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On the solution of riccati matrix differential equations by an improved variational iteration scheme

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Abstract
We consider the approximate solution to the iterative scheme and its convergence. Variational iteration method (VIM) is applied to the general solution form of the iterative approximate solutions and a new Variational iteration scheme is derived with an improved and higher rate of convergence to approximate solution after some few iterations. The modified method proved to accelerate the convergence to the exact solution, where a new correction functional is formulated by Lagrange multiplier.

Keywords: Riccati matrix differential equations, variational iteration, he's method, approximate solution

Introduction
The fundamental theories of Riccati equation with applications to engineering science with a newer application to economics and finance. Various researchers had made attempts to the derivation of solutions to the problems by using the classical approach. However, (see) [24] applied Adomian decomposition techniques in solving the nonlinear Riccati by analytical approach. Again, the work Tan and Abbasbandy (see) [25] applied the method of Homotopy Analysis Method (HAM) to solve quadratic Riccati equation. The works of He (see) [26] pioneered the rigorous research in the variational iteration method (see) [27, 28].
The application of VIM to some problems, proved to be simple to adopt and efficient in solving nonlinear problems.
Mathematical modelling of real-life problems application in control problems generates differential equations, integral equations, system of differential and algebraic equations. Solutions to such models are difficult to evaluate analytically, hence, numerical and approximate methods seem to be appropriate in solving such problems. Several researchers investigate Variational iteration methods (VIM) with other numerical and approximate methods, where it's shown by all that this method provide more accurate results and faster than the other methods.
However, a well-known Riccati Matrix Differential Equation (RMDE) has a vast range of applications. Various approaches can be used to solve RMDE with constant coefficients analytically. See Nguyen T. *et al*. The method of Nguyen T., *et al.* [1] is shown to be robust and numerically efficient.
Recently, an improvements were recorded in the application of VIM, see [2-4].
Over the years, there is a wide application of RMDE as a control model in which the analytical and theoretical results arising from matrix equation has been established.
Readers are referred the following papers (see) [5-11] for further application areas.
VIM is an improved general Lagrange's multiplier method see [10], which has shown to solve a large class of nonlinear problems accurately and efficiently
The novel contribution in our work is the construction of new variational iteration technique which solves the nonlinear terms to be differentiable with respect to the dependent variable and its derivatives. An improved of VIM to find an accurate approximate numerical solution to the problems of RMDEs.

Solution of RMDE by VIM

Consider a system of Riccati equation defined as stated in [15, 16]. The new proposed approach iteration by considering a linear operator where $I, J, and\ K\ are\ n \times n$ matrices such that $I\ and\ K$ are expressed as:

$$Q' + QJ + J^T Q - QIQ + K(t), 0 \leq t \leq 1 \tag{1}$$

By the application of correction functional with respect to the RMDE using VIM, it's possible to generate a sequence of iteration: $For\ n = 0,1,2, \dots,$

$$Q_{n+1}(t) = Q_n(t) + \int_0^t \lambda(t,s)[Q_n'(s) + Q_n(s)J + J^T Q_n(s) - Q_n(s)IQ_n(s) + K]ds \tag{2}$$

$where\ \lambda$ is the Lagrange multiplier.

Rewrite (1) as $Q'(t) + M\big(t, Q(t)\big) = 0$ (3)

Where $M\big(t, Q(t)\big) = Q(t)J + J^T Q(t) - Q(t)IQ(t) + K$ with the components of the nonlinear operator Λ necessary for the derivation of the sequence for the RMDE as:

$$\Lambda = \frac{d}{dt} \cdot + \cdot J + J \cdot^T - \cdot I \tag{4}$$

by decomposing the nonlinear operator (4) into two parts of linear and nonlinear respectively given by: $\Phi\ and\ \Pi$, where

$$\Phi \cdot := \frac{d}{dt} \cdot + \cdot J + J^T \text{ and } \Pi \cdot := - \cdot \mathrm{I} \tag{5}$$

The subsequent estimated/ generated sequence of solutions by the method with respect to the defined operators as in (5) nonlinear RMDE is defined by:

$$\Phi Q(t) + TQ(t) + K(t) = 0 \tag{6}$$

where Q is to be evaluated from the sequence.
The correction functional is expressed as:

$$Q_{n+1}(t) = Q_n(t) + \int_0^t \lambda(t,s) \left[\Phi(Q_n(s) + \Pi\left(\hat{Q}(s)\right) + K(s)\right] ds \tag{7}$$

Where $\hat{Q}$ is assumed as a restricted variation with $\partial \hat{Q}_n = 0$. (8)

Formulation of New VIM for Solving RMDEs

Consider the linear and nonlinear operators denoted by $\Phi\ and\ \Pi$ respectively.
Let Φ be a new linear operator introduced and stated as:

$$\Phi\big(Q(t)\big) + \Phi_1\big(Q(t)\big) - \Phi_1\big(Q(t)\big) + \Pi\big(Q(t)\big) + K(t) \tag{9}$$

(9) is used to construct the correction functional using the linear and nonlinear operator

Q as $\widehat{\Phi}\big(Q(t)\big) = \Phi\big(Q(t)\big) + \Phi_1(Q(t))$ (10)

And $\widehat{\Pi}\big(Q(t)\big) = -\Phi_1\left(\hat{Q}(t)\right) + \Pi(\hat{Q}(t))$ (11)

For any sequence generated for $n = 0,1,2,3, \dots,$ and using $\widehat{\Phi}\ and\ \widehat{\Pi}$ is given as:

$$Q_{n+1}(t) = Q_n(t) + \int_0^t \lambda(t,s) \begin{bmatrix} \Phi(Q_n(s) + \Phi_1(Q_n(s) - \Phi_1(\hat{Q}_n(s) + \\ \Pi\left(\hat{Q}_n(s)\right) + K(s) \end{bmatrix} ds \tag{12}$$

$$= Q_n(t) + \int_0^t \lambda(t,s)[\widehat{\Phi}(Q_n(s) + \widehat{\Pi}(\hat{Q}_n(s) + K(s)]ds \tag{13}$$

$Set\ \Phi_1(Q) = Q$, with the Lagrange multiplier λ and first variation δ with respect to $Q_n(t)$ and $\delta \hat{Q}_n(t) = 0\ and\ \delta K(t) = 0$, then

$$\delta Q_{n+1}(t) = \delta Q_n(t) + \int_0^t \lambda(t,s)[(Q_n'(s)) + Q_n(s) + \Psi(s, Q_n(s)]ds \tag{14}$$

$$where\ \Psi(s, Q_n(s)) = Q_n(s)J + J^T Q_n(s) - Q_n(s) - Q_n(s)IQ_n(s) + K \tag{15}$$

(15) is the nonlinear term such that Ψ can be reformulated in terms of the restricted variation $\hat{Q}_n(s)$, $\delta\hat{Q}_n(s) = 0, and\ \delta\Psi(s, \hat{Q}_n(s) = 0$.
By conjunction, (14) is reduced to:

$$\delta Q_{n+1}(t) = \delta Q_n(t) + \int_0^t \lambda(t,s)\delta Q_n'(s)ds + \int_0^t \lambda(t,s)\delta Q_n(s)ds + \int_0^t \lambda(t,s)\delta\Psi\left(s, \hat{Q}_n(s)\right)ds$$

$$= \delta Q_n(t) + \int_0^t \lambda(t,s)\delta Q_n'(s)ds + \int_0^t \lambda(t,s)\delta Q_n(s)ds \tag{16}$$

By integration by parts on (16), we have:

$$\delta Q_{n+1}(t) = [1 + \lambda(t,s)]Q_n(s)|_{\ s=t} \int_0^t [1 - \lambda'(t,s)]\delta Q_n(s)ds \tag{17}$$

By setting variation principal $\delta Q_{n+1}(t) = 0$, and The Euler-Lagrange result to the following differential equation: $1 + \lambda'(t, 0) = 0$ (18)

With boundary condition: $1 + \lambda(t,s)|_{s=0} = 0$ (19)

The solution of (18) and (19) is $\lambda(t,s) = s - t - 1$.
The iteration scheme is reduced to a sequence after the substitution λ in to (2)

$$Q_{n+1}(t) = Q_n(t) + \int_0^t \lambda(s - t - 1)[Q_n'(s) + Q_n(s)J + J^T Q_n(s) - Q_n(s)IQ_n(s) + K]ds \tag{20}$$

Convergence Criteria
Theorem 1: *Let* $Q_1, Q_2, \ldots . Q_n \in C^1[0,1]\ \forall n = 0,1, \ldots,$
And suppose $C_n(t) = Q_n(t) - Q(t), \forall\ 0 \le t \le 1$ such that the nonlinear operator $\Pi Q = -QIQ$ satisfies Lipschitz condition with constant $\ell < 2\|J\|$, then the sequence $\{Q_n(t)\}, n = 0,1, \ldots,$ generated by the approximate solutions converge to the exact solution $Q(t), \forall\ 0 \le t \le 1\ as\ t \to \infty$.

Proof
Equation (20) is the approximate solution of the iteration of (1), given that Q is the exact solution. It implies that Q is the VIM which follows that the solution is given by:

$$Q(t) = Q(t) + \int_0^t (s - t - 1)[Q'(s) + Q(s)J + J^T Q(s) - Q(s)IQ(s) + K]ds \tag{21}$$

Using (20) and (21), we have: $Q_{n+1} - Q(t) = Q_n(t) - Q(t) +$

$$\int_0^t (s - t - 1)\begin{bmatrix} Q_n'(t) - Q'(s) + \left(Q_n(s) - Q(s)\right) \cdot J \\ +J^T\left(Q_n(s) - Q(s)\right) - Q_n(s)IQ_n(s) \\ +Q(s)IQ(s) \end{bmatrix} ds \tag{22}$$

The error function ϵ_n of the iteration is given as:
$\epsilon_n(t) = (Q_n(t) - Q(t))$, and by conjunction, (22) is written in terms of the error function (ϵ_n) as

$$\epsilon_{n+1}(t) = \epsilon_n(t) + \int_0^t (s - t - 1)\epsilon_n'(s)ds + \int_0^t (s - t - 1)\epsilon_n(s)Jds +$$

$$\int_0^t (s - t - 1)J^T\epsilon_n(s)ds - \int_0^t (s - t - 1)[Q_n(s)IQ_n(s) - Q(s)IQ(s)]ds$$

Given that $0 \le t, s \le 1, then\ Sup\ value\ of\ s - t - 1 \le 1$

$$\Rightarrow\ \epsilon_{n+1}(t) \le \epsilon_n(t) + \int_0^t \epsilon_n'(s)ds + \int_0^t \epsilon_n(s)Jds + \int_0^t J^T\epsilon_n(s)ds - \int_0^t Q_n(s)IQ_n(s)ds$$

$$= \epsilon_n(t) - \epsilon(t) + \epsilon_n(0) + \int_0^t \epsilon_n Jds + \int_0^t J^T\epsilon_n(s)ds$$

$$- \int_0^t [Q_n(s)IQ_n(s) - Q(s)IQ(s)]ds \tag{23}$$

Obviously, $\epsilon_n = Q_n(0)Q(0) = 0$ the supremum norm of (23) yields

$$\|\epsilon_{n+1}(t)\|\ \le \int_0^t \|\epsilon_n(s)\|\|J\|ds + \int_0^t \|J^T\|\|\epsilon_n(s)\|ds + \int_0^t \|Q_n(s)IQ_n(s) - Q(s)IQ(s)\|ds$$

$\leq \|J\| \int_0^t \|\epsilon_n(s)\| ds + \|J^T\| \int_0^t \|\epsilon_n(s)\| ds + \ell \int_0^t \|Q_n(s) - Q(s)\| ds$

$\Rightarrow \|\epsilon_{n+1}(t)\| \leq \|J\| \int_0^t \|\epsilon_n(s)\| ds + \|J^T\| \int_0^t \|\epsilon_n(s)\| ds + \ell \int_0^t \|\epsilon_n(s)\| ds$

$$= (2\|J\| + \ell) \int_0^t \|\epsilon_n(s)\| ds \qquad (24)$$

By induction, if $= 0$, then

$$\|\epsilon_1(t)\| \leq (2\|J\| + \ell) \int_0^t \|\epsilon_0(s)\| ds \leq (2\|J\| + \ell) \sup_{s \in [t_0, 1]} |\epsilon_0(s)| \int_0^t ds$$

$\leq (2\|J\| + \ell) \cdot t \cdot Sup|\epsilon_0(t)|$

For $n = 1$, then;

$$\|\epsilon_2(t)\| \leq (2\|J\| + \ell) \int_0^t \|\epsilon_1(s)\| ds \leq (2\|J\| + \ell) \int_0^t [(2\|J\| + \ell)] \sup_{s \in [t_0, 1]} |\epsilon_0(s)| ds$$

$$= (2\|J\| + \ell)^2 \sup_{s\in[t_0,1]} |\epsilon_0(s)| \int_0^t ds \leq \frac{(2\|J\|)^2}{2} t^2 \sup_{s\in[t_0,1]} |\epsilon_0(s)| \qquad (25)$$

For $n = 2$, then;

$$\|\epsilon_3(t)\| \leq (2\|J\| + \ell) \int_0^t \|\epsilon_2(s)\| ds \leq (2\|J\| + \ell) \frac{\int_0^t (2\|J\|)^2 s^2 ds}{2} \sup_{s\in[t_0,1]} |\epsilon_0(s)|$$

$$\leq \frac{(2\|J\|)^3}{3!} t^3 \sup_{s\in[t_0,1]} |\epsilon_0(s)| \qquad (26)$$

For $n > 3, and\ \forall n,$

$$\|\epsilon_n(t)\| \leq \frac{(2\|J\|)^n}{n!} t^n \sup_{s\in[t_0,1]} |\epsilon_0(s)| \leq \frac{(2\|J\|)^n}{n!} t^n \sup_{s\in[t_0,1]} |\epsilon_0(s)| \int_{t_0}^1 ds$$

Since $(2\|J\| + \ell) < 1$ as $n \to \infty, then\ \frac{1}{n!} \to 0$ which implies that $\|\epsilon_n(t)\| \to 0\ as\ n \to \infty,$ implies that the sequence of the approximate solution using (13) converge to the exact solution as required.

Illustration of the new algorithm

We use example to demonstrate the efficiency of the new algorithm.

Consider the scalar RDE $y'(t) - 1 + y^2(t) - t^2 = 0, y = 1, 0 \leq t \leq 1$ (27)

$with\ the\ exact\ solution\ y(t) = t + \frac{e^{-t^2}}{1+\int_0^t e^{-u^2} du}$

The new scheme of VIM for (27), using iteration defined in (20) with the initialization point with $y_0(t) = 1$, then:

$$y_1(t) = y_0(t) + \int_0^t (s - t - 1)[y_0'(s) - 1 + y_0^2(s) - s^2] ds = 1 + \frac{t^3}{3} + \frac{t^4}{12} y_2(t)$$

$$y_2(t) = y_1(t) + \int_0^t (s - t - 1)[y_1'(s) - 1 + y_1^2(s) - s^2] ds = 1 + \frac{t^3}{3} + \frac{t^4}{6} - \frac{t^5}{12} - \frac{t^6}{180} - \frac{t^7}{63} - \frac{t^8}{112} + \frac{t^9}{648} - \frac{t^{10}}{12960}$$

Using the same iteration techniques for $y_3(t), y_4(t), y_5(t), \ldots y_9(t), y_{10}(t)$. The table below shows the exact solutions with the approximations solutions errors.

Steps of Iterations	Exact solution	Absolute errors at y_9	Absolute errors at y_{10}
0.1	1.00031731	2.11e-16	2.1105e-16
0.2	1.002419825	3.39e-13	1.68754e-15

0.3	1.007794588	4.8642e-11	3.39817e-11
0.4	1.017650879	1.6154e-9	1.51098e-10
0.5	1.032957576	2.5009e-8	2.93965e-9
0.6	1.05446681	2.4036e-7	3.4122e-7
0.7	1.082727481	1.6713e-6	2.78951e-7
0.8	1.118092545	9.2128e-6	1.77355e-6
0.9	1.160723973	4.2708e-5	9.34812e-4
1	1.2106	1.7329e-4	4.26587e-5

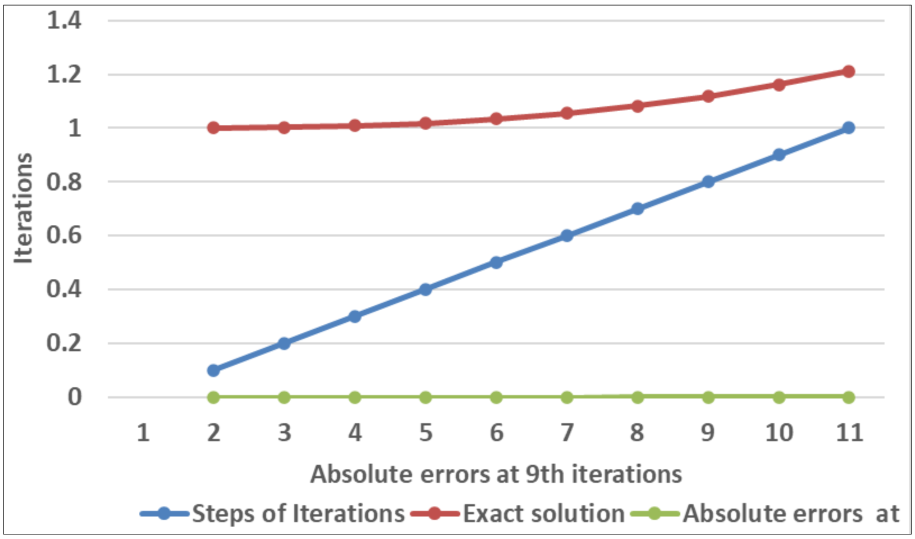

Fig 1: 9th Iteration, with exact solutions and absolute erros

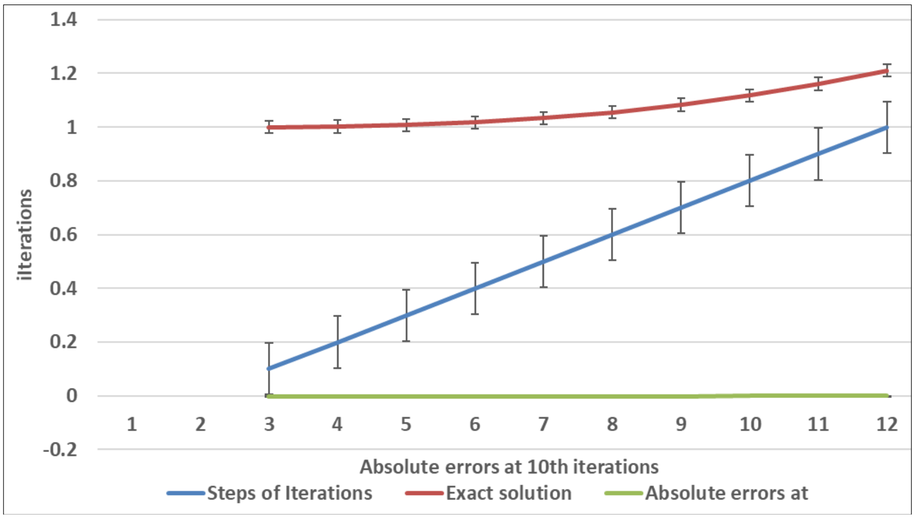

Fig 2: 10th Iterations, with exact solutions and absolute errors

Conclusion

The solution of Riccati matrix differential equations using the improve VIM were established with a higher rate of convergence to the exact solution with minimal error as it tends to zero with successive iterations.

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