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On the formulation of a stochastic model for an accumulated claim amount under renewal risk process

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Abstract
Traditionally an Insurance risk process is characterised by claim process using renewal process assuming claim amount is independent of inter claim time. It is usually modelled as a stochastic process such as Compound Poisson Process. It is also assumed that the premium amount is proportional to the time we refer with each claim. Depending upon the type of portfolio, the insurer can make a variety of different assumptions on the sequence of inter occurrence times and accumulated claim amount as well. In this paper we discuss a stochastic model for Renewal Risk model with different distributions to number of demands and Generalised Exponential distribution to the impact of each demand under insurance claim scenario. Assume that number of cases is independent of severity of each case throughout the model. We present the model when case frequency is Poison or Negative Binomial or Geometric and also present severity of each case with Generalised Exponential distribution.

Keywords: Generalised exponential distribution, poison distribution, negative binomial distribution, geometric distribution, maximum likelihood estimation, information matrix, aggregate loss model

1. Introduction
Insurance risk theory mainly concerns with the study of insurer's bankruptcy through the analysis of the level of reserves as a function of intermediate time and other important units such as the probability of ruin, the time of ruin, nature of various probability distributions associated with. Many papers have been discussed to the above mentioned features. Dickson and Hipp (2001) [18] has discussed the Erlangian (2) distribution for the time of ruin in Sparre Anderson so called the Classical model. From then onwards various works have been carried out by introducing various distributions such as Erlangian (n), Generalised Erlangian (n) based on Gerber-Siu Discounted Penalty function. Dickson and Willmort (2005) and Landriault and Willmort (2009) studied the ruin theory features using Laplace transforms and Inverse Laplace transforms. Heckman and Mayes (1983) presented Aggregate Loss Distribution in view of Collective Risk Theory for severity and count distributions. The Gamma, Beta, F, Pareto, Burr, Weibull and Logistic distributions have been used for representing accumulated distributions. Robertson (1992) [19] contributed an application of the fast Fourier transform to the computation of aggregate loss distribution. Bortolosso *et al*. (2009) aimed estimating claim size in the auto insurance by using zero adjusted Inverse Gaussian distribution.
Recently many works have been reported by using discrete/continuous phase-type distribution for inter arrival time distribution Hu Yang & Zhimin Zang (2010) risk models with Phase type claims have been considered by many researchers. Badsecu *et al*. (2005), Dickson and Li (2010), Derik *et al*. (2014) and Stanford *et al*. (2015) [20], Rebello *et al*. (2017) [14] considered the time to ruin or its Laplace transforms for the renewal risk model, where both the claim inter arrival time distribution and the claim size distribution are Phase types. In 2021, Zhang Lili discussed The Erlangian (n) risk model with two sided jumps and a constant dividend barrier.
Taking into account the works that have already carried out in risk theory the most significant goal is to achieve a satisfactory model for the probability distribution of the total claim amount.

Here we introduce a statistical distribution known as Generalised Exponential Distribution to represent the claim amount and its characteristics for applying it in actuarial studies. In this paper, we present the model claim severity under a Generalised Exponential Distribution, the problem of estimating the parameters of distribution by using maximum likelihood method, the aggregate loss model collective risk theory when the claim frequency distribution is Poisson, Negative Binomial and Geometric distribution. Finally, the numerical example is given to validate the result.

2. Model and Notation

One of the most widely discussed models of the evolution of the surplus of an insurance company is the Classical Compound Poisson model, known as Crammer-Lindberg model. Under this, premiums are assumed to arrive at a constant rate over time. Claims are modelled by a homogenous Compound Poisson process, which implies that inter claim times have Exponential distribution and claim amounts are independent and identically distributed. In the present paper, we use the same model but we negotiate the number of claims by various distributions such as negative Binomial, Geometric along with Poisson distribution.
The model is defined as follows;
Assume that the initial reserve of a particular insurance company is R≥0, the company charges a particular premium rate P≥0 at any point in time. In addition, events, such as claims occur randomly in time. Let Z_i be the time between $(i-1)^{th}$ event and i^{th} event. We assume that Z_1, Z_2, are independent and identically distributed positive random variables.
For t≥0, Define, N (t)=Sup[t≥1; $Z_1+Z_2+. Z_i \leq t$ } such that {N(t)} is a renewal counting process
X_1, X_2, represents the claim amount where $X_1, X_2, \ldots$. are i.i.d with common c.d. f

$$F_X(x) = 1 - \bar{F}_X(x), \mathrm{x}\, \varepsilon\, \mathrm{R}$$

Then company's reserve process is the given by

$$\mathrm{R}(t) = R + Pt + \sum_{i=1}^{\mathrm{N}(t)} X_i. \qquad (1)$$

2.1 Model under Generalised Exponential Distribution

Here we assume Generalised Exponential Distribution as our distribution for the aggregate claim $S = \sum_{i=1}^{N(t)} X_i$, The cumulative density function for G.E(α, β) is

$$F(x;\alpha,\beta) = (1-e^{-\beta x})^{\alpha}; \alpha,\beta, x > 0. \qquad (2)$$

If X has the distribution function (2) then the corresponding density function is

$$f(x;\alpha,\beta) = \alpha\beta(1-e^{-\beta x})^{\alpha-1} e^{-\beta x}$$

Where
α is a shape parameter and β is a scale parameter.
Then the Moment Generating Function is given by,

$$M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx}\alpha\beta(1-e^{-\beta x})^{\alpha-1} e^{-\beta x} dx$$

Making the substitution, $y=e^{\beta x}$,

$$M_Y(t) = \int_0^1 e^{t \log y^{1/\beta}} \alpha(1-y)^{\alpha-1} dy$$

$$M_Y(t) = \int_0^1 y^{\frac{-t}{\beta}} (1-y)^{\alpha-1} dy$$

$$M_Y(t) = \frac{\Gamma(1-t/\beta)\Gamma\alpha}{\Gamma(\alpha+1-t/\beta)}$$

Using moment generating function, we have

$$\mathrm{E}(X) = \frac{1}{\beta}(\psi(\alpha+1) - \psi(1))$$

$$Var(X) = -\frac{1}{\beta^2}(\psi^{'}(\alpha+1) - \psi^{'}(1))$$

Here $\psi(.)$ denotes the digamma function and $\psi^{'}(.)$ denotes the derivative of $\psi(.)$.

3. Maximum Likelihood Estimation of Parameters

In this section the maximum likelihood estimators of GE (α, β) are considered. We consider estimation of α and β when both are unknown. If $x_1, x_2, \ldots.., x_n$ is a random sample from GE (α, β), then the log-likelihood function L (α, β) is

$$L(\alpha,\beta) = n\log_e(\alpha) + n\log_e(\beta) + (\alpha-1)\sum_{i=1}^{n}\log_e(1-e^{-\beta x_i}) - \beta\sum_{i=1}^{n}x_i \tag{3}$$

The normal equations are

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log_e(1-e^{-\beta x_i}) = 0$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} + (\alpha-1)\sum_{i=1}^{n}\frac{x_i e^{-\beta x_1}}{(1-e^{-\beta x_i})} - \sum_{i=1}^{n}x_i = 0$$

We estimate the parameters by appropriate iterative technique. For this we obtain the Fisher Information matrix

$$I(\alpha,\beta) = -\frac{1}{n}\begin{bmatrix} E(\frac{\partial^2 L}{\partial \alpha^2}) & E(\frac{\partial^2 L}{\partial\alpha\partial\beta}) \\ E(\frac{\partial^2 L}{\partial\alpha\partial\beta}) & E(\frac{\partial^2 L}{\partial \beta^2}) \end{bmatrix}$$

The elements of the Fisher information matrix are as follows,
For α > 2;

$$E(\frac{\partial^2 L}{\partial \alpha^2}) = -\frac{n}{\alpha^2}$$

$$E(\frac{\partial^2 L}{\partial\alpha\partial\beta}) = \frac{n}{\beta}[\frac{\alpha}{\alpha-1}(\psi(\alpha)-\psi(1)) - (\psi(\alpha+1)-\psi(1)]$$

$$E(\frac{\partial^2 L}{\partial \beta^2}) = -\frac{n}{\beta}[1+\frac{\alpha(\alpha-1)}{\alpha-2}(\psi^{'}(1)-\psi^{'}(\alpha-1)+(\psi(\alpha-1)-\psi(1))^2] - \frac{n\alpha}{\beta^2}[(\psi^{'}(1)-\psi(\alpha)+(\psi(\alpha)-\psi(1))^2)]$$

and for 0 < α ≤ 2,

$$E(\frac{\partial^2 L}{\partial \alpha^2}) = -\frac{n}{\alpha^2}$$

$$E(\frac{\partial^2 L}{\partial\alpha\partial\beta}) = \frac{n\alpha}{\beta}\int_0^\infty x e^{-2x}(1-e^{-x})^{\alpha-2}dx$$

$$E(\frac{\partial^2 L}{\partial \beta^2}) = -\frac{n}{\beta^2} - \frac{n\alpha(\alpha-1)}{\beta^2}\int_0^\infty x^2 e^{-2x}(1-e^{-x})^{\alpha-3}dx$$

The MLE $\hat{\alpha}$ and $\hat{\beta}$ obtained by inverting the Fisher –Information matrix.

4. Accumulated Claim Amount under Collective Risk Theory

Suppose that portfolio has N claims in the past period of time in our experience and each unit has X_i is the claim size which is independent and identically distributed Generalised exponential with parameters α and β. i.e.GE (α, β) with respective pdf and c.d.f.

Then accumulated claim amount or aggregate loss is

$S = x_1 + x_2 + ...x_N$ also assume that individual claim amounts, X_i are independent on the annual loss frequency.

Then using convolution principle,

$$f_S(s) = \sum_{k=0}^{\infty} P_r(N=k) f_x^{*k}(s) \tag{4}$$

Where $f_x^{*k}(s)$ is the k^th fold convolution of $x_1 + x_2 + ...x_N$

In finding the initial moments of 'S', we use Panjer-Recursion formula as follows.

4.1 Moments of S

Mean of S

$$E(S) = E(ES/N=n) = EE(\sum_{i=1}^{N} X_i / N=n) = E\sum_{i=1}^{n} X_i = E(N).E(X_i)$$

$$or$$

$$E(S) = E(N).E(X)$$

Variance of S

$$V(S) = E(V(S/N=n)) + V(E(S/N=n))$$

$$Here$$

$$V(S/N=n) = V(\sum_{i=1}^{N} X_i / \mathrm{N=n}) = \mathrm{V}(\sum_{i=1}^{n} X_i)$$

$= \sum_{i=1}^{n} V(X_i) = n(m_2 - m_1^2)$, taking $E(X_i^r) = m_i^r$

$$\therefore V(S) = E(N(m_2 - m_1^2)) + V(Nm_1)$$

$$i.e$$

$$V(S) = E(N(m_2 - m_1^2)) + m_1^2 V(N)$$

$$finally$$

$$V(S) = E(N).V(X) + V(N).(E(X))^2$$

5. Estimation of Mean and Variance of Accumulated Claim Amount Distribution

We will consider Poisson, Negative Binomial and Geometric distribution for the number of cases /claims as follows

1. **Poisson Distribution:** suppose that the annual frequency of losses forms a Poisson distribution with parameter λ.

Then

$$\mathrm{E(S)} = \frac{\lambda}{\beta}(\psi(\alpha+1) - \psi(1))$$

and

$$Var(X) = -\frac{\lambda}{\beta^2}(\psi^{'}(\alpha+1) - \psi^{'}(1)) \tag{5}$$

2. **Negative Binomial distribution:** Suppose that the annual frequency of losses from a portfolio follows a Negative Binomial distribution with parameters r and p

Then

$$E(s)=\frac{(1-p)r}{p.\beta}(\psi(\alpha+1)-\psi(1))$$

$$Var(S)=-\frac{(1-p)r}{p^2\beta^2}(\psi^{'}(\alpha+1)-\psi^{'}(1)) \qquad (6)$$

3. Geometric Distribution: Suppose that the annual frequency of losses follows a Geometric distribution with parameter 'p' Then

$$E(S)=\frac{p}{(1-p).\beta}(\psi(\alpha+1)-\psi(1))$$

$$Var(S)=-\frac{p}{(1-p)^2\beta^2}(\psi^{'}(\alpha+1)-\psi^{'}(1)) \qquad (7)$$

6. Excess of Loss Insurance

Under this arrangement a claim is shared between insurer and reinsurer only if the claims exceed a fixed amount M, called retention level, otherwise the insurer pays the claim in full. Let Y be part of the claim paid by insurer, D be the part paid by the reinsurer for the claim amount S. then,

Y= min(S,M); D= max{0,S-M} such that Y+D = S

7. Numerical Method

In this section we present a numerical calculation of the maximum likelihood estimation for the parameters of α and β. We use Newton-Raphson method as an iterative procedure for the estimation. (Assuming initial values for each of the parameters α and β). The procedure is continued until either the number of iterations will be 200 or when $Y_{m+1}-Y_m < 0.00005$)

$$Y_{n+1}=Y_n-A_n^{-1}B_n$$

Where

$$Y_{n+1}=\begin{bmatrix}\hat{\alpha}_{n+1}\\ \hat{\beta}_{n+1}\end{bmatrix}, Y_n=\begin{bmatrix}\hat{\alpha}_{n}\\ \hat{\beta}_{n}\end{bmatrix}, B_n=\begin{bmatrix}\frac{\partial \log L}{\partial \alpha_n}\\ \frac{\partial \log L}{\partial \beta_n}\end{bmatrix}$$

and

$$B_n=\begin{bmatrix}\frac{\partial^2 \log L}{\partial \alpha_n^2} & \frac{\partial^2 \log L}{\partial \alpha_n \partial \beta_n}\\ \frac{\partial^2 \log L}{\partial \beta_n \partial \alpha_n} & \frac{\partial^2 \log L}{\partial \beta_n^2}\end{bmatrix}$$

7.1 Other Measures

Along with $\hat{\alpha}$ and $\hat{\beta}$ we find relative bias which is the absolute difference between the estimated parameter and its true value divided by its true value.

Relative bias= $\left|\frac{\hat{\alpha}-\alpha_0}{\alpha_0}\right|$ and the mean square error (MSE) which is the mean square of the difference between the estimated parameter are presented for all the estimated parameters considering initial points of the parameters.

MSE= $\sum\frac{(\hat{\alpha}-\alpha_0)^2}{N}$

Where N is the number of experiments did.

Table 1: Estimation of parameters of GE distribution, Relative Bias and MSE

α-initial	β-initial	Parameters	Estimator	Relative Bias	MSE
1	1	α	1.114	0.114	0.00006498
		β	1.2931	0.2931	0.0004295
5	5	α	5.0122	0.00244	0.00000003
		β	5.1388	0.02776	0.0000038
10	10	α	10.5847	0.05847	0.000017
		β	10.2674	0.02674	0.0000035
15	10	α	15.2903	0.01935	0.0000018
		β	11.8266	0.018266	0.000166
20	10	α	21.9748	0.09874	0.0000487
		β	12.3400	0.02340	0.0002738

Table (1) shows the estimators of the parameters of the model. Relative bias and MSE. We can notice that the absolute value of the difference between the true value of the parameter and its estimator is small value converges to zero, so these estimators are said to be consistent estimators.

7.2 Estimation of the mean and variance of Generalised Exponential distribution

By using the estimated value of α and β we get the mean and variance of GE as shown in the table (2).

Table 2: Mean and Variance of GE

$\hat{\alpha}$	$\hat{\beta}$	E(X)	V(X)
1.114	1.2931	0.8283	0.6238
5.0122	5.1388	0.4476	0.0554
10.5847	10.2674	0.2905	0.0147
15.2903	11.8266	0.2821	0.0113
21.9748	12.3400	0.2990	0.0151

From table (2) we could see some relationship between value of $\hat{\alpha}$ and expected mean and variance of X. Also, for $\hat{\beta}$.

7.3 Estimation of Mean and Variance of Aggregate loss distribution

When the claim numbers from the portfolio follows the Poisson distribution with Parameter λ= 5 (say) by substituting in equations (5) or the Negative binomial Distribution with Parameters r=10 and p=0.55 (say)in equations (6) or Geometric Distribution with parameter p=0.6(say) we get the following table (3) as:

Table 3: Estimation of Mean and Variance of Aggregate loss distribution

$\hat{\alpha}$	$\hat{\beta}$	Poisson Distribution		Negative Binomial Distribution		Geometric Distribution	
		E(S)	V(S)	E(S)	V(S)	E(S)	V(S)
1.114	1.2931	4.1415	3.119	6.7769	9.2796	1.2424	2.3393
5.0122	5.1388	2.238	0.277	3.6622	0.8241	0.6714	0.2078
10.5847	10.2674	1.4525	0.735	2.3768	0.2187	0.4358	0.0551
15.2903	11.8266	1.4105	0.0565	2.3081	0.1681	0.4232	0.0424
21.9748	12.3400	1.495	0.0755	2.4464	0.2246	0.4485	0.0566

8. Summary and Conclusion

In this study we emphasis the Generalised Exponential Distribution and its application in aggregate claim distribution. The maximum likelihood estimation is used for estimating the parameters of the distribution. Under the model we estimate the mean and variance of aggregate claim amount and mean and variance of claim frequency distribution Poisson distribution, Negative Binomial Distribution and Geometric Distribution. The analysis shows that the claim amount distribution can make differences in related descriptive measures. The classical process assumes Poisson distribution as incoming claim amount, but from above table shows that other distributions such as Binomial and Negative Binomial distributions could make differences in related measures. If we identify the distribution of the claim amount, we can make use it for analysing various components in connection with risk theory of insurance study.

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