \frac{\mu+\nu}{2}, \frac{\mu+\nu-1}{2}; \\ \frac{x+1}{x-1} \\ \frac{1}{2}, \mu, \nu; \end{matrix} \right] d\xi d\eta$$

Where, $T_n(x)$ are the Tchebicheffe Polynomials of First Kind

(iii) If we take $r = 0 = s = p = g = u_k = v_k; q = 1 = h = d = e = e_1 = \xi = \sigma = \lambda_1 = -\lambda = y; \mu_1 = \frac{1}{2} = \mu; = D_1 = \frac{3}{2} = b_1$; and writing $\frac{x-1}{x+1}$, for x , we achieve

$$U_n(x) = \frac{\Gamma(\mu+\nu)(n+1)!}{\Gamma(\mu)\Gamma(\nu)n!} \left(\frac{x-1}{2}\right)^n \int_0^1 \int_0^1 \frac{(1-\xi)^{\mu-1} \eta^\mu (1-\eta)^{\nu-1}}{(1-\xi\eta)^{\mu+\nu-1}} \times F \left[\begin{matrix} -n, \frac{-1}{2} - n, \frac{\mu+\nu}{2}, \frac{\mu+\nu-1}{2}; \\ \frac{x+1}{x-1} \\ \frac{3}{2}, \mu, \nu; \end{matrix} \right] d\xi d\eta$$

Where $U_n(x)$ are the Tchebicheffe Polynomials of Second Kind.

(iv) If we set $r = 0 = s = p = g = \alpha_{u_k} = \beta_{v_k}; q = 1 = h = y = \square = d = e = e_1 = -\lambda = \sigma = \lambda_1; \mu = \frac{1}{2} = \mu_1; D_1 = 1 + \alpha = b$; and writing $\frac{x+1}{x-1}$ for x , we obtain

$$P_n^{(\alpha,\alpha)}(x) = \frac{\Gamma(\mu+\nu)(1+\alpha)}{\Gamma(\mu)\Gamma(\nu)n!} \left(\frac{x+1}{2}\right)^n \int_0^1 \int_0^1 \frac{(1-\xi)^{\mu-1} \eta^\mu (1-\eta)^{\nu-1}}{(1-\xi\eta)^{\mu+\nu-1}} \times F \left[\begin{matrix} -n, -\alpha - n, \frac{\mu+\nu}{2}, \frac{\mu+\nu-1}{2}; \\ \frac{x-1}{x+1} \\ 1 + \alpha, \mu, \nu; \end{matrix} \right] d\xi d\eta$$

Where $P_n^{(\alpha,\alpha)}(x)$ are the Ultrahpherical Polynomials

(v) If we setting $r = 0 = s = p = g = \alpha_{u_k} = \beta_{v_k}; q = 1 = h = e = e_1 = d = \square = y = -\square = \square = \square\square_1; \mu = \frac{1}{2} = \mu_1; D_1 = 1 + \square\square$; and $b_1 = 1 + \square; x_1 = \frac{x+1}{x-1}$, we get

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\mu+\nu)(1+\alpha)_n}{\Gamma(\mu)\Gamma(\nu)n!} \left(\frac{x+1}{2}\right)^n \int_0^1 \int_0^1 \frac{(1-\xi)^{\mu-1} \eta^\mu (1-\eta)^{\nu-1}}{(1-\xi\eta)^{\mu+\nu-1}} \times F \left[\begin{matrix} -n, -\beta - n, \frac{\mu+\nu}{2}, \frac{\mu+\nu-1}{2}; \\ \left(\frac{x+1}{x-1}\right) \\ 1 + \alpha, \mu, \nu; \end{matrix} \right] d\xi d\eta$$

Where $P_n^{(\alpha, \beta)}(x)$ are the Jacobi Polynomials

(vi) For $r = 0 = s = p = g = ; q = 1 = h = e = e_1 = d = y = \sigma = -\lambda = \lambda_1 = y; \mu = \frac{1}{2} = \mu_1; D_1 = \frac{1}{2} + \lambda = b_1$ and writing $\frac{x+1}{x-1}$ for x_1 , we obtain

$$C_n^\lambda(x) = \frac{\Gamma(\mu+\nu)(2\lambda)_n}{\Gamma(\mu)\Gamma(\nu)n!} \left(\frac{x+1}{2}\right)^n \int_0^1 \int_0^1 \frac{(1-\xi)^{\mu-1} \eta^\mu (1-\eta)^{\nu-1}}{(1-\xi\eta)^{\mu+\nu-1}}$$

$$\times F \left[\begin{matrix} -n, \frac{1}{2} - \lambda - n, \frac{\mu+\nu}{2}, \frac{\mu+\nu-1}{2}; \\ \frac{x-1}{x+1} \\ \lambda + \frac{1}{2}, \mu, \nu; \end{matrix} \right] d\xi d\eta$$

Where $C_n^\lambda(x)$ are the Gegenbauer Polynomials

References

1. Burchinal JL, Chaundy TW. Expansions of Appell's double hyper geometric functions (ii), Quart. J. Math. Oxford ser. 1941;12:112-128.
2. Dutta M, More KL. A new class of generalized Legendre polynomials. Mathematica (Cluj). 1965;7(30):33-41.