International Journal of Statistics and Applied Mathematics 2023; 8(4): 21-25

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2023; 8(4): 21-25

https://www.mathsjournal.com
Received: 05-04-2023
Accepted: 11-05-2023

Dr. Amar Nath Kumar
Assistant Professor, Department of Mathematics, Kisan College, Sohsarai, Nalanda, Bihar, India

Corresponding Author:
Dr. Amar Nath Kumar
Assistant Professor, Department of Mathematics, Kisan College, Sohsarai, Nalanda, Bihar, India

A note on spaces of operators on Hilbert spaces

Dr. Amar Nath Kumar

Abstract
In this research paper, we explained and established the results for the space of operators in Hilbert space. Some of them are if the space of all operators on a Hilbert space H, B(H) is a Banach Space as well as here we construct a set of operators R on a Hilbert space H in the following way R = {T_i: $T_i^2 = 0$, $T_iT_j = 0$} and we have proved that R is a Banach Space. Also If R and R' be linear spaces and T: R → R' be a linear transformation then Kernel T is a linear subspace of R. We discussed on isomorphism concept also here if T be a linear transformation of R onto R' then T is an isomorphism if and only if Ker T = {0}. We have also discussed algebraic structure, we have seen that the set of all non-singular transformations on a linear space R forms a group with respect to multiplication.

Keywords: Space, homomorphism, operator, Hilbert space

1. Introduction

The title of this paper is the index of the study made in this paper in order to make it clear we have concentrated on the study to form different spaces of operators such as linear space, normed linear space and complete Normed Linear space that is Banach space. In previous studies we have studied a bit about a group but at length about the ring. Here in this paper, we have taken the help of the study of group and ring information of liner space of the set R of operators on a Hilbert space H. Our first effort in this paper has been to establish that B(H) the set of all operators on a Hilbert space H. is a Banach space which gently helped in establishing that the constructed set R of operators on a Hilbert space H also forms normed linear space and complete normed linear space also.

We have also studied linear transformation (or vector space homomorphism) for the Linear space R of operators on a Hilbert space H. The notion of isomorphism of transformation has also been given proper attention by establishing some of the analogous results for R. We have also studied Kernel of transformation, Kernel of identity Transformation and Kernel of zero transformation and with the notions of these, a few analogous results have been established to make these results as a means of progress. We have also witnessed a result which establish a connection between isomorphism and the Kernel of transformation. It has also been shown that the Kernel of transformation is a linear subspace of the linear space R. Thus as a whole in this paper we have brought forward a good number of things into harmony which in fact the resultant of our study in this paper.

2. Preliminaries and Definitions

For definitions we refer to preliminaries however we give below some of the definitions to serve as a ready reference.

2.1 Space: A set together with a structure defined is called a space.

2.2 Metric (or Distance Function)**:** Let M be a non-empty set. By a metric (or distance function) on M we understand a real-valued function d on M×M which satisfies the following conditions:

i) $d(x, y) \geq 0$
ii) $d(x, y) = 0 \Leftrightarrow x = y$
iii) $d(x, y) = d(y, x)$ (symmetry)

iv) d(x, z) ≲ d(x, y) + d(y, z) (the triangle inequality)

Here d(x, y) means the distance between x and y.

2.3 Metric Space: The set M together with the metric d defined on it is called a metric space.

2.4 NORM: Let E be a linear space over a given field K. By a norm on E we understand a mapping f: E → R^+ (the set of positive real numbers) satisfying the following conditions:

i) f(x) = 0 ⇔ x = 0
ii) f(λx) = |λ|f(x) for all $\lambda \in$ K, x ∈ E.
iii) f(x+y) ≲ f(x) + f(y) for all x, y ∈ E.

Now if there be no chance of confusion in writing || || for f and ||x|| for f(x) then all above conditions immediately take the form.

a) ||x|| = 0 ⇔ x = 0
b) ||λx|| = |λ| ||x||
c) ||x + y|| ≲ ||x|| + ||y||

2.5 Normed Linear Space: A linear space E together with a norm (|| ||) defined on it is called a normed linear space.
Here we recall a famous result that every normed linear space e is a metric space with respect to the metric d defined by d(x, y) = ||x-y|| for all x, y ∈ E.
(For proof we refer to Jha, (1), Simmons,(1).)

2.6 Cauchy Sequence: A sequence (x_n) of points of metric space M is said to be a Cauchy sequence if for each ε > 0, there exists a natural number n_ϵ such that m, n ∈ N and m, n ≥ n_ϵ ⇒ d(x_m, x_n) < ε

That is if $\lim_{m,n\to\infty} d(x_m, x_n) = 0$

2.7 Convergent Sequence: If a sequence has a definite finite limit, we say that the sequence is convergent. If a sequence has no limit then we say that it is divergent.
Also every convergent sequence (x_n) of points of a metric space (E, d) is a Cauchy sequence.
(For proof we refer to Jha, (1).)

2.8 Complete Metric Space: A metric space (M, d) is said to be complete if every Cauchy sequence in (M, d) is convergent in (M, d).

2.9 Banach Space: A normed linear space is said to be a Banach space if it is complete as a metric space.
Also Norm function is a continuous function. That is if x_n→x in M then ||x_n||→||x||.

2.10 Inner Product: Let E be a linear space over a field K. By an inner product (or scalar product) on E we mean a map. (x, y) → (x/y) of E × E into K.

Satisfying the following conditions namely.

i) (x/y) = $(\overline{y/x})$.
ii) (x/x) ≥ 0 ∀ x in E.
iii) (x/x) = 0 ⇔ x = 0.
iv) (λx + μ y/z) = λ(x/z) + μ(y/z) ∀ λ,μ ∈ K, x, y,z ∈ E.

How ever if K = R (Set of real numbers) then, the condition (iii) takes the form (x/y) = (y/x). Also, from conditions (iii) and (iv) we have (x/λy + μz) = $(\overline{\lambda y + \mu z/x})$ $\overline{\lambda(y/x) + \mu(z/x)}$ = $\bar{\lambda}\overline{(y/x)}$ + $\bar{\mu}\overline{(z/x)}$ = $\bar{\lambda}$(x/y) + $\bar{\mu}$(x/z).
(We refer to Simmons, G.F (1) (p-245))
It is worth much to note that an inner product space **E** is a normed linear space with respect to the norm-defined in term of an inner product given by ||x|| = +$\sqrt{x/x}$
(For verification we refer to Jha, K.K (1) p-270)
There is no chance of confusion then (x/y) is read as the inner product of x with y (or equivalently the dot product of x with y or the dot product of x and y.

2.11 Hilbert Space: A banach space is said to be a Hilbert space if its norm is or can be defined by means of an inner product.
Or, A Hilbert space is an inner product space which is a Banach space whose norm is or can be defined by means of an inner product.

2.12 Linear Transformations: Let R_1 and R_2 be two linear spaces over the same field K. A mapping f: $R_1 \to R_2$ is called a linear transformation (or a linear mapping) if it satisfies the following conditions. (i) f(u + v) = f(u) + f(v), ∀ u,v ∈ R_1 (ii) f(αu) = αf(u), ∀ α ∈ K, ∀ u ∈ R_1
We can express the conditions (i) and (ii) together as f(αu + βv) = αf(u) + βf(v) for all α,β ∈ K, u,v ∈ R_1
Here f preserves the origin and negatives, That is f(0) = f(0.0) = 0f(0) = 0, And f(-x)=f(-1.x)=-1f(x)=-f(x).

2.13 Non-Singular Linear Transformations; Let T be a linear transformation on a linear space E. Then T is called invertible or non-singular if T is one-one and onto otherwise T is called singular. When T is T(x) = y ⇔ x = T^{-1}(y). Also if I is the identity function on E then $TT^{-1} = I = T^{-1}T$.

2.14 Continuous Linear Transformation: Let E and E′ be normed linear spaces over the same field K. Let T: E → E′ be a linear transformation from E in to E′. Then T is said to be continuous if it is continuous as a mapping of the metric space E into the metric space E′ where the metric is defined in terms of norm.
Hence, T is continuous ⇔ x_n → x in E ⟹ T(x_n) →T(x) in E′.
It is worthy much to note that 1. For any normed linear space E the identity transformation: E → E defined by I(x) = x for every x ∈ E, is a continuous linear transformation. In fact x_n → x in E ⟹ Ix_n → I_x in E. 2. For normed linear spaces E and E′ the zero transformation O: E → E′ denoted by O(x) = O ∈ E for every x ∈ E is a continuous linear transformation. In fact x_n→x in E⟹O(x_n) = O→O(x)=O.

2.15 Continuous Linear Operator: If a continuous linear transformation T defined from E to E i.e. defined on E itself then this continuous linear transformation is said to be continuous linear operator.

2.16 Operator on A Hilbert Space H: By an operator on a Hilbert space H we shall mean a continuous linear transformation from H into self.

2.17 Set of Operators R On A Hilbert Space H: We construct a set of operators R on a Hilbert space H in the following way $R = \{T_i : T_i^2 = 0, T_iT_j = 0\}$

2.18 Kernel of F (or Null Space): The Kernel of the linear transformation f is denoted by Kernel of f = Kerf = $\{x \in R_1 : f(x) = 0\}$ Or equivalently Null space of f = $\{x \in R_1 : f(x) = 0$.

3. Results and Discussion

In this section, we establish some of the results using the definitions given in section 2 of this chapter and chapter one of this work.

Theorem (3, I): B(H), the space of all operators on a Hilbert space H is a Banach space.

Proof: Since for any $f_1, f_2 \in B(H)$, $\alpha \in K$ (the field of scalars) the point-wise addition and scalar multiplication are defined by $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ and $(\alpha f)(x) = \alpha f(x)$ for all $x \in H$
Clearly, $f_1 + f_2$ and αf_1 are linear transformations on H (that is linear operators on H)
Also, f is continuous if and only if T is bounded in the sense that $||f(x)|| \le m||x||$ for all $x \in H$, m be a real number.
[we refer to Jha,(1),p-213]

Also, $||(f_1 + f_2)(x)|| = ||f_1(x) + f_2(x)|| \le ||f_1(x)|| + ||f_2(x)|| \le ||f_1||.||x|| + ||f_2||.||x|| = (||f_1|| + f_2||)||x||$ …….(3.11)

Thus $f_1 + f_2$ is continuous. Hence $f_1 + f_2 \in B(H)$.

Also $||(\alpha f)(x)|| = ||\alpha f(x)|| = |\alpha|||f(x)|| \le |\alpha|.||f||.||x||$

Thus αf is continuous. Hence $\alpha f \in B(H)$.
Also, the zero element of B(H) is the continuous linear transformation o defined by o(x) =o for all $x \in H$.
Now it is easy to see that B(H) is a linear space
We now define a norm on B(H) in the following manner

$$||f|| = \underset{\substack{x\in H \\ x\neq 0}}{\text{Sup}}.\ \frac{||f||}{||x||},\ f \in B(H) \ \ldots\ldots\ldots\ (3.12)$$

From (3.12) it is clear that $||f|| \ge 0$,

Also, $||f||=0$ iff $\underset{\substack{x\in H \\ x\neq 0}}{\text{Sup}} \frac{||f||}{||x||} = 0$ iff $\frac{||f||}{||x||} = 0$, for all $x \neq 0$

$\Leftrightarrow ||f|| = 0$ for all $x \in H$
$\Leftrightarrow f = 0$ (that is f is zero transformation)

Thus $||f|| = 0 \Leftrightarrow f = 0$

Also $||\alpha f|| \underset{\substack{x\in H \\ x\neq 0}}{\text{Sup}} \frac{||(\alpha f)(x)||}{||x||} = \text{Sup} \frac{K||f(x)||}{||x||} = |\alpha|.\text{Sup} \frac{||f(x)||}{||x||} = |\alpha|\ ||f||$.

Also from (3.11), $||f_1 + f_2|| = \underset{\substack{x\in H \\ x\neq 0}}{\sup} \frac{||(f_1+f_2)(x)||}{||x||} \le ||f_1|| + ||f_2||$

There from (3.12) forms a Norm.
Thus B(H) is a Normed Linear space.

Since H is a Hilbert space so H is also a Banach space.
Let $\{f_n\}$ be a Cauchy sequence in B(H).
Then by definition of a cauchy sequence to give any $\epsilon > 0$, there exists a positive no such that

$$d(f_n, f_m) = ||f_n - f_m|| = \underset{\substack{x\in H \\ x\neq 0}}{\sup} \frac{||(f_n - f_m)(x)||}{||x||}$$

$$= \underset{\substack{x\in H \\ x\neq 0}}{\sup} \frac{||f_n(x) - f_m(x)||}{||x||} \le \varepsilon \text{ for all } n,m \ge n_0$$

Thus $||f_n(x) - f_n(x)|| \le \varepsilon||x||$ for all $n,m \ge n_0$ and all $x \in H$. ………(3.13)

Hence for each particular $x \in H$, $\{f_n(x)\}$ is a cauchy sequence in H and we recall that H is a Banach space also (as if is a Hilbert space)

Let $\{f_n(x)\} \to f(x)$ in H

Let $x, y \in H, \alpha \in K$

Then $f(x+y) = \lim_{n\to\infty} f_n(x+y) = \lim_{n\to\infty} (f_n(x) + f_n(y)) = f(x)+f(y)$

Also $f(\alpha x) = \lim_{n\to\infty} f_n(\alpha x) = \lim_{n\to\infty} \alpha f_n(x) = \alpha f(x)$

Thus f is a linear transformation from H into H itself.
But we already know that Norm is a continuous function.
Now in (3.13) lifting $m\to\infty$ we get,

$||f_n(x) - f_m(x)|| \le \varepsilon ||x||$, for all $n \ge n_0$ and all $x \in H$. ………(3.14)

Now $||f(x)|| \le ||f(x) - f_{n_0}(x)|| \le (\varepsilon + ||f_{n_0}||).||x||$ [by (3.14)]

Thus f is continuous [by a proposition p.213, Jha (1)]

Thus $f \in B(H)$.

Also $d(f_n, f) = ||fn - f|| = \underset{\substack{x\in H \\ x\neq 0}}{\sup} \frac{||f_n(0) - f(x)||}{||x||} \le \varepsilon$ for all $n \ge n_0$

Hence the Cauchy sequence $\{f_n\} \to f$ in B(H).
Therefore B(H) is complete normed linear space.
Hence B(H) is a Banach space.

Theorem (3, II): R is a Banach space.
Proof: We have seen in the previous paper that R is a ring and hence it is an additive Abelian group.

Also $(T_i + T_j)^2 = T_i^2 + T_j^2 + 2T_iT_j = 0$

Hence $T_i + T_j \in R$ for $T_i, T_j \in R$.

Also let α be any scalar. Let T_i belongs to R

Thus $T_i^2 = 0$
Thus $(\alpha T_i)^2 = \alpha^2 T_i^2 = \alpha^2.0 = 0$
Also $(\alpha T_i)T_j = \alpha T_iT_j = \alpha.0 = 0$
Thus αT_i is an element of R.
Thus vector addition and scalar multiplication are closed in R.

Hence R is a linear space in its own right, being a linear subspace of B(H), the space of all operators on a Hilbert space H.
Since we have also seen that B(H)is a Normed linear space so R is also a normed linear space.

Now, let T be an element of $\bar{R}$ then there exists a sequence $\{T_n\} \subseteq R$ such that $T_n \to T$.

Hence $T_n^2 = 0$, for all n = 1,2,3,……..
But $T_n \to T$, $T_n \to T \Rightarrow T_n^2 \to T^2$
Thus $T_n^2 = 0 \Rightarrow T^2 = 0$, Also $T_n \to T \Rightarrow T_nT_j \to TT_j$ as $n \to \infty$,
When $T_j \in R$

Since $T_nT_j = 0$ for all n, so $TT_j = 0$ therefore T belongs to R

Thus $T \in \bar{R} \Rightarrow T \in R$ and T is arbitrary.

Hence $\bar{R} \subseteq R \subseteq \bar{R}$.

This implies that $R = \bar{R}$.

Therefore R is a closed linear subspace of the Banach B(H).
Thus R is a Banach space in its own right.

Theorem (3.3, III): Let R and R' be linear spaces and T: R → R' be a linear transformation then Kernel T is a linear subspace of R.

Proof: Here R and R' are linear spaces of operators on a Hilbert space H.
By definition Ker T = $\{T_i \epsilon R: T(T_i)=0\}$, Here we have to show that KerT is a linear subspace of R.

For this, let $T_i, T_j \in$ KerT, then $T(T_i) = 0$, $T(T_j) = 0$

Let α, β be scalars in K.

Hence $\alpha T_i + \beta T_j \in R$

Also T: R → R' is a linear transformation

Thus $T(\alpha Ti + \beta T_j) = \alpha T(T_i) + \beta T(T_j) = \alpha.0 + \beta.0 = 0 + 0$.
Thus $\alpha T_i + \beta T_j \in$ KerT = 0.

Therefore KerT is a linear subspace of R.

Theorem (3.3, IV): Let R and R' be linear space of the operators on a Hilbert space H and T: R → R' be linear transformation. Then T is one to one if and only if KerT={0}

Proof: Here R and R' are linear spaces and T: R → R' be a linear transformation,

By definition Ker T = $\{T_i \in R: T(T_i) = 0\}$.

Let T be a one-one mapping and let $T_i \in$ KerT $\Rightarrow T(T_i) = 0$
Also T(0) = 0, thus $T(T_i) = T(0)$.

But T is one-one $\Rightarrow T_I = 0$ and T_i is arbitrary. Hence KerT = {0}.

Conversely let KerT ={0}. To show that T is one-one.

Also let $T(T_i) = T(T_j)$, also $T(T_I - T_j) = T(T_i) + T(-T_j) = T(T_i) - T(T_j) = 0$

[since above we have supposed $T(T_i) = T(T_j)$]

This implies that $T(T_i - T_j) = 0 \Rightarrow T_i - T_j \in$ KerT = {0}. Thus $T_i - T_j = 0 \Rightarrow T_i = T_j$

That is supposing $T(T_i) = T(T_j)$ implies that $T_i = T_j$, Thus T is one-one.
Thus the necessary and sufficient conditions are observed.
Theorem (3.3,V): Let T be a linear transformation of R onto R' then T is isomorphism if and only if KerT = {0}.

Proof: Let Ker T = {0}. To prove T is an isomorphism.
For by hypothesis T is onto.
By the above theorem in this situation T is one-one.
Also, T is linear that is T is a homomorphism
Thus T is an isomorphism.
Clearly Let T be an isomorphism.
Thus T is one to one and hence by the above theorem (3.3,IV) Ker T = {0}.
Thus the result is established.

Theorem (3.3, VI): Let T: R → R' be a linear transformation from R into R'.
Then the range of T is a linear subspace of R'.

Proof: Let T: R → R' be a linear transformation from R into R'.
Let R(R) be the range of T. Then $T(R) = \{T(T_i): T_i \in R\}$. To show that T(R) is a subspace of R'.

For this, let $T_i, T_j \in R$. Then $T(T_i), T(T_j) \in T(R)$.

Also let $\alpha T_i + \beta T_j \in R$ then $T(\alpha T_i + \beta T_j) \in T(R)$ …(3.15)

Since T is a linear transformation

Thus $T(\alpha T_i + \beta T_j) = \alpha T(T_i) + \beta T(T_j)$ **……….**(3.16)

Thus from [(3.15) and (3.16)], $\alpha T(T_i) + \beta T(T_j) \in T(R)$

Thus $T(R) = \{T(T_i): T_i \in R\}$ is a linear subspace of R'.

Theorem (3.3, VII): Let (i) R be a linear space (ii) T be a non-singular linear transformation on R
(iii) T^{-1} be the inverse of T Then T^{-1} is also a linear transformation on R.

Proof: Let $T_i', T_j' \in R$ and α,β be scalar in K.
T is given to be non-singular
Thus T is one-one and onto.
Hence there must exist unique vectors T_i, T_j in R such that

$T(T_i) = T_i'$, And $T(T_j) = T_j'$.

But by hypothesis, T^{-1} is the inverse of T

Therefore $T_i = T^{-1}(T_i')$, and $T_j = T^{-1}(T_j')$

As R is a linear space, T_i, $T_j \epsilon R$, α,β are scalars
Thus $\alpha Ti + \beta T_j \epsilon R$

Also, T is supposed to be linear

Hence $T(\alpha T_i + \beta T_j) = \alpha T(T_i) + \beta T(T_j) = \alpha T_i' + \beta T_j'$

Also $T^{-1}(\alpha T_i' + \beta T_j') = T^{-1}[\alpha T(T_i)] + T^{-1}[\beta T(T_j)] = \alpha T^{-1}T(T_i) + \beta T^{-1}T(T_j) = \alpha T_i + \beta T_j = \alpha T^{-1}(T_i') + \beta T^{-1}(T_j')$

That $T^{-1}(\alpha T_i' + \beta T_j') = \alpha T^{-1}(T_i') + \beta T^{-1}(T_j')$.

Which explicitly shows the fact that T^{-1} is a linear transformation.

Hence the result is established.

Theorem (3.3, VIII): The set of all non-singular transformations on a linear space R forms a group with respect to multiplication.

Proof: Let R is the set $\{T_i : T_i^2 = 0,\ T_iT_j = 0$ for all i,j $\}$ of operators on a Hilbert space H.
Then R forms a linear space which we have already seen earlier.
Let G be the set of all non-singular transformations defined on R.
To show that (G,) forms a group.
Let $S,T \in G$ then S, T are one-one and onto.

Also, let $ST(T_i) = ST(T_j) \Rightarrow S(T(T_i)) = S(T(T_j)) \Rightarrow T(T_i) = T(T_j)$ [since S is one-one] $\Rightarrow T_i = T_j$ [since T is one-one] $\Rightarrow$ ST is one-one

Again if S, T are onto and $ST: R \rightarrow R$, let $T_i \in R$ be any element.

Since S is onto so we get $T_j \in R$ such that $S(T_j) = T_i$.

Also as $T: R \rightarrow R$ is onto so we must get T_p in R such that $T(T_p) = T_j$

Thus $T(T_p) = T_j \Rightarrow S(T(T_p)) = S(T_j) \Rightarrow ST(T_p) = T_i \Rightarrow$ ST is onto

Therefore ST is one-one, onto.
Hence $ST \in G$ implies that G is closed with respect to multiplication.

Also from the associativity of composites function we already know that for $T,S,P \in G \Rightarrow P(ST) = (PS)T$

Also, if $T \in G$ then T is a non singular linear transformation on vector space R.

Then $T: R \rightarrow R$ is one-one and onto

Thus the inverse of T exists. Let it be T^{-1} then $T^{-1}: R \rightarrow R$ such that $T(T_i) = T_i'$ if and only if $T_i = T^{-1}(T_i')$

Also $TT^{-1} = I = T^{-1}T$, where I is the identity function on R.
Also T^{-1} is also a linear transformation [for which we refer to theorem (3.3,VII)].
Here we call this group a linear group.

4. References

1. Edward RE. (i) Functional analysis Holt Richart and Winston, Inc., New York; c1965.
2. Halmos PR. (i) Finite dimensional vector space, Van Nostrand, Princeton, NJ; c1958. (ii) A Hilbert space, Problems Book, springer international student edition, Narosa Publishing House New Delhi; c1978. (iii) Naïve set theory, D.Van nostrand, Princeton, New Jersey, Toronto, New York, London; c1964 Jun.
3. Hille E. (i) Functional Analysis and semigroups. Americal Math. Soc. colloquium publications, Vol.31, New York; c1948.
4. Hildebrandt JH. (i) on bounded linear transformation, Trans., Amer., Math. Soc. 1934;36:868-875.
5. Hoffman, Kenneth. (I) Linear Algebra (pearson education, Dorling Kindersley India, Pvt LTD, Pancheel Park, New Delhi, India.
6. Horvath John. (i) Topological Vector spaces and Distributions, Volume 1. (Addison-Wesley publishing company, Palo Alto, London; c1966.
7. Herstein IN. (i) Topics in Algebra, 2nd edition, (John wiley & sons, New York, chichester, Brisbane, Toronto, Singapore)
8. Hyers DH. (i) Linear Topological spaces, Boll. Amer. Math. soc. 1945;51:1-24.
9. Jacobson N. (i) Lectures in Abstract Algebra., 1. D.Van Nostrand company Inc.
10. Jha KK. (i) Advanced course in modern Algebra (New Bharti Prakashan patna, Delhi; c2002. (ii) Functional Analysis (students Friends, Patna; c1986.
11. Maddox IJ. (i) Elements of Functional analysis, Cambridge University Press; c1970.
12. Royden HL. (i)Real Analysis, the Macmillan Company, New York; c1964.
13. Rudin W. (i) Functional Analysis, Tata Mc. Graw Hill Publishing company Ltd., New Delhi; c1974.
14. Simmons GF. (i) Introduction to Topology and Modern Analysis. (Mc Graw-Hill Book company, Inc., New York; c1963.