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Calculation of optimum premium values

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Abstract
The present paper deals with optimizing the maximum premium values using Hamiltonian-Jacobi-Bellamann equation. Here it is considering that quadratic and fractional power utility functions for different loss distributions of discrete analogues of continuous distributions. It has been seen that the quadratic utility function is more beneficial to the insured and the fractional power utility function is more beneficial to insurer. Numerical illustrations for discrete analogues of continuous distributions using these two utility functions are given.

Keywords: Utility function, life insurance, optimum premium, HJB equation and loss distributions

1. Introduction
Here we consider the subject of the calculus of variations by analogy with the classical topic of maximization and minimization in calculus. Richard Bellmann (1957) [1] in his book of dynamic programming states that an *optimal policy* as the property that, whatever the initial state or initial decision are, the remaining decision must constitute optimal policy with regard to the outcome resulting from the first decision. Gelfand and Fomin (1963) [5], Young (1969) [19], and Leitmann (1981) [12] have given rigorous treatments of the subject. The problem of the calculus of variations is that of determining a function that maximizes a given functional, the objective function. An analogous problem in calculus is that of determining a point (using the first-order condition for a maximum) at which a specific function, the objective function, is maximum. A similar procedure will be employed to derive the first-order condition for the variational problem. The analogy with classical optimization extends also to the second-order maximization condition of calculus. For more rigorous derivations of maximum principle by dynamic programming approach we refer Pontryagin *et al*. (1962) [17], Berkovitz (1961) [2], Halkin (1967) [6], Hartberger (1973) [7].
The dynamic programming tool is a problem in which a random element enters into the movement of the system; the optimal control must be stated in feedback form, in terms of the system, rather than in terms of time alone (because the state that has to be obtained cannot be known in advance, due to the stochastic disturbance). Karatzas and Shreve (1997) [11] cover most of stochastic optimal control problems and methods (see also Fleming and Rishel, 1975; Fleming-Zariphopoulou, 1991; Merton, 1992; Hipp and Plum, 2000; Oksendal, 2003; Hipp and Plum, 2003 and Li, 2017) [3, 4, 8, 15, 16, 9, 13]. Sethi and Thompson (2005) [18] discussed many concepts related optimal control theory such as HJB equation, Pontryagin's principle etc. In the present paper we optimize the maximum premium values using stochastic optimal control theory of dynamic programming problems such as Hamiltonian-Jacobi-Bellaman (HJB) equation. Finally we obtain the solution, using the relationship between the maximum principle of optimal control theory and the necessary conditions of the calculus of variations (Sethi and Thompson, 2005) [18].

2. Computation of optimum premium values for Quadratic utility function:
The maximum premium (P_{max}) that an insured is willing to pay can be determined using utility theory. Kapoor and Jain (2011) [10] determined P_{max} by considering different forms of the utility functions assuming the loss random variable to follow different forms of continuous statistical distributions. We determine optimum premium values using HJB equation for some discretized distributions.

The first two moments of some discretized distributions are given in the following table. See Mallappa and Talawar (2020) [14] for maximum premiums of various discrete loss distributions with different utility functions.

Table 1: First two moments of some discretized distributions

Discrete Distribution	μ_1'	μ_2'
Exponential(λ)	$\frac{e^{-\lambda}}{(1-e^{-\lambda})}$	$\frac{e^{-\lambda}(1+e^{-\lambda})}{(1-e^{-\lambda})^2}$
Gamma(m,λ)	$\frac{me^{-\lambda}}{(1-e^{-\lambda})}$	$\frac{me^{-\lambda}(1+me^{-\lambda})}{(1-e^{-\lambda})^2}$
Weibull (λ)	$\sum_{k=0}^{\infty} e^{-k^{\lambda}}$	$2\sum_{k=1}^{\infty} e^{-(k+1)^{\lambda}} + \sum_{k=1}^{\infty} e^{-k^{\lambda}}$
Burr (α,β)	$\sum_{k=1}^{\infty} \frac{1}{(1+k^{\alpha})^{\beta}}$	$\sum_{k=1}^{\infty} \frac{(2k-1)}{(1+k^{\alpha})^{\beta}}$
Pareto(β)	$\sum_{k=1}^{\infty} \frac{1}{(1+k)^{\beta}}$	$\sum_{k=1}^{\infty} \frac{(2k-1)}{(1+k)^{\beta}}$

The maximum premium (P_{max}) for quadratic utility function.

$$P_{max} = -(a-w) \pm \sqrt{(a-w)^2 + 2(a-w)\mu_1' + \mu_2'} \quad (1)$$

We optimize the above equation subject to

$$t = -(a-w)^2 \; a \geq w \quad (2)$$

The HJB equation is given by

$$H = -\frac{\sqrt{t}}{i} + \sqrt{-t + \frac{2\sqrt{t}}{i}\mu_1' + \mu_2'} + \lambda t \quad (3)$$

Where λ is Hamiltonian constant and its value is obtained using the following condition.

$\frac{\partial H}{\partial t} = 0$. This gives

$$\hat{\lambda} = \frac{1}{2i\sqrt{t}} - \frac{-1+\frac{1}{i\sqrt{t}}\mu_1'}{2\sqrt{-t+\frac{2\sqrt{t}}{i}\mu_1'+\mu_2'}} \quad (4)$$

And

$$\frac{\partial^2 H}{\partial t^2} = \frac{1}{2i} . \frac{1}{2t^{3/2}} - \frac{\partial}{\partial t}\left(\frac{y}{z}\right)$$

Where $y = -1 + \frac{1}{i\sqrt{t}}\mu_1'$ and $z = 2\sqrt{-t + \frac{2\sqrt{t}}{i}\mu_1' + \mu_2'}$

$$\frac{\partial}{\partial t}\left(\frac{y}{z}\right) = \frac{\left[2\sqrt{-t + \frac{2\sqrt{t}}{i}\mu_1' + \mu_2'} . \left(\frac{-1}{2it^{3/2}}\mu_1'\right) - \frac{\left(-1+\frac{1}{i\sqrt{t}}\mu_1'\right)^2}{4\sqrt{-t+\frac{2\sqrt{t}}{i}\mu_1'+\mu_2'}}\right]}{\left[2\sqrt{-t + \frac{2\sqrt{t}}{i}\mu_1' + \mu_2'}\right]^2}$$

$$\frac{\partial}{\partial t}\left(\frac{y}{z}\right) = \frac{\left[\left(\frac{-2}{it^{3/2}}\mu_1'\right)z - y^2\right]}{[2z^3]}$$

Finally we see that

$$\frac{\partial^2 H}{\partial t^2} = \frac{1}{4it^{3/2}} + \left[\frac{\left(\frac{2}{it^{3/2}}\mu_1'\right)z + y^2}{[2z^3]}\right] \leq 0 \quad (5)$$

This is the second-order condition for the maximization of the Hamiltonian. Hence optimal premium equation by HJB is given in the following form

$$P_{opt} = -\frac{\sqrt{t}}{i} + \sqrt{-t + \frac{2\sqrt{t}}{i}\mu_1' + \mu_2'} + \hat{\lambda}t \quad (6)$$

We compute and plot the optimum premium values for different values of quadratic utility function i.e., $u(w) = -16, -36, -64, -100, -144 \; and \; 249001$ against the different parametric values of discretized probability distributions. Following are the plots of optimum premiums using quadratic utility function assuming loss follows some descritized probabilty distributions.

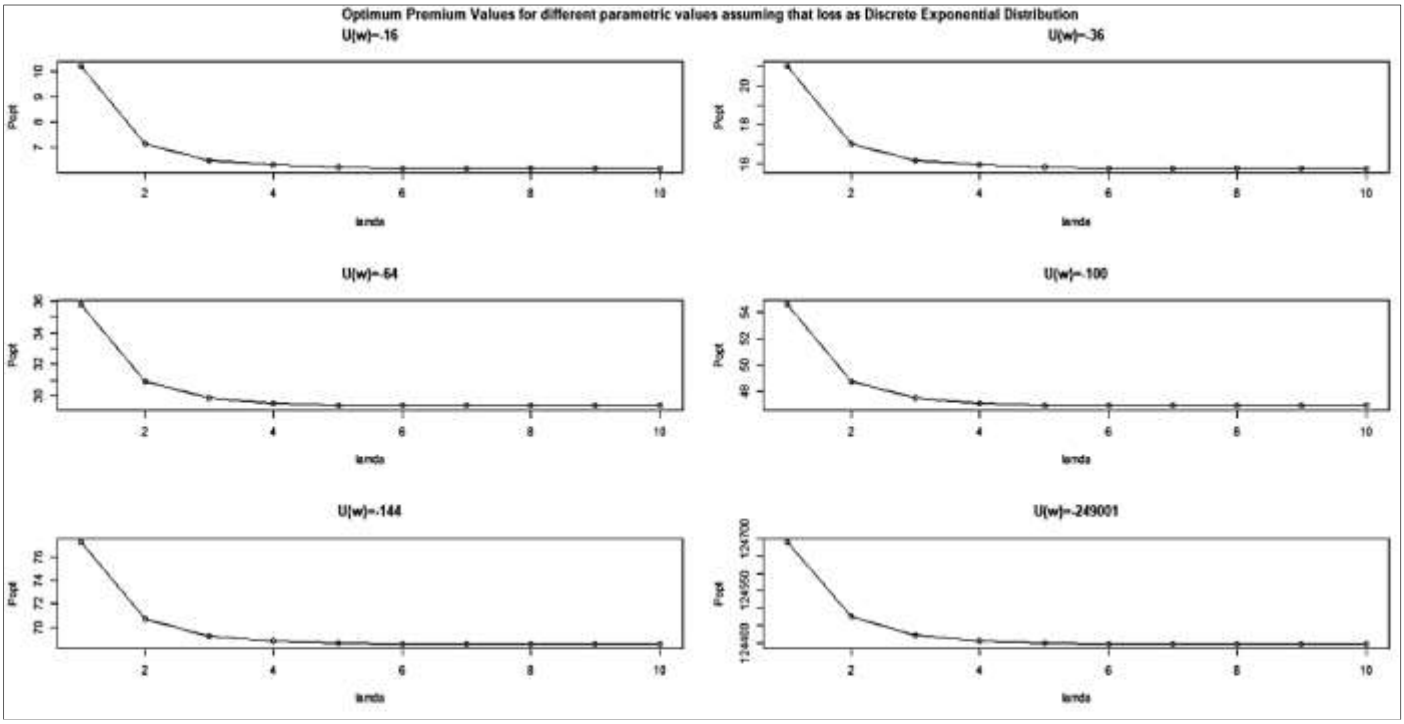

Fig 1: Optimum premiums using quadratic utility function for discrete exponential distribution.

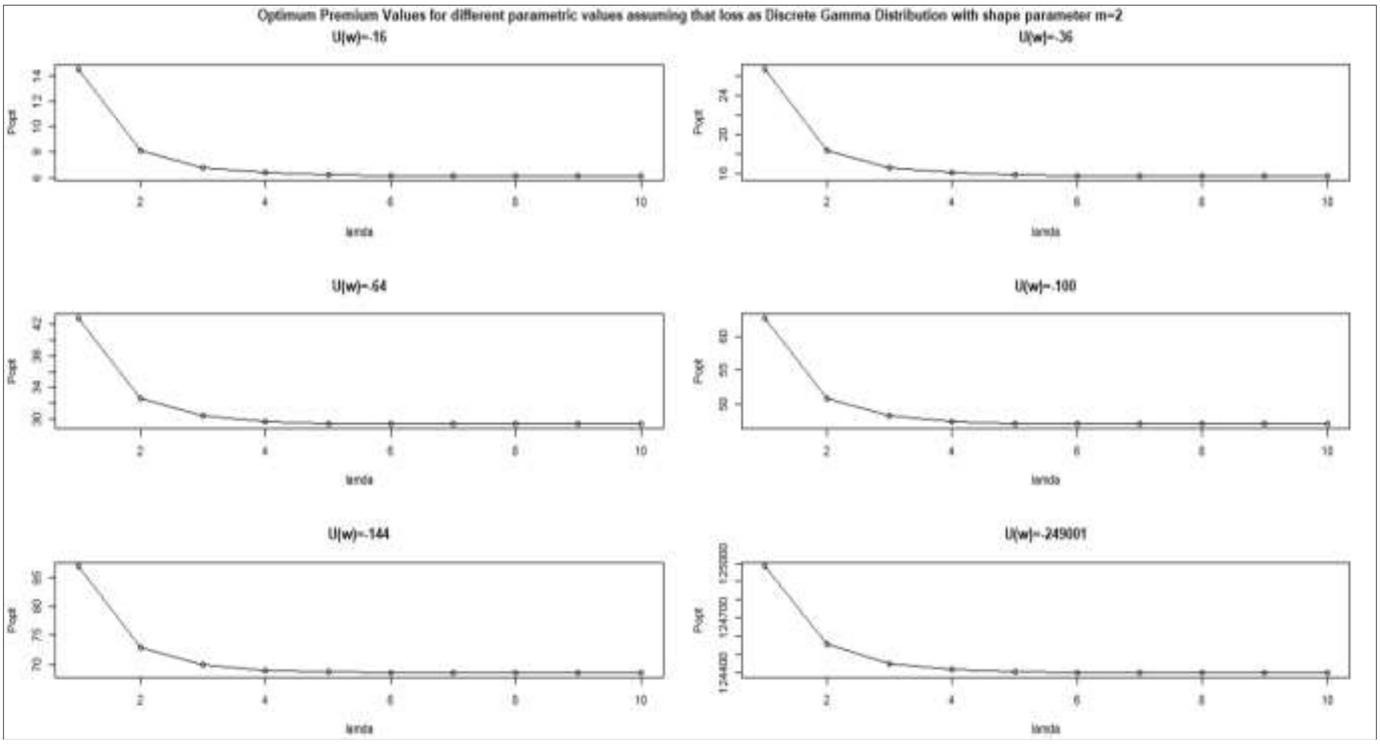

Fig 2: Optimum premiums using quadratic utility function for discrete Gamma distribution with parameter $m = 2$.

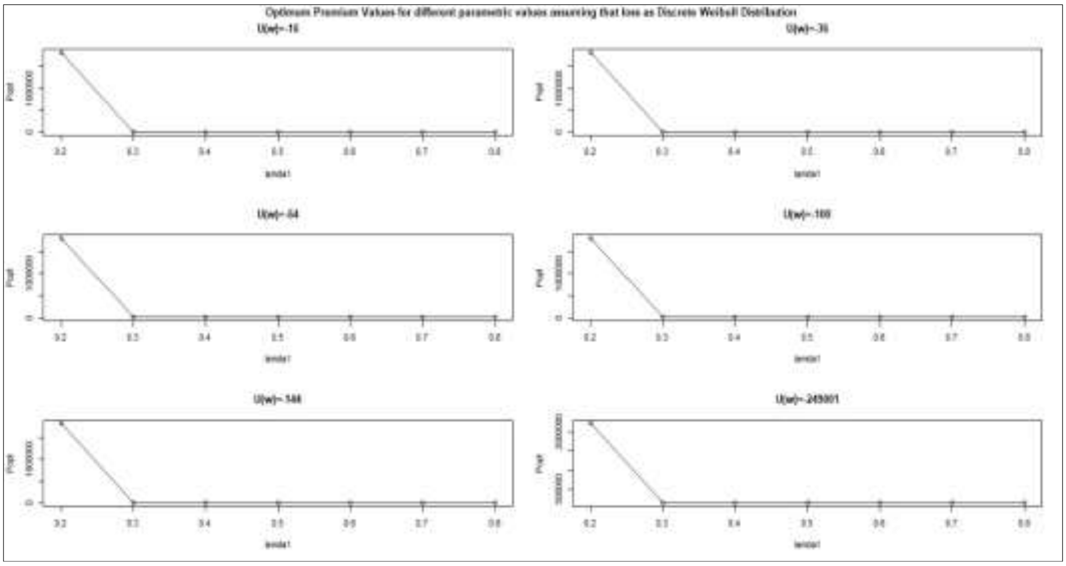

Fig 3: Optimum premiums using quadratic utility function for discrete Weibull distribution.

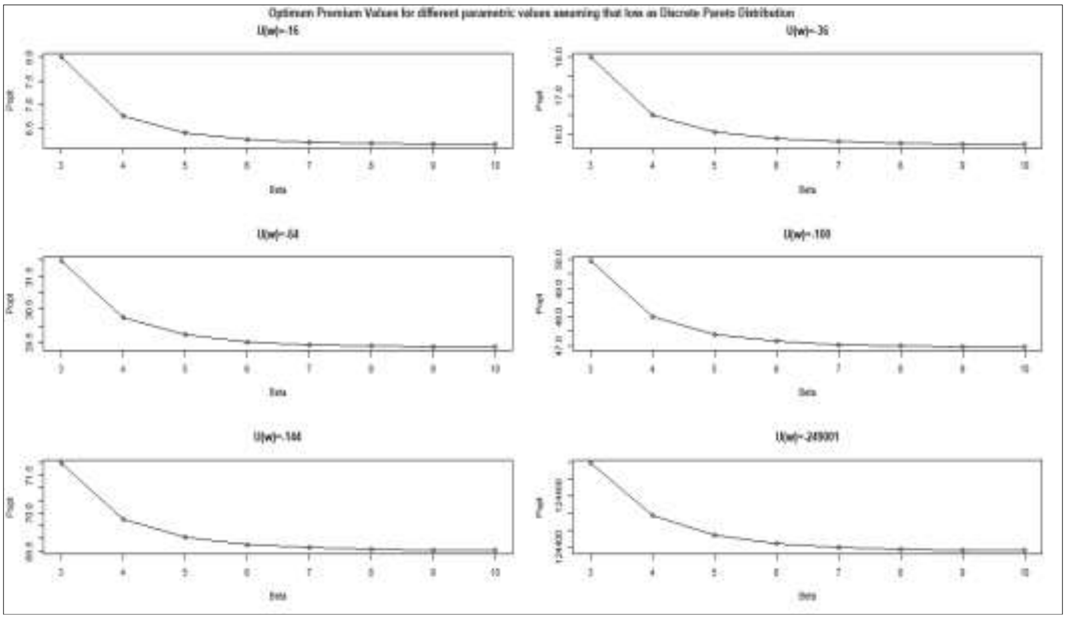

Fig 4: Optimum premiums using quadratic utility function for discrete Pareto distribution.

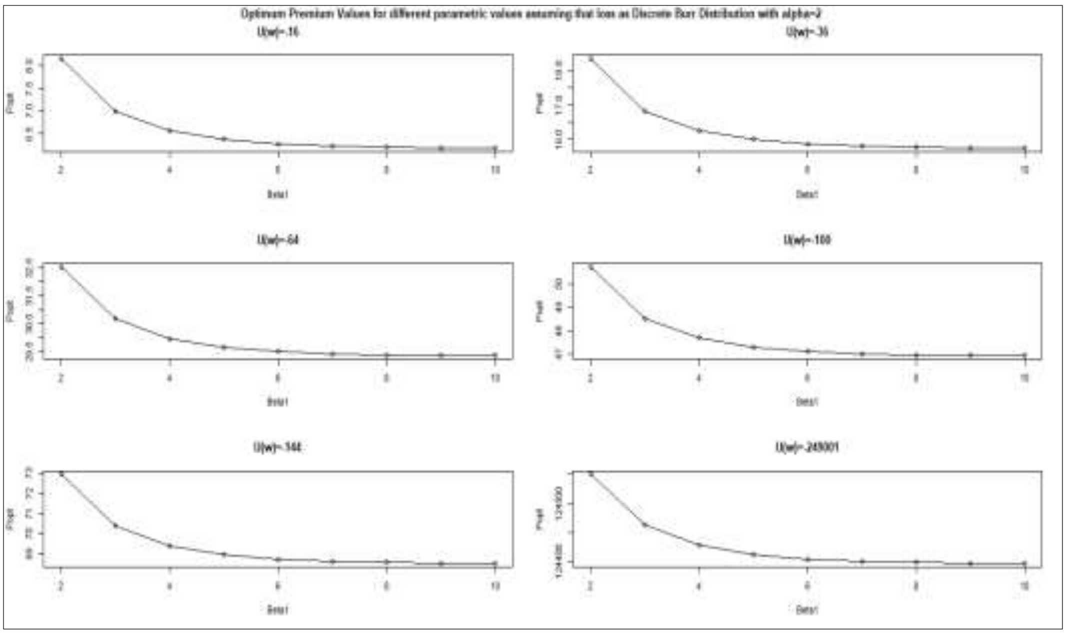

Fig 5: Optimum premiums using quadratic utility function for discrete Burr distribution with $\alpha = 2$.

3. Computation of optimum premium values for Fractional power utility function:

The maximum premium equation for fractional power utility function

$$P_{max} = \frac{w}{c-1}\left(\sqrt{1-2\left(\frac{c-1}{w}\right)\mu_1' + \left(\frac{c-1}{w}\right)^2 \mu_2'}\right) \quad (7)$$

We optimize the above equation subject to

$$t = w^c, 0 < c < 1 \quad (8)$$

Thus the HJB equation is given by

$$H = \frac{t^{1/c}}{c-1}\left(\sqrt{1-2\left(\frac{c-1}{t^{1/c}}\right)\mu_1' + \left(\frac{c-1}{t^{1/c}}\right)^2 \mu_2'}\right) + \lambda t \quad (9)$$

Where λ is Hamiltonian constant and its value is obtained using the following condition, $\frac{\partial H}{\partial t} = 0$

$$\frac{\partial H}{\partial t} = \frac{1}{c-1}\left\{\frac{1}{c}t^{\left(\frac{1}{c}-1\right)}.\sqrt{1-2\left(\frac{c-1}{t^{1/c}}\right)\mu_1' + \left(\frac{c-1}{t^{1/c}}\right)^2 \mu_2'} + t^{1/c}.\frac{1}{2\sqrt{1-2\left(\frac{c-1}{t^{1/c}}\right)\mu_1' + \left(\frac{c-1}{t^{1/c}}\right)^2 \mu_2'}}.\left(2\left(\frac{c-1}{c}\right)\frac{1}{t^{\left(\frac{1}{c}+1\right)}}\mu_1' - 2\frac{(c-1)^2}{c}.\frac{1}{t^{\left(\frac{2}{c}+1\right)}}\mu_2'\right)\right\} + \lambda$$

$$\frac{\partial H}{\partial t} = \frac{1}{c-1}\left\{\frac{1}{c}t^{\left(\frac{1}{c}-1\right)}y + t^{1/c}y'\right\} + \lambda$$

Where

$$y = \sqrt{1-2\left(\frac{c-1}{t^{1/c}}\right)\mu_1' + \left(\frac{c-1}{t^{1/c}}\right)^2 \mu_2'}$$

$$y' = \frac{\left(\frac{c-1}{c}\right)\frac{1}{t^{\left(\frac{1}{c}+1\right)}}\mu_1' - \frac{(c-1)^2}{c}.\frac{1}{t^{\left(\frac{2}{c}+1\right)}}\mu_2'}{y}$$

Therefore we get that, $\hat{\lambda} = \frac{-1}{c-1}\left\{\frac{1}{c}t^{\left(\frac{1}{c}-1\right)}y + t^{1/c}y'\right\}$ (10)

and hence the second-order condition for the maximization of the Hamiltonian is

$$\frac{\partial^2 H}{\partial t^2} = \frac{1}{c-1}\left\{\frac{1}{c}\left(\frac{1}{c}-1\right).t^{\left(\frac{1}{c}-2\right)}.y + \frac{1}{c}t^{\left(\frac{1}{c}-1\right)}y' + \frac{1}{c}t^{\left(\frac{1}{c}-1\right)}y' + t^{1/c}y''\right\} \leq 0$$

Where

$$y' = \frac{\left(\frac{c-1}{c}\right)\frac{1}{t^{\left(\frac{1}{c}+1\right)}}\mu_1' - \frac{(c-1)^2}{c}.\frac{1}{t^{\left(\frac{2}{c}+1\right)}}\mu_2'}{y}$$

$$y'' = \frac{\partial y'}{\partial t} = \frac{\left\{y\left[\frac{(c-1)^2}{c}\left(\frac{2}{c}+1\right)\frac{1}{t^{\left(\frac{2}{c}+1\right)}}\mu_2' - \left(\frac{c-1}{c}\right)\frac{1}{t^{\left(\frac{1}{c}+1\right)}}\mu_1'\right] - y'\left[\left(\frac{c-1}{c}\right)\frac{1}{t^{\left(\frac{1}{c}+1\right)}}\mu_1' - \frac{(c-1)^2}{c}.\frac{1}{t^{\left(\frac{2}{c}+1\right)}}\mu_2'\right]\right\}}{y^2}$$

$$P_{opt} = \left(\frac{t^{1/c}}{c-1}\right)y + \hat{\lambda}t \quad (11)$$

Following are the plots of optimum premiums using fractional power utility function assuming loss follows some probabilty distributions.

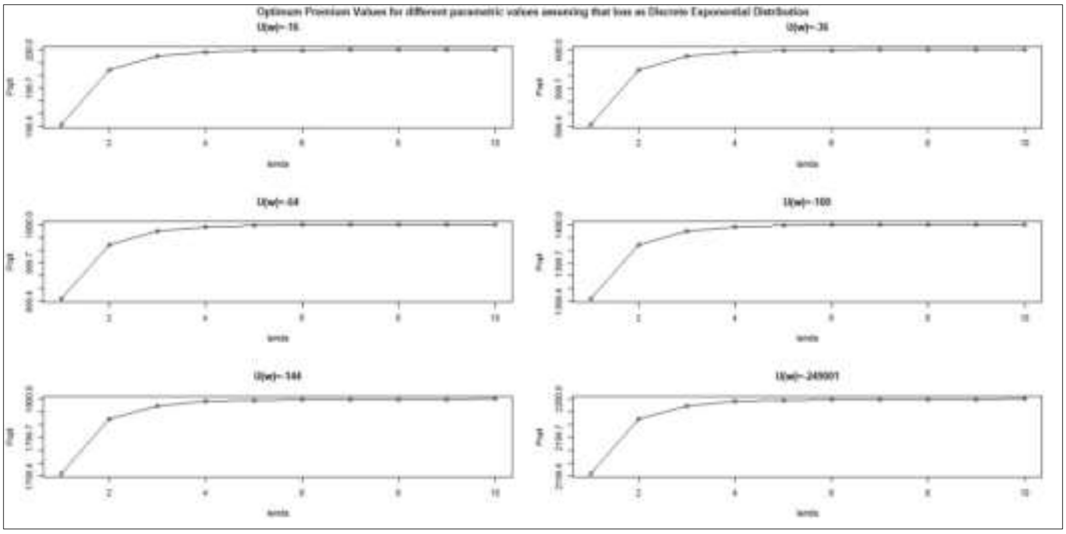

Fig 6: Optimum premiums using fractional power utility function for discrete Exponential distribution.

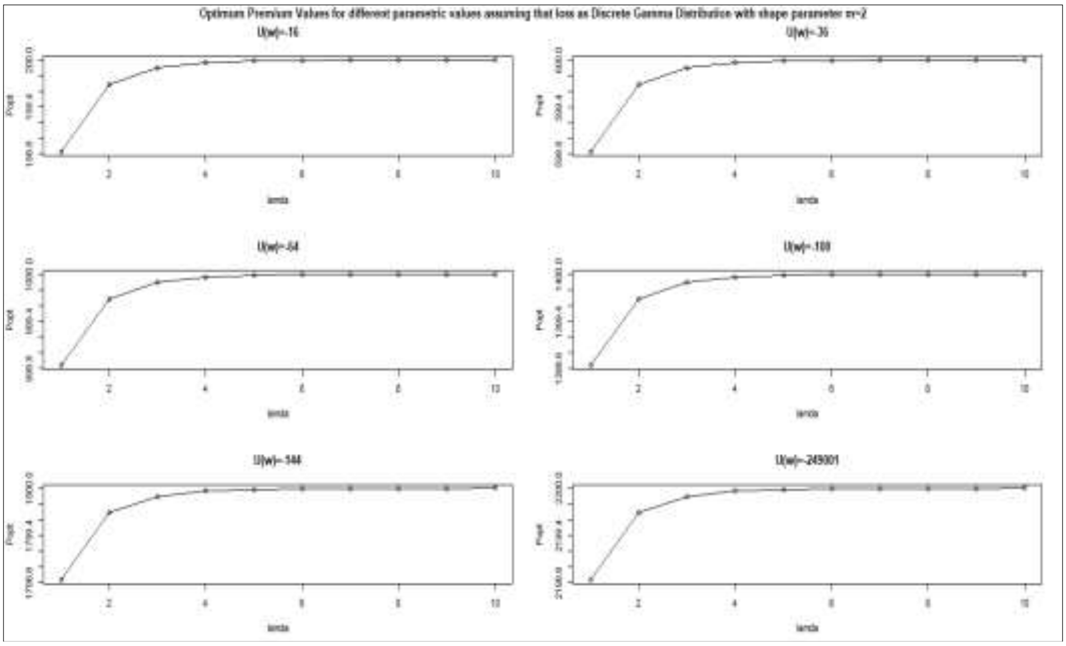

Fig 7: Optimum premiums using fractional power utility function for discrete Gamma distribution with paramter $m = 2$.

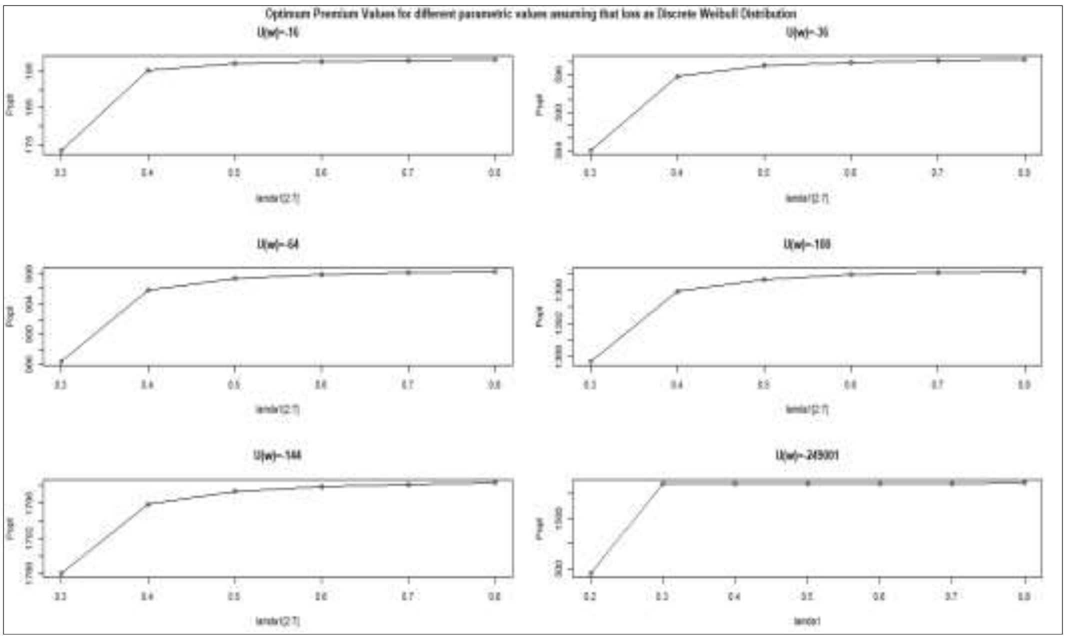

Fig 8: Optimum premiums using fractional power utility function for discrete Weibull distribution.

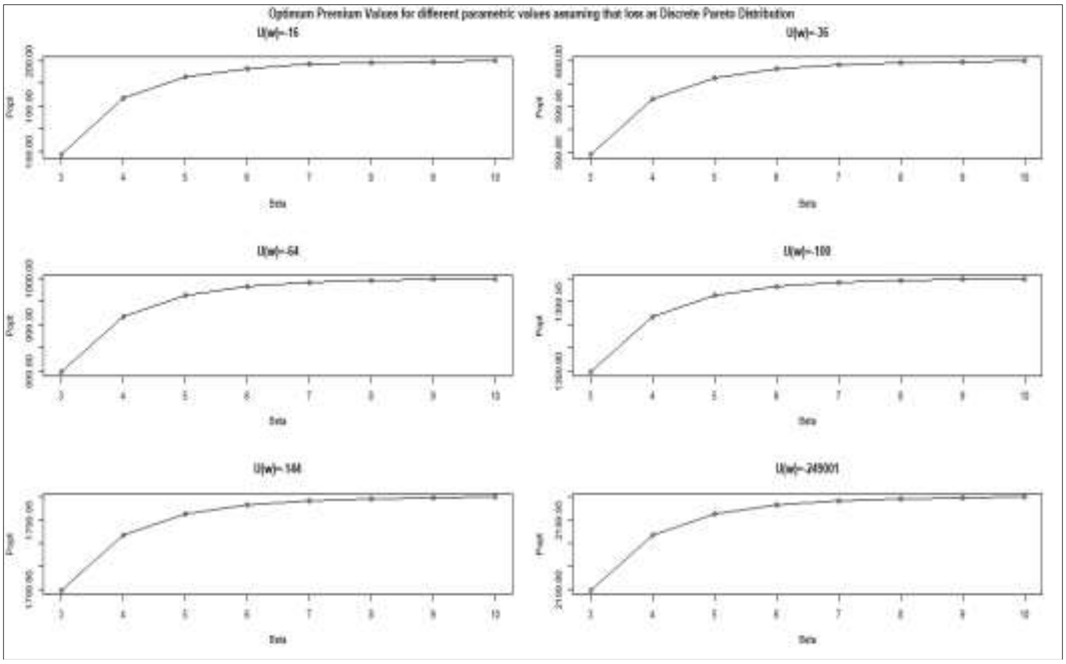

Fig 9: Optimum premiums using fractional power utility function for discrete Pareto distribution.

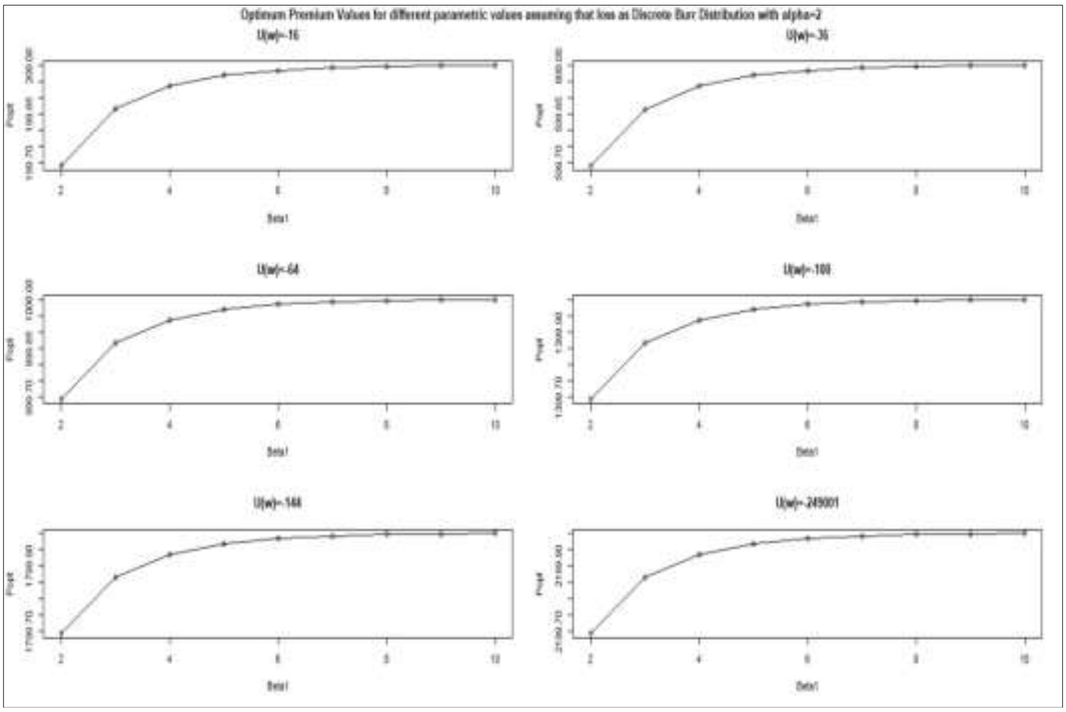

Fig 10: Optimum premiums using fractional power utility function for discrete Burr distribution with parameter $\alpha = 2$.

4. Conclusion

If the insured adopts a quadratic and/or fractional power utility function with numerical values namely $u(w) = -16, -36, -64, -100, -144\ and -249001$ assuming that loss random variable follows some discretized probability distributions namely discrete exponential, Gamma with varying shape parameters, Weibull, Pareto and Burr distribution with different shape parameter value α. The above figures give plotted optimum premium values against varying parametric values for some discretized distributions. Therefore in the case of quadratic utility function and conclude the following.

- For discrete exponential and discrete gamma distributions the optimum values decrease as the parametric value increases. The optimum values are somewhat more as in discrete gamma as compared to discrete exponential distribution. The optimum premium values increase along with shape parameter value of the distribution.
- For discrete Weibull distribution, the optimum premium values suddenly decrease from the parameter values 0 .2 to 0.3. Later the optimum premium values stabilize as the parameter value increases more.
- For discrete Pareto and Burr distribution with various shape parameter $\alpha = 2, 3, 4$ the optimal premium values are low compared to discrete exponential and gamma distribution.

Thus the quadratic utility function is more beneficial to the insured because the optimum premium values are high compared to maximum premium values (Mallappa and Talawar, 2020) [14].

And in the case of fractional power utility function, we observe the following

- For discrete Exponential, Gamma, Weibull, Pareto and Burr distributions the optimum premium value increase as the parameter λ increase and the values are higher slightly than the maximum premium values.
- It is also observed that even for the smaller values of utility function, the optimum premium values are very high.

  Hence fractional power utility function is more beneficial to insurer.

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