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Use of Orstein: Uhlenbeck model in deriving black: Scholes equation

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Abstract
One of the diffusion processes that is used to model the velocity of a particle that undergoes Brownian motion is the Orstein - Uhlenbeck, which can also be used to model the volatility of an underlying process. Black - Scholes model gives a very good approximation to the analysis of market derivatives despite its various assumptions. One benefit of the Orstein - Uhlenbeck model is that it leads to tractable solutions to a number of financial challenges. This study, therefore, formulates the Black - Scholes equation using Orstein - Uhlenbeck model, this will help solve some of the financial challenges faced in the Black - Scholes equation like constant volatility. This study uses the analysis of the Black - Scholes equation to derive a new model of the Black - Scholes equation using Orstein - Uhlenbeck model.

Keywords: Geometric brownian motion, black-scholes equation, volatility, orstein-uhlenbeck model

Introduction
The standard Black-Scholes model [3] has been a breakthrough in financial literature as option pricing is concerned. Though it undertakes various assumptions in its derivation as option modelling is concerned. The major assumptions as listed by Black – the Scholes model are;

- The measure of volatility is constant.
- There is no dividend payment.
- The interest rate is known to be constant.
- The returns are log-normally distributed.
- There is no transactional cost and no commission.

Several models have been proposed to address these shortcomings of Black - the Scholes model since in real practice, the assumptions cannot be applied in real market structure. The option price of the assets is given as some unction of time X_t which follows a diffusion process presented by Geometric Brownian motion given by;

$$dX_t = \mu X_t dt + \boldsymbol{\sigma} dW_t \tag{1}$$

where μ is the rate of growth of the underlying option, $X_t$is the option pricing process at time t, $\boldsymbol{\sigma}$ is the volatility which is constant and dW_t is the Wiener process depicted by a random walk known as Brownian Motion.

Mean reverting process and jump-diffusion processes have been incorporated in equation (1), the Geometric Brownian motion to address the shortcomings of the Black - Scholes model. Hull and White [5], Stein and Stein [17], and Heston [5] among others are some mean reverting models that help address the shortcomings of the Black – Scholes model.
Vasicek model has also been one of the standard mean reverting processes model where the pricing process reverts back to a certain equilibrium price. Orstein – Uhlenbeck model extended the Vasicek model by applying a maximum likelihood estimation to derive a stochastic differential equation given by;

$$dX_t = \lambda(\mu - X_t)dt + \boldsymbol{\sigma} dW_t, \tag{2}$$

Where dW_t is the Wiener process, λ>0 is the mean reverting speed, µ is the long–term equilibrium price (long-term mean) with time, $\boldsymbol{\sigma}$>0 is the volatility process and X_t is the price of the underlying option that fluctuates randomly over a long period of time but reverts back to the equilibrium price µ. Therefore this study built a Black - Scholes process that has tractable solutions obtained from Orstein - Uhlenbeck on a number of financial challenges.

Preliminaries

Stochastic process

These are processes that follows law of probability. Mathematically a stochastic process is expressed as $\{X_t: t \in (0, \infty)\}$ where random variables have some collections in each t in some index set$(0, \infty)$. There are two different parts of stochastic processes; discrete-time and continuous-time stochastic processes. The discrete-time stochastic process is where change of variables is at a certain fixed - point in time and the continuous-time stochastic process is where change of the value of variable is over a given range in time.

Markov Stochastic Process

This is where the behaviour of the current price of an asset is not influenced by the past history as it is always believed that all the relevant information contained in the past history is already contained in the present price of the asset.

Wiener Process

This is a type of Geometric Brownian motion in which the mean - rate of change is zero and the variance - rate of change is one. Its properties are given by;

Property 1: If we have a small change of time - period, ΔZ we define

$$\Delta Z = \varepsilon\sqrt{\Delta t} \tag{3}$$

Where $\varepsilon \sim N(0,1)$

Property 2: If we have two - distinct short - time periods, where ΔZ are independent, we define;

$$cov(\Delta Z_i, Z_j) = 0 \tag{4}$$

Where $i \neq j$.

Hull [7]

Options

It is an obligation to buy or sell an underlying derivative before the expiry date or at a certain price from a contract between two investors. The two different types of options are the call option and the put option.

Call option

It is where underlying: Derivative has a contract that can allow one to buy it at expiry date or at certain price.

Put option

It is where underlying: Derivative has a contract that can allow one to sell it at expiry date or at certain price.

Itȏ Process

It is a generalized: Wiener diffusion process, mathematically expressed as;

$$dX = a(X,t)dt + b(X,t)dW_t, \tag{5}$$

Where $b$is the rate of variance and $a$is the expected drift rate both are function defined by variables X and t. The discrete version of equation (5) is given by;

$$\Delta X = a(X,t)\Delta t + b(X,t)\varepsilon\sqrt{\Delta t} \tag{6}$$

Itȏ Lemma

A function $P(X,t)$ that is differentiable twice in $X$and once in t is an Itȏ Lemma obtained from an Itȏ process given by;

$$dP = \left(\frac{\partial P}{\partial X}a + \frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial X^2}b^2\right)dt + \frac{\partial P}{\partial X}b dW_t \tag{7}$$

Where dW_t is the standard - Wiener diffusion process, the drift rate P is given by $\left(\frac{\partial P}{\partial X}a + \frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial X^2}b^2\right)$ and the variance rate is given by $\left(\frac{\partial P}{\partial X}\right)^2 b^2 dt$

Main results

Formulation of Black-Scholes equation using Orstein-Uhlenbeck model

We look at an extended Vasicek model that is suitable for applying maximum likelihood estimation. The Orstein – Uhlenbeck satisfies the following stochastic differential equation of the form;

$$dX_t = \lambda(\mu - X_t)dt + \boldsymbol{\sigma}dW_t, \tag{8}$$

Where dW_t is the Wiener process, λ>0 is the mean reverting speed, µ is the long–term equilibrium price (long-term mean) with time Orstein – Uhlenbeck tends to drift towards the mean, $\boldsymbol{\sigma}$>0 is the volatility process.

We consider P that is a function of X and t that forms an optimal price for any call – option which is differentiable twice in X and once in$t$. Using Itô lemma, we have;

$$dP = \left(\frac{\partial P}{\partial X}\lambda(\mu - X_t) + \frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial X^2}\sigma^2\right)dt + \frac{\partial P}{\partial X}\sigma dW_t \tag{9}$$

Using the discrete version of equations (8) and (9) which are given as;

$$\Delta X_t = \lambda(\mu - X_t)\Delta t + \sigma\Delta W_t, \tag{10}$$

And

$$\Delta P = \left(\frac{\partial P}{\partial X}\lambda(\mu - X_t) + \frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial X^2}\sigma^2\right)\Delta t + \frac{\partial P}{\partial X}\sigma\Delta W_t \tag{11}$$

Using the definition of Π taken as the price of the portfolio whereby the portfolio holder can take both short and long-term option positions in acquiring the number of shares. By definition;

$$\Pi = -P + \frac{\partial P}{\partial X}X \tag{12}$$

Which has the discrete version of $\Delta\Pi$ from the price of the portfolio over a given intervalΔt. This is given as;

$$\Delta\Pi = -\Delta P + \frac{\partial P}{\partial X}\Delta X \tag{13}$$

We substitute equation (10) and equation (11) in equation (13) to get;

$$\Delta\Pi = -\left\{\left(\frac{\partial P}{\partial X}\lambda(\mu - X_t) + \frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial X^2}\sigma^2\right)\Delta t + \frac{\partial P}{\partial X}\sigma\Delta W_t\right\} + \frac{\partial P}{\partial X}\{\lambda(\mu - X_t)\Delta t + \sigma\Delta W_t\} \tag{14}$$

Equation (14) simplifies to;

$$\Delta\Pi = \left(-\frac{\partial P}{\partial t} - \frac{1}{2}\frac{\partial^2 P}{\partial X^2}\sigma^2\right)\Delta t \tag{15}$$

From equation (15) we see that it does not contain ΔW_t hence it's no longer stochastic. This makes the value of the portfolio to be risk – free in a given duration of time Δt. This assumption is listed in the Black - Scholes Model which shows that the price of the portfolio is possibly able to earn a standard rate of return during the short – term life of the risk – less asset. When it earns more, the arbitrageurs can make a risk – free profit when borrowing from financial institutions by buying this free risk portfolio. In case it earns less, one can make a risk – free profit by shortening the life of the portfolio and buying a risk - free asset. From this we have;

$$\Delta\Pi = r\Pi\Delta t, \tag{16}$$

Where r is the rate of risk – free interest given by $\lambda(\mu - X_t)$ which is from Orstein – Uhlenbeck model, where $\lambda>0$ is the mean reverting speed, μ is the long – term equilibrium price (long term mean) and X_t is the price of the underlying option.

When we substitute equation (12) and equation (15) into equation (16) we get

$$\left(-\frac{\partial P}{\partial t} - \frac{1}{2}\frac{\partial^2 P}{\partial X^2}\sigma^2\right)\Delta t = \lambda(\mu - X_t)\left(-P + \frac{\partial P}{\partial X}X\right) \tag{17}$$

Which simplifies to;

$$\frac{\partial P}{\partial t} + \lambda(\mu - X_t)\frac{\partial P}{\partial X}X + \frac{1}{2}\frac{\partial^2 P}{\partial X^2}\sigma^2 = \lambda(\mu - X_t)P \tag{18}$$

This is the Black – Scholes equation derived using the Orstein- Uhlenbeck equation.

Conclusion

From this paper we have derived a Black – Scholes equation using Orstein-Uhlenbeck equation that can lead to tractable solutions for very many problems in finance that can help investors in making viable decisions when setting up their strategies.

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