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A study of weighted Lindley distribution: Bayesian approach

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Abstract
A weighted version of Lindley Distribution has been presented in this paper, offering a new single-parameter lifetime distribution. The expressions for the moment generating function (MGF), cumulant generating function (CGF), characteristic function, moments, reliability and hazard rate functions of the new weighted Lindley distribution have also been obtained. Tierney and Kadane approximation method has been used to derive Bayes estimators for its parameter (θ), reliability function R(t), and hazard rate function h(t) under three priors namely uniform, exponential and gamma. The findings have been illustrated using various randomly produced data sets from the proposed model using simulation technique, with each sample replicated 10,000 times. The Bayes Risks have been estimated under Squared Error Loss Function (SELF). Two real life data sets have further been used to illustrate its utility. It is finally concluded that gamma prior outperforms uniform and exponential priors for computing the Bayes estimates of the parameter θ, reliability function R(t) and hazard rate function h(t) of the proposed weighted Lindley distribution.

Keywords: Lindley distribution, weighted Lindley distribution, moments, maximum likelihood estimation, tierney and kadane approximation, reliability function, hazard rate function, uniform prior, exponential prior, gamma prior, simulation technique

1. Introduction
Bayesian inference is a statistical analytic tool that relies on the simple notion that probability is the only way to adequately describe uncertainty. It involves the use of Bayes Theorem through which the probability of an event can be found, given that, another event is known to have already occurred. This theorem lays the foundation of Bayesian approach. It makes use of the information, we already have about the unknown parameter (commonly known as prior information), allowing us to get better estimates for the unknown population and hence obtain better outcomes. Biostatistics, Actuarial Sciences, Business analytics and modelling and testing of lifetime experiments are some major fields of research where Bayesian Estimation plays an important role. The Bayesian estimation approach is a non-classical estimation technique in statistical inference and is very useful in real world situation.

Bayesian analysis has been widely employed in lifetime distributions in recent years, but its implementation is difficult because it can result in analytically inflexible posterior models, which are difficult to compute numerically. The statistical analysis of life time and failure time data has become extremely important in recent years. The primary goal of life time model analysis is to obtain information on failure. This information is then used for quantifying reliability, improving product reliability, and determining whether safety and reliability objectives have been met. Exponential, Weibull, normal, gamma, modified gamma, and other key probability distributions are utilized as life testing models. The Exponential and Gamma distributions are the most widely used distributions in lifetime modelling as they possess several structural characteristics. For example, Exponential distribution shows memoryless property as well as constant hazard rate. Also, the Exponential distribution, which is a special case of the gamma distribution, has been used in modelling time-to-event data or modelling waiting times, and their various extensions can be found in the literature for describing the uncertainty behind real-world phenomena in the areas of survival modelling and reliability engineering. In many practical situations it has been observed that the extensions of many distributions fit better than their corresponding standard ones. Bhattacharya (1967) [20] pioneered the use of Bayesian analysis in reliability and life testing, as well as in estimating the parameter and reliability of a one-parameter exponential distribution with type II censoring. Several modifications and extensions have been carried out in various distributions till date but none of them provides universal accuracy for real-life examples in the fields of survival analysis. For this reason, research for several new probability distributions is being carried out on a regular basis.

Lindley first proposed the Lindley distribution as a one-scale parameter distribution (1958). Lindley distribution has received a lot of attention in recent years because of its usefulness in modelling complex real-life data. Some academics pursued a more in-depth investigation of the Lindley distribution and its features. It has been discovered that in modelling the life testing experiments, finite mixed distributions perform better than standard ones. The utility of finite mixed distribution for research work in numerous domains is increasing nowadays. The Lindley distribution is generally considered to be a mixture of Exponential distribution (with parameter θ) and Gamma distribution (with shape parameter 2 and rate parameter θ).

Many scholars have expressed interest in this distribution, which has been generalized multiple times by various authors. First, Sankaran (1970) [15] employed its original pdf to develop the discrete Poisson-Lindley Distribution (PLD) where the parameter follows a Poisson Law. Later, Asgharzadeh *et al*. (2013) [1], Ghitany *et al*. (2008a) [7], and Ghitany *et al*. (2008b) [8] presented Zero-truncated Poisson-Lindley and Pareto Poisson-Lindley distributions, which are limited to the original pdf of the Lindley distribution. In addition, Zeghdoudi and Nedjar (2016) [19] presented the Gamma-Lindley distribution, which is based on combinations of gamma (2, θ) and one-parameter Lindley distributions.

When standard distributions are not applicable, weighted distributions are a stepping stone towards effective statistical data modelling and prediction. They are frequently utilized for the building of appropriate statistical models in various domains, including medicine, ecology, and dependability, to mention a few. A large number of researches on weighted distributions have been reported in the literature. The sampling frames in many observational studies were not clearly specified, and the collected observations were skewed. They did not have an equal opportunity to be recorded. Such data do not follow the original distribution unless each observation is given an equal probability of being recorded, and hence their modelling gave rise to the notion of weighted distributions. Fisher (1934) [6] established the notion of weighted distributions to investigate the influence of ascertainment methods on frequency estimates. Later, C.R. Rao (1965) [21] identified the situations that can be modelled using weighted distributions with a unified approach and introduced the use of weighted distributions as a method of adjustment applicable to various statistical situations.

We have proposed the weighted version of Lindley Distribution for survival analysis as well as the statistical analysis of life testing experiments. We have obtained the Bayes estimators of the parameter, its reliability function and its hazard rate function under three different priors namely Uniform, Exponential, and Gamma. The obtained estimators have further been compared using Squared Error Loss Function (SELF). The study has been illustrated with the help of simulated as well as real data sets.

This paper contains the following sections: Section 2 completes the presentation of Weighted Lindley Distribution by expressing some functions of interest and important properties. Section 3 deals with the Classical and Bayesian Estimation of parameter (θ), along with its MSE, Reliability function (R(t)) and Hazard Rate function (h(t)). Section 4 shows the application of proposed model to 2 real data sets. In Section 5, a simulation study is shown. Section 6 graphically depicts MLE for R(t) and h(t) for different values of t. The conclusions have been reported in Section 7.

2. Weighted Lindley Distribution

Let us suppose that the original observation, say X, has $f_o(x)$ as the Probability Density Function and the probability of recording the observation x is $0 < w(x) < 1$, then According to Patil and Rao (1978) [11], the pdf of X^w (the recorded observation) is

$$f(x) = \frac{w(x) f_o(x)}{E(w(x))} \tag{1}$$

Where, w(x) is a non-negative weight function and $f_o(x)$ is the pdf of original distribution.

The Probability Density Function (PDF) of Lindley Distribution with parameter θ is given by

$$f_o(x) = \frac{\theta^2}{(1+\theta)}(1+x)\exp(-\theta x) \tag{2}$$

Taking weight function, $w(x) = \frac{1}{\theta}\exp(-\theta x)$ and original distribution $f_o(x)$ as (2), the Weighted Lindley density with parameter θ is obtained as:

$$f(x) = \frac{4\theta^2}{(2\theta+1)}(1+x)\exp(-2\theta x); \theta > 0, x > 0 \tag{3}$$

2.1 Some functions of interest
2.1.1 Cumulative Density Function

$$F(x) = 1 - \exp(-2\theta x)\left[1 + \frac{2\theta x}{(2\theta+1)}\right] \tag{4}$$

2.1.2 Reliability Function

R(t)=1 - F(t)

$$R(t) = \exp(-2\theta t)\left[1 + \frac{2\theta t}{(2\theta+1)}\right] \tag{5}$$

2.1.3 Hazard rate function

$$h(t) = \frac{f(t)}{S(t)}$$

$$h(t) = \frac{4\theta^2(1+t)}{(2\theta t + 2\theta + 1)} \tag{6}$$

2.2 Moments and some associated measures

The moment generating function (MGF) of X is given by $M_X(t)$

$$M_X(t) = \frac{4\theta^2}{(2\theta+1)}\frac{1}{(2\theta-t)}\left[1 + \frac{1}{(2\theta-t)}\right]; \quad |t| < 2\theta \tag{7}$$

The characteristic function (CF) of X is given by

$$\phi_X(t) = E\left(e^{itX}\right) = \frac{4\theta^2}{(2\theta+1)}\frac{1}{(2\theta-it)}\left[1 + \frac{1}{(2\theta-it)}\right]; \theta > 0, t \in R \tag{8}$$

Cumulant Generating Function (CGF) of X is given by

$$K_X(t) = \log M_X(t) = \log 4\theta^2 - \log(2\theta+1) - 2\log(2\theta-t) + \log(2\theta-t+1); \theta > 0, t \in R \tag{9}$$

The r^{th} order raw moment is given by:

$$E(X^r) = \int_0^\infty \frac{4\theta^2}{(2\theta+1)}(1+x)\exp(-2\theta x)x^r\,dx$$

$$E(X^r) = \frac{r!}{(2\theta+1)}\frac{[2\theta+(r+1)]}{(2\theta)^r} \tag{10}$$

Mean= $$E(X) = \frac{(\theta+1)}{\theta(2\theta+1)} \tag{11}$$

Variance = $$V(X) = \frac{(2\theta^2+4\theta+1)}{2\theta^2(2\theta+1)^2} = \mu_2 \tag{12}$$

Now, Coefficient of skewness (β_1) is given by

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} \qquad \beta_1 = \frac{2(4\theta^3 + 12\theta^2 + 6\theta + 1)^2}{(2\theta^2 + 4\theta + 1)^3} \tag{13}$$

Coefficient of kurtosis (β_2) is given by

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

$$\beta_2 = \frac{6(6\theta^4 + 8\theta^3 + 30\theta^2 + 16\theta + 1)}{(2\theta^2 + 4\theta + 1)^2} \tag{14}$$

3. Classical Estimation and Bayesian Estimation under 3 different priors

3.1 Classical Estimation

3.1.1 Method of moment estimator

Let $X_1, X_2, X_3, ..., X_n$ denote a random sample having size n which has been drawn from the Weighted Lindley distribution. If $\overline{X}$ is the mean of the sample, then by the application of Method of Moments, we get,

$$\overline{X} = \frac{(\theta + 1)}{\theta(2\theta + 1)}$$

Now, Let $\hat{\theta}_M$ be the method of moment estimator of θ, then

$$\hat{\theta}_M = \frac{-(\overline{X} - 1) + \sqrt{\overline{X}^2 + 6\overline{X} + 1}}{4\overline{X}} \tag{15}$$

3.1.2 Maximum Likelihood Estimator

Let the sample observations on $X_1, X_2, ..., X_n$ be denoted as $\tilde{x}' = (x_1, x_2, ..., x_n)$. Then the Likelihood function of θ given $\tilde{x}$ can be written as:

$$l(\theta \mid \tilde{x}) = \prod_{i=1}^{n} \frac{4\theta^2}{(2\theta + 1)}(1 + x_i)\exp(-2\theta x_i)$$

$$= \frac{(4\theta^2)^n}{(2\theta + 1)^n}\exp(-2\theta\sum_{i=1}^{n} x_i)\prod_{i=1}^{n}(1 + x_i) \tag{16}$$

The log-likelihood function hence becomes,

$$L(\theta \mid \tilde{x}) = n\log 4\theta^2 - n\log(2\theta + 1) - 2\theta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\log(1 + x_i) \tag{17}$$

Differentiating it with respect to θ and equating it to 0, we get

$$\frac{\partial}{\partial\theta}L(\theta \mid \tilde{x}) = \frac{2n}{\theta} - \frac{2n}{(2\theta + 1)} - 2\sum_{i=1}^{n} x_i = 0 \tag{18}$$

Again differentiating it with respect to θ, we have

$$\frac{\partial^2}{\partial\theta^2}L(\theta \mid \tilde{x}) = \frac{-2n}{\theta^2} + \frac{4n}{(2\theta + 1)^2} \tag{19}$$

It is not possible to solve this equation analytically, thus we find the maximum likelihood estimate $(\hat{\theta})$ of θ with the help of Newton-Raphson method. The MLE's $(\hat{R}(t), \hat{h}(t))$ of reliability function R(t) and hazard rate function h(t) are obtained by using the invariance property.

3.2 Bayesian Estimation

We obtain the Bayes estimators of parameter θ, reliability function and hazard rate function under Uniform, Exponential and Gamma priors using Tierney and Kadane method of approximation.

3.2.1 Uniform prior

$$g_1(\theta) = \frac{1}{\theta}, \theta > 0 \tag{20}$$

3.2.2 Exponential prior

$$g_2(\theta) = e^{-\theta} \tag{21}$$

3.2.3 Gamma prior

$$g_3(\theta) = \frac{1}{\Gamma b}\theta^{b-1}\exp(-\theta) \tag{22}$$

3.2.1(a) Bayes estimation of θ under Uniform prior

The Posterior distribution of θ is given by,

$$g_1(\theta \mid \tilde{x}) = \frac{l(\theta \mid \tilde{x}).g_1(\theta)}{\int_0^\infty l(\theta \mid \tilde{x}).g_1(\theta)d\theta}$$

$$g_1(\theta \mid \tilde{x}) = \frac{\frac{\theta^{2n-1}}{(2\theta+1)^n}\exp(-2\theta\sum_{i=1}^n x_i)\prod_{i=1}^n(1+x_i)}{\int_0^\infty \frac{\theta^{2n-1}}{(2\theta+1)^n}\exp(-2\theta\sum_{i=1}^n x_i)\prod_{i=1}^n(1+x_i)d\theta}$$

The integration obtained in the denominator of the above equation is not in the closed form. Hence, the Bayes estimator of θ under Squared Error Loss Function (SELF) can be obtained by solving:

$$\tilde{\theta} = E(\theta) = \int_\theta \theta.g_1(\theta \mid \tilde{x})d\theta$$

$$\tilde{\theta} = \frac{\int_0^\infty \frac{\theta^{2n}}{(2\theta+1)^n}\exp(-2\theta\sum_{i=1}^n x_i)\prod_{i=1}^n(1+x_i).\,d\theta}{\int_0^\infty \frac{\theta^{2n-1}}{(2\theta+1)^n}\exp(-2\theta\sum_{i=1}^n x_i)\prod_{i=1}^n(1+x_i)\,d\theta}$$

The integral thus obtained is the ratio of two integrals and it cannot be directly solved.
Hence, It has been solved with the help of Tierney and Kadane method.
Tierney and Kadane method consists of the following steps:

$$\delta(\theta) = \log 4\theta^2 - \log(2\theta+1) - 2\theta\frac{\sum_{i=1}^n x_i}{n} + \frac{1}{n}\sum_{i=1}^n \log(1+x_i) - \frac{1}{n}\log\theta$$

$$\delta_\theta^*(\theta) = \log 4\theta^2 - \log(2\theta+1) - 2\theta\frac{\sum_{i=1}^n x_i}{n} + \frac{1}{n}\sum_{i=1}^n \log(1+x_i)$$

$$\delta_{\theta^2}^*(\theta) = \log 4\theta^2 - \log(2\theta+1) - 2\theta\frac{\sum_{i=1}^n x_i}{n} + \frac{1}{n}\sum_{i=1}^n \log(1+x_i) + \frac{1}{n}\log\theta$$

Now,

$$\frac{\partial^2}{\partial\theta^2}\delta(\theta) = -\frac{2}{\theta^2} + \frac{4}{(2\theta+1)^2} + \frac{1}{n\theta^2}$$

$$\frac{\partial}{\partial\theta}\delta^*_{\theta^2}(\theta)=\frac{2}{\theta}-\frac{2}{(2\theta+1)}-\frac{2\sum_{i=1}^{n}x_i}{n}+\frac{1}{n\theta}$$

$$\frac{\partial^2}{\partial\theta^2}\delta^*_{\theta^2}(\theta)=-\frac{2}{\theta^2}+\frac{4}{(2\theta+1)^2}-\frac{1}{n\theta^2}$$

In the next step, we find

$$\delta(\hat{\theta}),\delta^*_{\theta}(\hat{\theta}),\delta^*_{\theta^2}(\hat{\theta})$$

and $\Sigma,\Sigma^*_{\theta},\Sigma^*_{\theta^2}$ as:

$$\Sigma=\left|\frac{1}{\frac{\partial^2}{\partial\theta^2}\delta(\theta)}\right|_{\theta=\hat{\theta}}\quad \Sigma^*_{\theta}=\left|\frac{1}{\frac{\partial^2}{\partial\theta^2}\delta^*_{\theta}(\theta)}\right|_{\theta=\hat{\theta}}\quad \Sigma^*_{\theta^2}=\left|\frac{1}{\frac{\partial^2}{\partial\theta^2}\delta^*_{\theta^2}(\theta)}\right|_{\theta=\hat{\theta}}$$

We obtain the Bayes estimator of θ as $\tilde{\theta}=E(\theta)=\hat{\theta}\sqrt{\frac{|\Sigma^*_{\theta}|}{|\Sigma|}}$

Mean square error of $\tilde{\theta}$, $MSE(\tilde{\theta})=E(\tilde{\theta}-\theta)^2$

Now, the Bayes estimator of θ^2, $E(\theta^2)=\hat{\theta}^2\sqrt{\frac{|\Sigma^*_{\theta^2}|}{|\Sigma|}}$

$$Risk(\tilde{\theta})=\tilde{\theta}^2-(\tilde{\theta})^2$$

3.2.1(b) Bayes estimator of Reliability function under Uniform prior

Bayes estimator of Reliability function can be evaluated as:

$$\delta^*_{R(t)}(\theta)=\log 4\theta^2-\left(\frac{n+1}{n}\right)\log(2\theta+1)-\frac{2\theta\sum_{i=1}^{n}x_i}{n}+\frac{1}{n}\sum_{i=1}^{n}\log(1+x_i)-\frac{\log\theta}{n}-\frac{2\theta t}{n}+\frac{1}{n}\log(2\theta t+2\theta+1)$$

$$\delta^*_{R^2(t)}(\theta)=\log 4\theta^2-\left(\frac{n+2}{n}\right)\log(2\theta+1)-\frac{2\theta\sum_{i=1}^{n}x_i}{n}+\frac{1}{n}\sum_{i=1}^{n}\log(1+x_i)-\frac{\log\theta}{n}-\frac{4\theta t}{n}+\frac{2}{n}\log(2\theta t+2\theta+1)$$

Now,

$$\frac{\partial^2}{\partial\theta^2}\delta^*_{R(t)}(\theta)=-\frac{2}{\theta^2}+\frac{4(n+1)}{n(2\theta+1)^2}+\frac{1}{n\theta^2}-\frac{4(t+1)^2}{n(2\theta t+2\theta+1)^2}$$

$$\frac{\partial^2}{\partial\theta^2}\delta^*_{R^2(t)}(\theta)=-\frac{2}{\theta^2}+\frac{4(n+2)}{n(2\theta+1)^2}+\frac{1}{n\theta^2}-\frac{8(t+1)^2}{n(2\theta t+2\theta+1)^2}$$

Bayes estimator of *R(t)*, $\tilde{R}(t)=E(R(t))=\sqrt{\frac{|\Sigma^*_{R(t)}|}{|\Sigma|}}\left[1+\frac{2\hat{\theta}t}{(2\hat{\theta}+1)}\right]\exp(-2\hat{\theta}t)$

$$MSE(\tilde{R}(t))=E(\tilde{R}(t)-R(t))^2$$

Next, we find the Bayes estimator of $R^2(t)$ by using

$$\tilde{R}^2(t)=E\left(R^2(t)\right)=\sqrt{\frac{\left|\Sigma^*_{R^2(t)}\right|}{|\Sigma|}}\left[1+\frac{2\hat{\theta}t}{\left(2\hat{\theta}+1\right)}\right]^2\exp\left(-4\hat{\theta}t\right)$$

$$Risk\ \tilde{R}(t)=\tilde{R}^2(t)-\left(\tilde{R}(t)\right)^2$$

3.2.1(c) Bayes estimator of Hazard Rate function under Uniform Prior
Bayes estimator of Hazard Rate function can be evaluated as:

$$\delta^*_{h(t)}(\theta)=\frac{(n+1)}{n}\log 4\theta^2-\log(2\theta+1)-\frac{2\theta\sum_{i=1}^{n}x_i}{n}+\frac{1}{n}\sum_{i=1}^{n}\log(1+x_i)-\frac{\log\theta}{n}+\frac{1}{n}\log(1+t)-\frac{1}{n}\log(2\theta t+2\theta+1)$$

$$\delta^*_{h^2(t)}(\theta)=\frac{(n+2)}{n}\log 4\theta^2-\log(2\theta+1)-\frac{2\theta\sum_{i=1}^{n}x_i}{n}+\frac{1}{n}\sum_{i=1}^{n}\log(1+x_i)-\frac{\log\theta}{n}+\frac{2}{n}\log(1+t)-\frac{2}{n}\log(2\theta t+2\theta+1)$$

Now,

$$\frac{\partial^2}{\partial\theta^2}\delta^*_{h(t)}(\theta)=-\frac{2(n+1)}{n\theta^2}+\frac{4}{(2\theta+1)^2}+\frac{1}{n\theta^2}+\frac{4(t+1)^2}{n(2\theta t+2\theta+1)^2}$$

$$\frac{\partial^2}{\partial\theta^2}\delta^*_{h^2(t)}(\theta)=-\frac{2(n+1)}{n\theta^2}+\frac{4}{(2\theta+1)^2}+\frac{1}{n\theta^2}+\frac{8(t+1)^2}{n(2\theta t+2\theta+1)^2}$$

Bayes estimator of *h(t)*,

$$\tilde{h}(t)=E\left(h(t)\right)=\sqrt{\frac{\left|\Sigma^*_{h(t)}\right|}{|\Sigma|}}\frac{4\hat{\theta}^2(1+t)}{\left(2\hat{\theta}t+2\hat{\theta}+1\right)}$$

$$MSE\left(\tilde{h}(t)\right)=E\left(\tilde{h}(t)-h(t)\right)^2$$

Now, we find the estimator of $h^2(t)$ as,

$$\tilde{h}^2(t)=E\left(h^2(t)\right)=\sqrt{\frac{\left|\Sigma^*_{h^2(t)}\right|}{|\Sigma|}}\frac{16\hat{\theta}^2(1+t)^2}{\left(2\hat{\theta}t+2\hat{\theta}+1\right)^2}$$

$$Risk\ \tilde{h}(t)=\tilde{h}^2(t)-\left(\tilde{h}(t)\right)^2$$

3.2.2(a) Bayes estimator of θ under Exponential prior
The Posterior distribution of θ is given by,

$$g_2(\theta\,|\,\tilde{x})=\frac{l(\theta\,|\,\tilde{x}).g_2(\theta)}{\int_0^\infty l(\theta\,|\,\tilde{x}).g_2(\theta)d\theta}$$

$$g_2(\theta\,|\,\tilde{x})=\frac{\frac{\theta^{2n}}{(2\theta+1)^n}\exp(-\theta(1+2\sum_{i=1}^{n}x_i))\prod_{i=1}^{n}(1+x_i)}{\int_0^\infty\frac{\theta^{2n}}{(2\theta+1)^n}\exp(-\theta(1+2\sum_{i=1}^{n}x_i))\prod_{i=1}^{n}(1+x_i)d\theta}$$

Bayes estimator of θ under Squared Error Loss Function (SELF) can be obtained by solving

$$\tilde{\theta}=E(\theta)=\int_\theta \theta.g_1(\theta\,|\,\tilde{x})d\theta$$

$$\tilde{\theta} = \frac{\int_0^\infty \frac{\theta^{2n+1}}{(2\theta+1)^n}\exp(-\theta(1+2\sum_{i=1}^n x_i))\prod_{i=1}^n(1+x_i)d\theta}{\int_0^\infty \frac{\theta^{2n}}{(2\theta+1)^n}\exp(-\theta(1+2\sum_{i=1}^n x_i))\prod_{i=1}^n(1+x_i)d\theta}$$

The integral thus obtained is the ratio of two integrals and it cannot be directly solved.
Hence, It has been solved with the help of Tierney and Kadane method.

$$\delta(\theta) = \log 4\theta^2 - \log(2\theta+1) - 2\theta\frac{\sum_{i=1}^n x_i}{n} + \frac{1}{n}\sum_{i=1}^n \log(1+x_i) - \frac{\theta}{n}$$

$$\delta_\theta^*(\theta) = \log 4\theta^2 - \log(2\theta+1) - 2\theta\frac{\sum_{i=1}^n x_i}{n} + \frac{1}{n}\sum_{i=1}^n \log(1+x_i) + \frac{1}{n}\log\theta - \frac{\theta}{n}$$

$$\delta_{\theta^2}^*(\theta) = \log 4\theta^2 - \log(2\theta+1) - 2\theta\frac{\sum_{i=1}^n x_i}{n} + \frac{1}{n}\sum_{i=1}^n \log(1+x_i) + \frac{2}{n}\log\theta - \frac{\theta}{n}$$

Now,

$$\frac{\partial^2}{\partial\theta^2}\delta(\theta) = -\frac{2}{\theta^2} + \frac{4}{(2\theta+1)^2}$$

$$\frac{\partial^2}{\partial\theta^2}\delta_\theta^*(\theta) = -\frac{2}{\theta^2} + \frac{4}{(2\theta+1)^2} - \frac{1}{n\theta^2}$$

$$\frac{\partial^2}{\partial\theta^2}\delta_{\theta^2}^*(\theta) = -\frac{2}{\theta^2} + \frac{4}{(2\theta+1)^2} - \frac{2}{n\theta^2}$$

In the next step, we find

$$\delta(\hat{\theta}), \delta_\theta^*(\hat{\theta}), \delta_{\theta^2}^*(\hat{\theta})$$

and $\Sigma, \Sigma_\theta^*, \Sigma_{\theta^2}^*$ as:

$$\Sigma = \left|\frac{1}{\frac{\partial^2}{\partial\theta^2}\delta(\theta)}\right|_{\theta=\hat{\theta}} \quad \Sigma_\theta^* = \left|\frac{1}{\frac{\partial^2}{\partial\theta^2}\delta_\theta^*(\theta)}\right|_{\theta=\hat{\theta}} \quad \Sigma_{\theta^2}^* = \left|\frac{1}{\frac{\partial^2}{\partial\theta^2}\delta_{\theta^2}^*(\theta)}\right|_{\theta=\hat{\theta}}$$

Next, we find the estimator of θ as,

$$\tilde{\theta} = E(\theta) = \hat{\theta}\sqrt{\frac{|\Sigma_\theta^*|}{|\Sigma|}}$$

$$MSE(\tilde{\theta}) = E(\tilde{\theta} - \theta)^2$$

Now, the Bayes estimator of θ^2 can be obtained by using

$$E(\theta^2) = \hat{\theta}^2\sqrt{\frac{|\Sigma_{\theta^2}^*|}{|\Sigma|}}$$

$$Risk\left(\tilde{\theta}\right)=\tilde{\theta}^{2}-(\tilde{\theta})^{2}$$

3.2.2(b) Bayes estimator of Reliability function under Exponential prior
Bayes estimator of Reliability function can be evaluated as:

$$\delta^{*}_{R(t)}(\theta)=\log 4\theta^{2}-\left(\frac{n+1}{n}\right)\log(2\theta+1)-\frac{2\theta\sum_{i=1}^{n}x_i}{n}+\frac{1}{n}\sum_{i=1}^{n}\log(1+x_i)-\frac{\theta}{n}-\frac{2\theta t}{n}+\frac{1}{n}\log(2\theta t+2\theta+1)$$

$$\delta^{*}_{R^2(t)}(\theta)=\log 4\theta^{2}-\left(\frac{n+2}{n}\right)\log(2\theta+1)-\frac{2\theta\sum_{i=1}^{n}x_i}{n}+\frac{1}{n}\sum_{i=1}^{n}\log(1+x_i)-\frac{\theta}{n}-\frac{4\theta t}{n}+\frac{2}{n}\log(2\theta t+2\theta+1)$$

Now,

$$\frac{\partial^2}{\partial\theta^2}\delta^{*}_{R(t)}(\theta)=-\frac{2}{\theta^2}+\frac{4(n+1)}{n(2\theta+1)^2}-\frac{4(t+1)^2}{n(2\theta t+2\theta+1)^2}$$

$$\frac{\partial^2}{\partial\theta^2}\delta^{*}_{R^2(t)}(\theta)=-\frac{2}{\theta^2}+\frac{4(n+2)}{n(2\theta+1)^2}-\frac{8(t+1)^2}{n(2\theta t+2\theta+1)^2}$$

Bayes estimator of *R(t)*,

$$\tilde{R}(t)=E(R(t))=\sqrt{\frac{\left|\Sigma^{*}_{R(t)}\right|}{|\Sigma|}}\left[1+\frac{2\hat{\theta}t}{\left(2\hat{\theta}+1\right)}\right]\exp\left(-2\hat{\theta}t\right)$$

$$MSE\left(\tilde{R}(t)\right)=E\left(\tilde{R}(t)-R(t)\right)^{2}$$

Next, we find the Bayes estimator of $R^2(t)$ by using

$$\tilde{R}^{2}(t)=E\left(R^{2}(t)\right)=\sqrt{\frac{\left|\Sigma^{*}_{R^2(t)}\right|}{|\Sigma|}}\left[1+\frac{2\hat{\theta}t}{\left(2\hat{\theta}+1\right)}\right]^{2}\exp\left(-4\hat{\theta}t\right)$$

$$Risk\ \tilde{R}(t)=\tilde{R}^{2}(t)-\left(\tilde{R}(t)\right)^{2}$$

3.2.2(c) Bayes estimator of Hazard Rate function under Exponential prior
Bayes estimator of Hazard Rate function can be evaluated as:

$$\delta^{*}_{h(t)}(\theta)=\frac{(n+1)}{n}\log 4\theta^{2}-\log(2\theta+1)-\frac{2\theta\sum_{i=1}^{n}x_i}{n}+\frac{1}{n}\sum_{i=1}^{n}\log(1+x_i)-\frac{\theta}{n}+\frac{1}{n}\log(1+t)-\frac{1}{n}\log(2\theta t+2\theta+1)$$

$$\delta^{*}_{h^2(t)}(\theta)=\frac{(n+2)}{n}\log 4\theta^{2}-\log(2\theta+1)-\frac{2\theta\sum_{i=1}^{n}x_i}{n}+\frac{1}{n}\sum_{i=1}^{n}\log(1+x_i)-\frac{\theta}{n}+\frac{2}{n}\log(1+t)-\frac{2}{n}\log(2\theta t+2\theta+1)$$

Now,

$$\frac{\partial^2}{\partial\theta^2}\delta^{*}_{h(t)}(\theta)=-\frac{2(n+1)}{n\theta^2}+\frac{4}{(2\theta+1)^2}+\frac{4(t+1)^2}{n(2\theta t+2\theta+1)^2}$$

$$\frac{\partial^2}{\partial\theta^2}\delta^{*}_{h^2(t)}(\theta)=-\frac{2(n+1)}{n\theta^2}+\frac{4}{(2\theta+1)^2}+\frac{8(t+1)^2}{n(2\theta t+2\theta+1)^2}$$

Bayes estimator of *h(t),* $\tilde{h}(t) = E(h(t)) = \sqrt{\frac{\left|\Sigma^{*}_{h(t)}\right|}{|\Sigma|}} \frac{4\hat{\theta}^2(1+t)}{(2\hat{\theta}t + 2\hat{\theta} + 1)}$

$$MSE(\tilde{h}(t)) = E(\tilde{h}(t) - h(t))^2$$

Now, we find the estimator of $h^2(t)$ as,

$$\tilde{h}^2(t) = E(h^2(t)) = \sqrt{\frac{\left|\Sigma^{*}_{h^2(t)}\right|}{|\Sigma|}} \frac{16\hat{\theta}^2(1+t)^2}{(2\hat{\theta}t + 2\hat{\theta} + 1)^2}$$

$$Risk\,\tilde{h}(t) = \tilde{h}^2(t) - (\tilde{h}(t))^2$$

3.2.3(a) Bayes estimator of θ under Gamma prior

The Posterior Distribution of θis given by

$$g_3(\theta|\tilde{x}) = \frac{l(\theta|\tilde{x}).g_3(\theta)}{\int_0^\infty l(\theta|\tilde{x}).g_3(\theta)d\theta}$$

$$g_3(\theta|\tilde{x}) = \frac{\frac{\theta^{2n+b-1}}{(2\theta+1)^n}\exp\left(-\theta\left(2\sum_{i-1}^{n} x_i + 1\right)\right)\prod_{i=1}^{n}(1+x_i)}{\int_0^\infty \frac{\theta^{2n+b-1}}{(2\theta+1)^n}\exp\left(-\theta\left(2\sum_{i-1}^{n} x_i + 1\right)\right)\prod_{i=1}^{n}(1+x_i)d\theta}$$

The integration obtained in the denominator of the above equation is not in the closed form.
Bayes estimator of θ under Squared Error Loss Function (SELF) can be obtained by solving

$$\tilde{\theta} = E(\theta) = \int_\theta \theta.g_3(\theta \mid \tilde{x})d\theta$$

$$\tilde{\theta} = E(\theta) = \frac{\int_0^\infty \frac{\theta^{2n+b}}{(2\theta+1)^n}\exp\left(-\theta\left(2\sum_{i-1}^{n} x_i + 1\right)\right)\prod_{i=1}^{n}(1+x_i)d\theta}{\int_0^\infty \frac{\theta^{2n+b-1}}{(2\theta+1)^n}\exp\left(-\theta\left(2\sum_{i-1}^{n} x_i + 1\right)\right)\prod_{i=1}^{n}(1+x_i)d\theta}$$

To solve the integral so obtained, Tierney Kadane method of approximation has been applied again.

$$\delta(\theta) = \log 4\theta^2 - \log(2\theta+1) - 2\theta\frac{\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n}\log(1+x_i) - \frac{\theta}{n} - \frac{1}{n}\log\Gamma b + \frac{(b-1)}{n}\log\theta$$

$$\delta^{*}_{\theta}(\theta) = \log 4\theta^2 - \log(2\theta+1) - 2\theta\frac{\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n}\log(1+x_i) - \frac{\theta}{n} - \frac{1}{n}\log\Gamma b + \frac{b}{n}\log\theta$$

$$\delta^{*}_{\theta^2}(\theta) = \log 4\theta^2 - \log(2\theta+1) - 2\theta\frac{\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n}\log(1+x_i) - \frac{\theta}{n} - \frac{1}{n}\log\Gamma b + \frac{(b+1)}{n}\log\theta$$

$$\frac{\partial^2}{\partial\theta^2}\delta(\theta) = -\frac{2}{\theta^2} + \frac{4}{(2\theta+1)^2} + \frac{(1-b)}{n\theta^2}$$

$$\frac{\partial^2}{\partial\theta^2}\delta_\theta^*(\theta) = -\frac{2}{\theta^2} + \frac{4}{(2\theta+1)^2} - \frac{b}{n\theta^2}$$

$$\frac{\partial^2}{\partial\theta^2}\delta_{\theta^2}^*(\theta) = -\frac{2}{\theta^2} + \frac{4}{(2\theta+1)^2} - \frac{(b+1)}{n\theta^2}$$

Next, We obtain the Bayes estimator of θ as $\tilde{\theta} = E(\theta) = \hat{\theta}\sqrt{\frac{|\Sigma_\theta^*|}{|\Sigma|}}$

Mean square error of $\tilde{\theta}$, $MSE(\tilde{\theta}) = E(\tilde{\theta} - \theta)^2$

Now, the Bayes estimator of θ^2 can be obtained by using

$$E(\theta^2) = \hat{\theta}^2\sqrt{\frac{|\Sigma_{\theta^2}^*|}{|\Sigma|}}$$

$$Risk(\tilde{\theta}) = \tilde{\theta}^2 - (\tilde{\theta})^2$$

3.2.3(b) Bayes estimator of Reliability function under Gamma prior

Bayes estimator of Reliability function can be evaluated as:

$$\delta_{R(t)}^*(\theta) = \log 4\theta^2 - \frac{(n+1)}{n}\log(2\theta+1) - \frac{2\theta\sum_{i=1}^n x_i}{n} + \frac{1}{n}\sum_{i=1}^n \log(1+x_i) - \frac{\theta}{n} - \frac{\log\Gamma b}{n} + \frac{(b-1)}{n}\log\theta - \frac{2\theta t}{n} + \frac{1}{n}\log(2\theta t + 2\theta + 1)$$

$$\delta_{R^2(t)}^*(\theta) = \log 4\theta^2 - \frac{(n+2)}{n}\log(2\theta+1) - \frac{2\theta\sum_{i=1}^n x_i}{n} + \frac{1}{n}\sum_{i=1}^n \log(1+x_i) - \frac{\theta}{n} - \frac{\log\Gamma b}{n} + \frac{(b-1)}{n}\log\theta - \frac{4\theta t}{n} + \frac{2}{n}\log(2\theta t + 2\theta + 1)$$

Now,

$$\frac{\partial^2}{\partial\theta^2}\delta_{R(t)}^*(\theta) = -\frac{2}{\theta^2} + \frac{4(n+1)}{n(2\theta+1)^2} + \frac{(1-b)}{n\theta^2} - \frac{4(t+1)^2}{n(2\theta t + 2\theta + 1)^2}$$

$$\frac{\partial^2}{\partial\theta^2}\delta_{R^2(t)}^*(\theta) = -\frac{2}{\theta^2} + \frac{4(n+2)}{n(2\theta+1)^2} + \frac{(1-b)}{n\theta^2} - \frac{8(t+1)^2}{n(2\theta t + 2\theta + 1)^2}$$

Bayes estimator of *R(t)*, $\tilde{R}(t) = E(R(t)) = \sqrt{\frac{|\Sigma_{R(t)}^*|}{|\Sigma|}}\left[1 + \frac{2\hat{\theta}t}{(2\hat{\theta}+1)}\right]\exp(-2\hat{\theta}t)$

$$MSE(\tilde{R}(t)) = E(\tilde{R}(t) - R(t))^2$$

Next, we find the Bayes estimator of $R^2(t)$ by using

$$\tilde{R}^2(t) = E(R^2(t)) = \sqrt{\frac{|\Sigma_{R^2(t)}^*|}{|\Sigma|}}\left[1 + \frac{2\hat{\theta}t}{(2\hat{\theta}+1)}\right]^2 \exp(-4\hat{\theta}t)$$

$$Risk\ \tilde{R}(t) = \tilde{R}^2(t) - \left(\tilde{R}(t)\right)^2$$

3.2.3(c) Bayes estimator of Hazard Rate function under Gamma prior

Bayes estimator of Hazard Rate function:

$$\delta^*_{h(t)}(\theta) = \frac{(n+1)}{n}\log 4\theta^2 - \log(2\theta+1) - \frac{2\theta\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n}\log(1+x_i) - \frac{\theta}{n} - \frac{\log\Gamma b}{n} + \frac{(b-1)\log\theta}{n} + \frac{1}{n}\log(1+t) - \frac{1}{n}\log(2\theta t + 2\theta + 1)$$

$$\delta^*_{h^2(t)}(\theta) = \frac{(n+2)}{n}\log 4\theta^2 - \log(2\theta+1) - \frac{2\theta\sum_{i=1}^{n} x_i}{n} + \frac{1}{n}\sum_{i=1}^{n}\log(1+x_i) - \frac{\theta}{n} - \frac{\log\Gamma b}{n} + \frac{2}{n}\log(1+t) - \frac{2}{n}\log(2\theta t + 2\theta + 1) + \frac{(b-1)}{n}\log\theta$$

Now,

$$\frac{\partial^2}{\partial\theta^2}\delta^*_{h(t)}(\theta) = -\frac{2(n+1)}{n\theta^2} + \frac{4}{(2\theta+1)^2} + \frac{(1-b)}{n\theta^2} + \frac{4(t+1)^2}{n(2\theta t + 2\theta + 1)^2}$$

$$\frac{\partial^2}{\partial\theta^2}\delta^*_{h^2(t)}(\theta) = -\frac{2(n+2)}{n\theta^2} + \frac{4}{(2\theta+1)^2} + \frac{(1-b)}{n\theta^2} + \frac{8(t+1)^2}{n(2\theta t + 2\theta + 1)^2}$$

Bayes estimator of *h(t)*,

$$\tilde{h}(t) = E(h(t)) = \sqrt{\frac{\left|\Sigma^*_{h(t)}\right|}{|\Sigma|}}\ \frac{4\hat{\theta}^2(1+t)}{\left(2\hat{\theta}t + 2\hat{\theta} + 1\right)}$$

$$MSE\left(\tilde{h}(t)\right) = E\left(\tilde{h}(t) - h(t)\right)^2$$

Now, we find the estimator of $h^2(t)$ as,

$$\tilde{h}^2(t) = E\left(h^2(t)\right) = \sqrt{\frac{\left|\Sigma^*_{h^2(t)}\right|}{|\Sigma|}}\ \frac{16\hat{\theta}^2(1+t)^2}{\left(2\hat{\theta}t + 2\hat{\theta} + 1\right)^2}$$

$$Risk\ \tilde{h}(t) = \tilde{h}^2(t) - \left(\tilde{h}(t)\right)^2$$

4. Real life Applications

Data set I: The first real life data set of size 72, depicts the life (survival time) of guinea pigs that were contagious with virulent tubercle bacilli. This data set was observed and recorded by Bjerkedel (1960). The survival time (in days) of the 72 guinea pigs are:

10, 33, 44, 56, 59, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115, 116, 120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196, 197, 202, 213, 215, 216, 222, 230, 231, 240, 245, 251, 253, 254, 254, 278, 293, 327, 342, 347, 361, 402, 432, 458, 555.

Data set II: The following data set represents the waiting times (in min.) just before the service of 100 bank customers. Ghitany *et al*. (2008) [7] initially examined and analyzed this data for fitting the Lindley distribution. The data of waiting times (in min.) are as follows:

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5

The proposed model is a good fit for both the above-mentioned data sets. The values of data set I were grouped into 12 class intervals, each of class width 50. The calculated chi-square value was found to be 17.4346 while the tabulated value for 11 degrees of freedom at 1% level of significance is 24.725. The values of data set II were grouped into 8 class intervals each of class width 5. The calculated chi-square value was found to be 1.622 while the tabulated value for 7 degrees of freedom at 1% level of significance is 18.475.

The MLE of θ for the first data set is found to be 0.00575 and the MLE of θ for the second data set is found to be 0.09328. Using these values of θ, we have calculated Bayes estimates of θ, R(t) and h(t) along with their risks under three different priors for both the data sets which are shown in Table 4.1.

Table 1: Estimated values of θ, R(t) and h(t) along with their risks

		Data set I			Data set II		
			R(t)	h(t)		R(t)	h(t)
Uniform prior	Estimate	0.005730	0.999749	0.000282	0.095088	0.948204	0.056065
	Risk	7.92E-10	5.62E-12	7.55E-12	1.16E-07	4.43E-08	1.50E-07
Exponential prior	Estimate	0.005730	0.999749	0.000282	0.095089	0.948205	0.056066
	Risk	7.81E-10	5.54E-12	7.45E-12	1.15E-07	4.38E-08	1.48E-07
Gamma prior	Estimate	0.005730	0.999749	0.000282	0.095092	0.948206	0.056072
	Risk	7.60E-10	5.39E-12	7.25E-12	1.13E-07	4.30E-08	1.42E-07

It can be observed from the above Table that Gamma prior is superior to Uniform prior and Exponential prior and Exponential prior is superior to Uniform prior for obtaining the Bayes estimates of θ, R(t) and h(t). This establishes the real life use of our proposed model. This model can further explain several other real life data sets.

5. Simulation Study

The Bayes estimates of the Weighted Lindley distribution's parameter, θ, have been computed. Its reliability function R(t), hazard rate function h(t), and some other associated measures have also been derived for random samples of various sizes. These random samples have been generated using R studio. Each random sample has been replicated 10,000 times to significantly enhance the size of the sample. At starting time t=1.2, the values of Bayes estimates of θ, R (t) and h(t) have also been acquired.

Using R studio, initially, the random samples of different sizes (n = 40, 80, 120, 160) were selected from the derived Weighted Lindley distribution taking the values of parameter θ as 1.0, 1.5, and 2.0. Each of the random samples was then replicated 10,000 times. For the different values of sample size n, the maximum likelihood estimates of reliability function R(t), and hazard rate function h(t) along with their Mean Square Errors were assessed. Furthermore, the Bayes estimates of θ, reliability function R(t), and hazard rate function h(t), as well as their Mean Square Errors (MSE) and Risks were calculated under Squared Error Loss Function (SELF) for three different priors. Tables 5.1, 5.2, and 5.3 demonstrate the values of Bayes estimates of θ, R(t), and h(t) for θ = 1.0, 1.5, and 2.0, respectively.

Table 2: Estimated values of θ, R(t) and h(t) along with their MSEs and risks when θ=1.0

θ=1.0	MLE Estimates			Bayes Estimates								
				Prior 1 (Uniform)			Prior 2 (Exponential)			Prior 3 (Gamma)		
	θ	R(t)	h(t)	θ	R(t)	h(t)	θ	R(t)	h(t)	θ	R(t)	h(t)
n=40	1.035353	0.163914	1.699841	1.018389	0.163357	1.653826	1.018928	0.163375	1.655266	1.019909	0.163376	1.666511
MSE	0.039879	0.004032	0.150191	0.037457	0.004005	0.137075	0.037533	0.004005	0.137458	0.037671	0.004005	0.140968
Risk				0.000577	7.03E-07	0.004159	0.000541	6.60E-07	0.00391	0.000479	6.60E-07	0.002202
n=80	1.038138	0.162776	1.705201	1.021120	0.162223	1.659026	1.021661	0.162241	1.660472	1.022645	0.162243	1.671767
MSE	0.039538	0.003977	0.148912	0.037041	0.003951	0.135420	0.037119	0.003952	0.135814	0.037261	0.003951	0.139404
Risk				0.000580	6.93E-07	0.004182	0.000544	6.51E-07	0.003932	0.000482	6.51E-07	0.002214
n=120	1.039091	0.162450	1.707046	1.022055	0.161898	1.660815	1.022596	0.161916	1.662263	1.023582	0.161917	1.673576
MSE	0.039447	0.003989	0.148537	0.036925	0.003964	0.134917	0.037003	0.003965	0.135315	0.037147	0.003964	0.138930
Risk				0.000581	6.91E-07	0.00419	0.000545	6.49E-07	0.00394	0.000483	6.49E-07	0.002218
n=160	1.035871	0.163418	1.700798	1.018898	0.162863	1.654759	1.019438	0.162881	1.656199	1.020419	0.162882	1.667450
MSE	0.039860	0.003948	0.150166	0.037277	0.003921	0.136220	0.037358	0.003922	0.136628	0.037505	0.003924	0.140332
Risk				0.000576	6.98E-07	0.004156	0.000541	6.55E-07	0.003907	0.000478	6.55E-07	0.002201

Table 3: Estimated values of θ, R(t) and h(t) along with their MSEs and risks when θ=1.5

θ=1.5	MLE Estimates			Bayes Estimates								
				Prior 1 (Uniform)			Prior 2 (Exponential)			Prior 3 (Gamma)		
	θ	R(t)	h(t)	θ	R(t)	h(t)	θ	R(t)	h(t)	θ	R(t)	h(t)
n=40	1.557052	0.055956	2.719068	1.529419	0.055770	2.640746	1.530368	0.055776	2.643392	1.532082	0.055780	2.665472
MSE	0.098150	0.001143	0.381113	0.091743	0.001134	0.345454	0.091959	0.001134	0.346569	0.092350	0.001134	0.357045
Risk				0.001530	9.57E-08	0.012001	0.001429	8.94E-08	0.011228	0.001254	8.62E-08	0.005702
n=80	1.561655	0.055150	2.728071	1.533929	0.054966	2.649466	1.534881	0.054972	2.652123	1.536601	0.054976	2.674296
MSE	0.094680	0.001116	0.367532	0.088189	0.001108	0.331494	0.088406	0.001108	0.332615	0.088800	0.001108	0.343113
Risk				0.001536	9.31E-08	0.012049	0.001435	8.70E-08	0.011273	0.001259	8.39E-08	0.005726
n=120	1.558878	0.055652	2.722642	1.531208	0.055467	2.644207	1.532158	0.055473	2.646858	1.533875	0.055477	2.668975
MSE	0.096698	0.001137	0.375412	0.090262	0.001128	0.339628	0.090478	0.001128	0.340744	0.090870	0.001128	0.351218
Risk				0.001533	9.48E-08	0.012019	0.001431	8.86E-08	0.011245	0.001256	8.54E-08	0.005712
n=160	1.556198	0.056116	2.717390	1.528583	0.055929	2.639124	1.529531	0.055936	2.641769	1.531244	0.055940	2.663828
MSE	0.096699	0.001184	0.375302	0.090405	0.001175	0.340304	0.090617	0.001175	0.341394	0.091000	0.001175	0.351647
Risk				0.001527	9.71E-08	0.011971	0.001426	9.07E-08	0.011201	0.001251	8.75E-08	0.005691

Table 4: Estimated values of θ, R(t) and h(t) along with their MSEs and risks when θ=2.0

θ=2.0	MLE Estimates			Bayes Estimates								
				Prior 1 (Uniform)			Prior 2 (Exponential)			Prior 3 (Gamma)		
	θ	R(t)	h(t)	θ	R(t)	h(t)	θ	R(t)	h(t)	θ	R(t)	h(t)
n=40	2.081431	0.019389	3.754442	2.042386	0.019327	3.641282	2.043800	0.019329	3.645313	2.046340	0.019331	3.679756
MSE	0.180286	0.000266	0.708391	0.167763	0.000263	0.638029	0.168204	0.000263	0.640309	0.168999	0.000264	0.661838
Risk				0.003047	1.31E-08	0.024906	0.002834	1.22E-08	0.023222	0.002471	1.15E-08	0.011063
n=80	2.078512	0.019527	3.748675	2.039530	0.019465	3.635707	2.040941	0.019468	3.639731	2.043478	0.019470	3.674105
MSE	0.183266	0.000267	0.720219	0.170815	0.000265	0.650200	0.171255	0.000265	0.652476	0.172047	0.000265	0.673974
Risk				0.003041	1.33E-08	0.02486	0.002829	1.24E-08	0.023178	0.002466	1.17E-08	0.011041
n=120	2.086825	0.019365	3.765167	2.047655	0.019303	3.651624	2.049074	0.019306	3.655672	2.051625	0.019308	3.690256
MSE	0.189048	0.000270	0.742996	0.175736	0.000268	0.667997	0.176207	0.000268	0.670452	0.177056	0.000268	0.693514
Risk				0.003073	1.32E-08	0.025134	0.002858	1.23E-08	0.023433	0.002491	1.16E-08	0.011151
n=160	2.081132	0.019341	3.753836	2.042094	0.019280	3.640700	2.043507	0.019282	3.644730	2.046048	0.019284	3.679163
MSE	0.178631	0.000261	0.701886	0.166204	0.000259	0.632094	0.166641	0.000259	0.634352	0.167429	0.000259	0.655684
Risk				0.003044	1.30E-08	0.024883	0.002832	1.21E-08	0.0232	0.002469	1.14E-08	0.011054

The above tables clearly demonstrate that for θ=1.0, 1.5 and 2.0, the Bayes risks of θ, R(t) and h(t) under SELF are lowest for Gamma prior (prior 3) compared to Uniform prior (prior 1) and Exponential prior (prior 2), while Exponential prior shows lower risks than Uniform prior for different sample sizes. This validates the conclusions drawn from the real data sets. Ultimately, it is concluded that Gamma prior is superior to Exponential prior and Exponential prior is superior to Uniform prior for all values of θ and all sample sizes for finding Bayes estimates of θ, R(t) and h(t). It can further be observed that with increase in the value of θ, R(t) decreases while h(t) increases.

6. MLE for R(t) and h(t) for different values of t for θ = 1

MLE of reliability function R(t) decreases continuously with the increase in values of time (t). At initial time 0, R(t) acquires value 1 and at time 4.0, it gradually decreases down to 0.019881. Furthermore, the MLE of hazard rate function h(t) increases continuously with the increasing values of time (t). At initial time 0, h(t) acquires value 1.333333 while at time 4.0, it gradually increases up to 1.818182. This has been shown graphically through Figures 6.1(a) and 6.1(b) respectively.

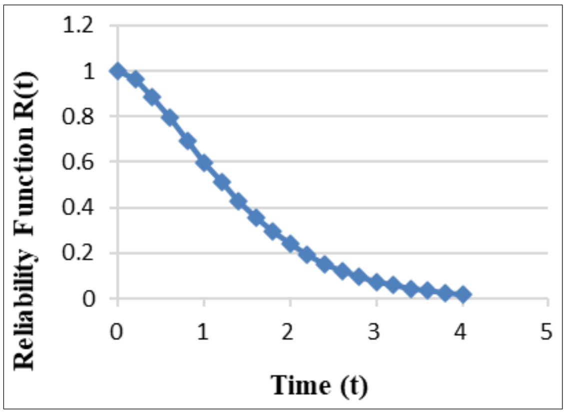

Fig 1(a): Reliability curve

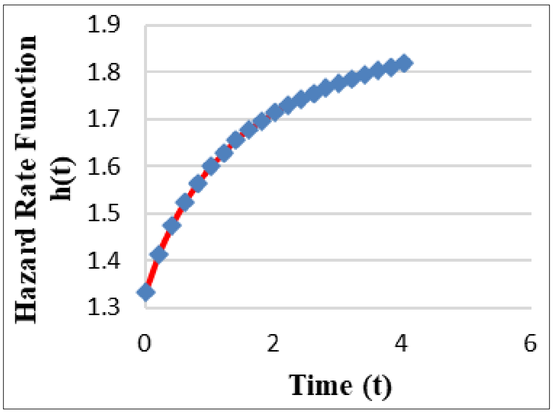

Fig 2(b): Hazard rate curve

7. Conclusion

It is concluded that the proposed model is very useful for describing many real world situations and can be of utility against various standard lifetime models already discovered. Furthermore, It is also concluded that gamma prior (prior 3) should be preferred over uniform prior (prior 1) and exponential prior (prior 2) for the computation of Bayes estimation of its parameter θ, reliability function R(t) and hazard rate function h(t) for the proposed Weighted Lindley Distribution.

8. References

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