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Abdullah Kurudirek
Department of Mathematics Education, Tishk International University, Erbil, Iraq

Corresponding Author:
Abdullah Kurudirek
Department of Mathematics Education, Tishk International University, Erbil, Iraq

Illuminating high school students with some interesting non-euclidean geometries in the plane

Abdullah Kurudirek

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Abstract
Global trends in mathematics education show that modern teaching methods are rapidly evolving at the national, regional, and global levels. In this regard, teaching non-Euclidean geometry concepts in schools can be essential in developing students' spatial imagination and enhancing scientific inquiry competencies. This paper aims to engage and increase students' interest in geometry science by introducing the fundamental concepts of several non-Euclidean geometries. With this aim, we will first give you a modern definition of geometry and move towards the exciting and fun world of non-Euclidean geometry. Of course, remember that the target audience we will talk about these issues is talented students in secondary and high schools.
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Introduction

Geometry is a science founded on a human desire to perceive the world, recognizing and presenting his notion. The name of this science is derived from two Greek words: geo- "earth," and metry- "measurement." Geometry, which dates back to the 25th century, has its foundation in Euclid's book 'Elements.' As is generally known, at the beginning of the nineteenth century, the axiomatic known as 'non-Euclidean' and researched by N.I. Lobachevsky revitalized, improved, and created a new science. In the 21st century and in other disciplines, it was necessary to give a new explanation of the geometry subject to meet modern requirements [5]. In particular, the article [3] is called "modern view of ancient science" and explains the description of geometry which was given by [13]. Therefore, today, geometry improvement occurred through 3-dimensional manifold geometry classifications formed by Thurston. We will describe the geometry that W. Thurston has improved. Then we will introduce you to some of the suitable geometries to this description in 2-dimensional planes [2, 4, 13, 14].

Let x be any set A and B are the elements of this set. We can define the elements of the $x -$ sum as points. We may add the 'the distance between two points' concept to the $x -$ sum. So, the function $S(A, B)$ is called the distance between the two points A and B. It also satisfies the condition,

$$S(A, B) = S(B, A)$$

(x, S) is the symbol that can be obtained by increasing the distance $S(A, B)$ to the x sum. Let us the sums (x, S) and (y, S^*) be given. The function f corresponds with any value between the sum (x, S) and (y, S^*). Therefore, $f: (x, S) \to (y, S^*)$.

Definition 1

If $S(A, B) = S^*(A', B')$ for any $A, B \in x$ and $A', B' \in y$ such that $A' = f(A)$ and, $B' = f(B)$ then f is called isometric transformation and denoted by $y = isomx$. Let us consider the isometric sum of $x -$ sum and all $y = isomx$ that is isometric to $x -$ sum.

Definition 2

The science that teaches the properties of the components of the sum $(x, isomx)$ is called geometry.

So that there is a geometry of "x" in total, these total elements must be used to define the concept of "distance" and to reflect the mirror that holds these distances. To explain these concepts to the students, they must use the period when familiar with the Cartesian coordinate system and the distance between two points. Therefore, it is necessary to tell students the following points. We will explain that geometry thought in schools parallels the definition given above.

Let Oxy be the Cartesian coordinate system which is given on the plane R_2. In this plane (Euclidean), the distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is calculated as follows:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \textbf{(1)}$$

In the formula, R_2, points are shown by x and any other plane points are shown by y. If $Ox'y'$ Descartes's coordinate system is given on y plane, and if distance is calculated by the formula (1),

$$f: \begin{cases} x' = xcos\alpha - ysin\alpha + a \\ y' = xsin\alpha + ycos\alpha + b \end{cases} \qquad \textbf{(2)}$$

The f transformation shows that these planes are isometric. The transformation formula (2), as is widely known, generates motion in the plane. This motion involves shifting its coordinates parallel to the $C(a, b)$ vector and rotating the coordinates' arrows as the measure of the angle α.

As a result, planimetry has its place in this formulation, as does Euclidean geometry. Is there any other geometry except Euclidean geometry parallel to the above definition? This question will be addressed further below.

Minkowskian Geometry

H. Minkowski proposed Minkowskian geometry in the first decade of the twentieth century, based partly on Einstein's work on special relativity. In many aspects, Minkowskian geometry is similar to Euclidean geometry. If only Euclidean geometry corresponds to the new interpretation of geometry, then what is the significance of the new explanation? The significance of that is that the students should know that there are geometries other than Euclidean geometry. The axiomatic construction of non-Euclidean geometries is common [8, 14]. But using deep axiomatic science in learning is hard to give a good result. However, it is easy for students to understand the geometrical facts used by the cardinal system and the analytical expressions in this system of cardinals. We recommend the Minkowskian geometry [4], which is easiest on the plane and for students to understand.

Students need to be familiar with the Descartes coordinate system, the distance between two points, the concept of an abstract number, and hyperbolic trigonometry in order to grasp Minkowskian geometry. The plane in which Descartes's coordinate system is placed reminds us of an affine plane. We will use it in the following sections as well.

Let's take the points in A_2 at $x -$ sum. The distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is calculated as a generalization of the Euclidean distance.

$$d = \sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2} \qquad \textbf{(3)}$$

Then the distance is preserved by the given transformation below.

$$f: \begin{cases} x' = xch\alpha + ysh\alpha + a \\ y' = xsh\alpha + ych\alpha + b \end{cases} \qquad \textbf{(4)}$$

In this situation, the plane has a new geometry corresponding to Einstein's relativity theory. By the way, the distance determined by the formula (3) may not be considered positive. This distance can be abstract and zero for places that do not coincide.

We'd like to introduce you to the concept of distance, which is divided into components that differ from the previously discussed concept. Showing how to keep track of distance (3) and change (4) is an excellent task for talented students. It is possible for students to have an idea that the distance given by the formula can take an abstract value and that the distance can be reduced 0. This geometry can be recommended for senior students.

Giving students the opportunity to see how this geometry is related to the theory of relativity and to give them the idea of replacing Lorentz's transformation will be a great exercise for talented students. The requirements for understanding Minkowskian geometry are the primary task of high school students. Although it appears to be less well known, tools for Minkowskian geometry are far easier to develop than tools for entire hyperbolic geometry. Because Minkowskian geometry is much more closely related to Euclidean geometry, it is, in fact, much easier to introduce.

Galilean Geometry

Galilean geometry is one of the simplest non-Euclidean geometries on the plane. It is sufficient for students to understand the fundamental ideas of Euclidean geometry and the properties of the coordinate system to visualize Galilean geometry. In Galilean geometry, the concepts of point, straight line, and parallelism are the same as in Euclidean geometry. The only difference between these geometries is how the distance between two points is defined.

The distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ in G^2 will be calculated as follows.

Firstly, we check $d_1 = |x_2 - x_1|$, if $d_1 = 0$ then check $d_2 = |y_2 - y_1|$. If $d_1 = 0$ and $d_2 = 0$ both are zero at the same time, then $A = B$. Therefore, it may be concluded that these points coincide.

So, if we can teach new conceptions of "distance" and "movement" without affecting the essential concepts introduced in school geometry, we will have a unique geometry.

Theorem 1

$$\begin{cases} x' = x + a \\ y' = hx + y + b \end{cases} \qquad \textbf{(5)}$$

The plane preserves the distance between the points A and B under one unit-valued transformation where h denotes the measure of the angle.

Proof: Let the points $A(x_1, y_1)$ and $B(x_2, y_2)$ be transformed to points $A'(x_1', y_1')$ and $B'(x_2', y_2')$ respectively by transformation shown as (5). It is necessary to show that $|AB| = |A'B'|$ is true.

In fact,

$|A'B'| = |x_2' - x_1'| = |x_2 + a - (x_1 + a)| = |x_2 - x_1| = |AB|$

if $x_2' - x_1' = 0$ then $|A'B'| = |y_2' - y_1'| = |hx_2 + y_2 + b - (hx_1 + y_1 + b)| = |h(x_2 - x_1) + y_2 - y_1| = |h(x_2 - x_1) + y_2 - y_1| = |y_2 - y_1| = |AB|$

As a result, the transformation given by (5) preserves the distance in the new appearance, i.e., the existing isometric transformation. Galilean geometry refers to geometry in the affine plane. At this moment, we informed you about the affine plane's Euclidean, Minkowskian, and Galilean geometries [8].

Limited (Finite) Geometry

Finite geometry is any geometric system with only a finite number of points. The familiar Euclidean geometry is not finite because a Euclidean line contains infinitely many points. The modern definition of geometry given by [13] extends the imagination of geometry. According to the description, the concept of geometry in which the x − total can be any combination of finite or infinite elements. When we say geometry on the plane, we consider the sum of the points on the entire plane as x and this is exactly a sum of unlimited points. But for the new geometry, the sum of x can be composed of limited points. The geometry of limited points is described in the article [10].

So far, we've only studied plane geometry. As seen in figure 1, we may see the same notions in limited point (Finite) geometry without the necessity for a plane.

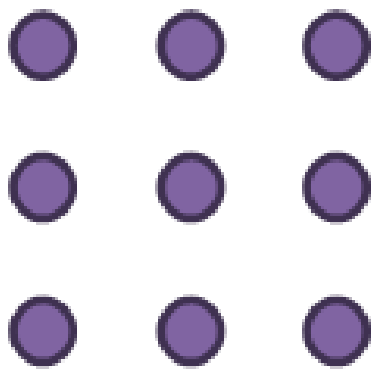


Figure 1

In this geometry, distance is defined as the shortest path which is taken by the number of points, and it allows (→,←) horizontal, (↓,↑) vertical and (,↘↖)diagonal motion, as one unit.

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

e.g., $d(a_{11}, a_{23}) = 2$ (Passing through $a_{11} \to a_{22} \to a_{23}$) and $d(a_{21}, a_{33}) = 2$. Totally, we have $s = \frac{1}{2} C_9^2 = 18$ different distances. We may explain the distance on different examples as follows:

Figure 2 shows a circle sketched out by a moving point so that its distance from a given point is constant. So a_{22} is at center and the other points are equidistant to a_{22}.

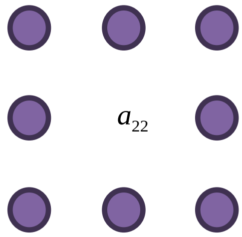
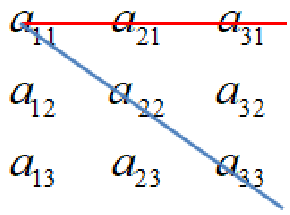
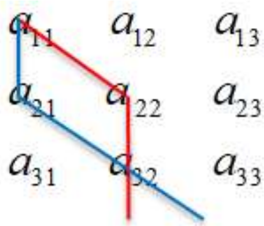


Figure 2

What about a square? The answer is obvious, and we get the same figure 2 as before. What about a line in this geometry? As previously stated, a line is defined as the shortest distance between two points.



The red line passes through a_{11} and a_{21}, the blue one passes through a_{11} and a_{22}.



And these lines pass through a_{11} and a_{32}. In this geometry, the Euclidian axiom is inconsistent. Because it says, "just a line passes through two points"! Is it really so? This axiom may be invalid in non-Euclidean geometry. Anyway, let us attempt to depict a triangle in this geometry, as seen in fig. 3.

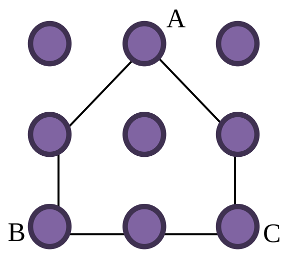


Figure 3

We may depict other triangles in this manner, and triangle inequality is easily apparent in figure 4, i.e., $BC \leq AB + AC$.

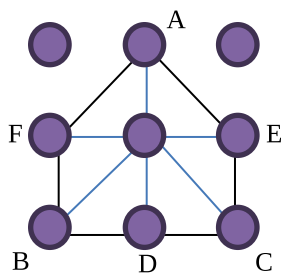


Figure 4

The AD median (in the ABC equilateral triangle), the BE median (a straight line), and the CF median (a straight line). We've introduced the distance, and now we need the kinematics that preserves it, which can be shown in [7, 14]. Hence, there is an isometry.

Diagonally			**Horizontally**			**Vertically**		
a_{11}	a_{21}	a_{31}	a_{31}	a_{32}	a_{33}	a_{13}	a_{12}	a_{11}
a_{12}	a_{22}	a_{32}	a_{21}	a_{22}	a_{23}	a_{23}	a_{22}	a_{21}
a_{13}	a_{23}	a_{33}	a_{11}	a_{12}	a_{13}	a_{33}	a_{32}	a_{31}

We may do the same operations by using 16 points, and for this one we have $s = \frac{1}{2}C_{16}^{2} = 60$ different distances.
Finally, we can utilize a chess board with 64 points, as illustrated in figure 5, and some of the kinematics on the chess board will be as follows:

1) King : → horizontal, ↓ vertical, and diagonal.

2) Rook : → horizontal and ↓vertical.

… and so on.

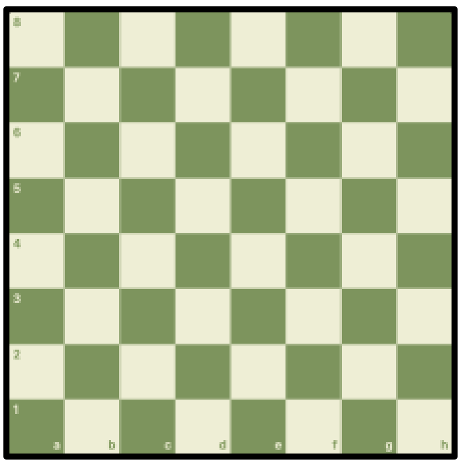


Figure 5

Triangle, square, and circle can be shown using the same definitions.
Attention! The shapes that exist can't be the ones we have previously seen! An intriguing point here is that we cannot discuss the area. If we were to ask why? We can utilize this fundamental question as an interactive educational activity with students.

Taxicab Geometry
Let's get acquainted with Taxicab geometry now. Taxicab geometry, on the other hand, is one of the geometries that can only be explained by the coordinate system on the plane and is recommended as an engaging study for secondary and high school students. This geometry is discovered by [12]. The core notions of this geometry will be demonstrated.

Taxicab geometry (T) is a form of geometry in which the distance between two points is calculated as the sum of their absolute coordinate differences rather than the length of the line segment, as in Euclidean geometry.
In other words, the sum of the distances d_1 and d_2 in Galilean plane. So, taxicab distance formula between two points is $d_T = |x_2 - x_1| + |y_2 - y_1|$. By the way, we can demonstrate that the taxicab distance formula meets metric criteria [4].

T-Line Segment. In Euclidean geometry, one line segment equals one length. However, in Taxicab geometry, several line segments equal one size. Figure 6 illustrates this. Both lines are nine units long, and this new sort of line is known as a *taxicab-line* or *t-line* [6].

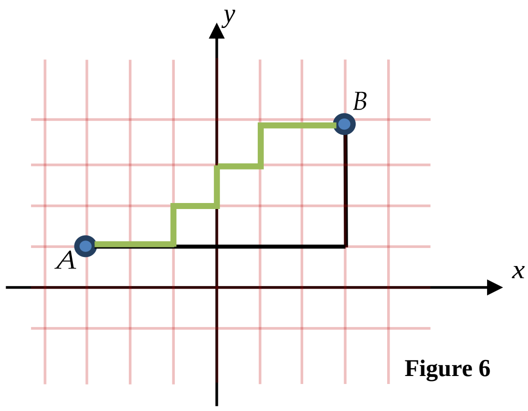


Figure 6

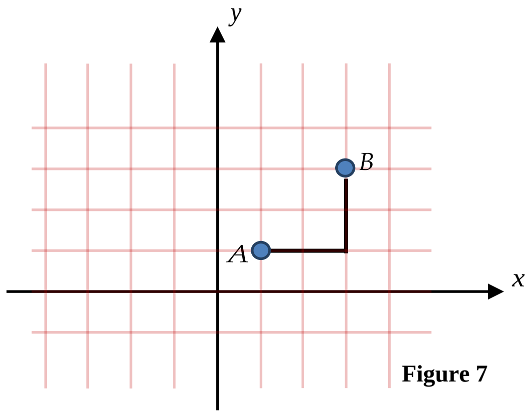
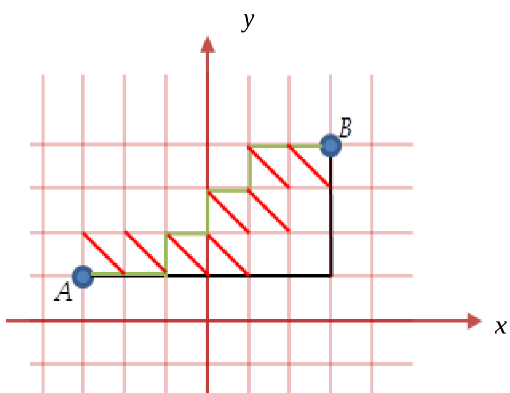


Figure 7

Example: In figure 7, how many *t-line* segments connect points A and B?

We can count them and get six *t-line* segments as a result. If we systematically examine the number of T-lines, we may find a rule. The number of *t-line* segments on a crossing is always the total of the parts at the preceding intersections. This leads to Pascal's triangle, whose rows are diagonal. Point B lies on the ninth diagonal and so on the ninth row of Pascal's triangle, connected to A, and in the fourth place [9].
There are 84 t-lines for instance, 6 to the right & 3 to the top to move from A to B in figure 8.



In general, the number is "$n+m$" choosing "m" for the case to the right, "n" to the top [11]. Taxicab geometry provides a more realistic perspective on geometry. A straight line is the shortest distance between two points in Euclidean geometry. This process is, in theory, faultless.
However, in practice, it works differently than expected. This is where taxicab geometry comes into play in math. The shortest distance between two points in taxicab geometry is not always a straight line.
Taxicab geometry is a valuable urban geography model. Taxicab distance is the "real" distance for people. Thus, taxicab geometry has many applications and is quite simple to examine. Taxicab allows kids to conduct arithmetic in a way that offers a more realistic idea of what they are trying to conclude. Therefore, taxicab geometry gives pupils more possibilities than Euclidean geometry to make the real world meaningful [1].

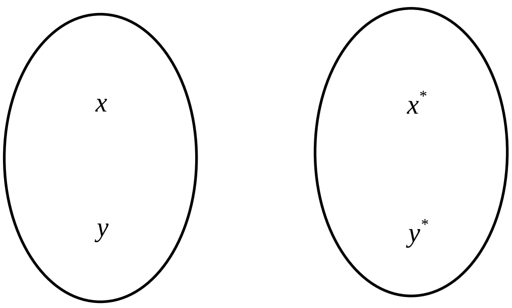
Results and Discussion

There is no doubt that non-Euclidean geometry has a special place, as it can be understood from the fact that the information obtained and existing so far has been handled and expressed differently from time to time. We tried to explain some, if not all, of them at the high school level and in a way that talented students could understand.
As a result of our work in which we have made progress in the form of club activities, the question of whether teaching Euclidean at the high school level is enough! Anymore cannot teach non-Euclidean geometries to students will always seem to occupy our minds.
Even if non-Euclidean geometry is not included in the school curriculum, teachers must be familiar with it to teach Euclidean geometry effectively.
However, not all colleges and institutions include non-Euclidean geometry in their mathematics programs for future teachers.
As a result, in-service courses must be devised to allow instructors to study topics in mathematics, such as non-Euclidean geometry, which are on the perimeter of the existing school curriculum but fundamentally linked to it. This will allow teachers to teach more confidently if they want to depend on their knowledge and expertise rather than coercive approaches to ensure that students learn well [7].

Conclusion

My paper intends to raise awareness of distance, kinematics, and isometry themes among teachers and high school students and provide them with a new perspective. In this regard, the study's findings could be useful for learning and teaching some geometric concepts. We can conclude our investigation by using the contemporary definition of geometry provided by [3] as taking $d(x,y)$ distance such that

$d(x,y)=d(f(x),f(y))$ and $d(x,y)=d^*(x^*,y^*)$



and if this condition is satisfied $d(x,y)=d^*(x^*,y^*)=d(f(x),f(y))$ then $y=isomx$ called isometry. Geometry is a branch of science that teaches the characteristics of the components of the sum $(x, isomx)$.
Students in high school can identify geometric concepts and figures based on their overall appearance. They recognize circles, triangles, squares, and so on because of their shapes; now, they will expressly identify the impression from a different viewpoint.
They also begin to analyze the properties of geometric concepts and figures and learn the terminology for describing them. Following this article, they may look for an interrelated technical language to describe figures or properties of figures.

As a result, the following points are described in this article.

- Basic concepts of non-Euclidean geometry, which are determined by using the affine coordinate system in the plane, are given.
- Using the modern definition of geometry, the basic ideas of non-Euclidean geometries in the plane are shown.
- Apparently, conveying non-Euclidean geometric ideas in this style is more accessible than the axiomatic method.
- The concepts of limited (finite) points and taxicab geometry were also simple and easy for secondary and high school students to learn.
- In addition, Minkowskian and Galilean geometries can be more interesting for high school-level students.

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