Extension of geometric series to hypergeometric function in Hindu mathematics

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Abstract

The eastern society is rich in terms of science and technology. Mathematics is considered as the base of science. The eastern history shows that the Hindu society is rich in mathematics. The evolution of mathematics can be studied from the time of ‘patiganita’ to the latest form of science and technology. Geometric series is an important tool in arithmetic which is now developed to hypergeometric function. Hypergeometric function is an advance function used to solve differential equations of second order. The purpose of this paper is to find the linkage between the ancient geometric series (Gunanka sreni) to the modern Hypergeometric function and to expose the work of ancient Hindu mathematicians which is believed to be narrow among the mathematics researchers of the present period. In this paper, some forms of geometric series that were used in Hindu mathematics are interpreted in terms of hypergeometric series.

Keywords: Hindu Mathematics, Bijaganita, geometric series, hypergeometric function

1. Introduction

1.1 History of Hindu mathematics

The Hindu society has a great history in mathematics. The history of mathematics in this region is as old as human civilization. It reflects the noblest thoughts of countless generations [23]. The mathematics did not have the independent existence at the time of its origination in this region. Mathematics was developed as a Vedanga, the wing of the Vedas, and an art to support the religious rituals in the Sindhu valley. Veda is the basis for development of arts and science and consequently the foundation of modern civilization. Thus India can be considered as the birthplace of modern Algebra and Arithmetic and he branches of mathematics; Algebra, Arithmetic and Geometry are believed to be started from Veda itself.

The advancement of technology in the world can be considered from the time of Ramayan about 1,000 BC [14]. Later Mahavira and Buddha taught the unique system of religious and moral philosophy of life. Mathematics was a key aspect in religious affairs and calculations for wealth, love, music and medicine. It was considered significantly important in the study of the moving heavenly bodies relating to ellipse and the planets. Mathematics was used to count the numbers, measure the diameter, position and perimeter of the islands oceans and Planets.

Ganita is the Hindu name for the mathematics and it means the science of calculations [18]. Ganita is found abundantly in vedic literature. The Vedanga Jyotish (1200 BC) gives the highest values to mathematics among all science which forms the Vedanga. Hindu manuscript describes the importance of Ganita as “As the crest on the heard of peacocks, as the germs in the hood of the snakes, so is ganita on the top of the sciences.”

The Ganita consists of Parikrama (Fundamental operations), Vyabahara (Determinations), Rajjyu, (Geometry), Rasi (Rule of three), Kalasavarna (Operation with fractions), Yarat tavat (Simple equations), Varga (Square or Quadratic equations), Ghana (Cube or Cubic equation), Varga-varga (Biquadratic equation), and Vikalpa (Permutation and Combination) in the early renaissance period [1]. The work dealt with algebra was named as Bijaganita and the chapter of Bijaganita was called Kuttaka (628 A. D.) in his work Brahma-Sput Siddhant, Sridharacarya.
1.1 Word numerals without Place Value
Arya Bhatta I (499 A. D.) developed the idea of place value in decimal system. After the invention of zero and the place value system the same numerical symbols from 1 to 9 continued to be employed with the zero to denote the numbers. The invention of number system may be assigned to the period of 1000 BC to 6000 BC.

1.1.2 The Decimal Value System
The decimal value system is an important feature of Hindu mathematics. Its existence can be found in ancient documents written by Lekhaka and Lipikara then later Divira, Karana and Kayastha. In this system there are 10 symbols called the anka for all numbers 1-9 and the zero is called Sunya.

1.1.3 Zero or the Sunya
Zero was invented in Indian region in the beginning of the Christian era to write the numbers in decimal scale. According to Chandah-Sutra, the zero symbol was used in matrices by Pingala before 200 BC. It is used in Panca Siddhanta (505 A. D.). Jinabhadrach Ghani, used zero as a distinct numerical symbol. The zero was also included as the other numbers in Bakhshali Manuscript before the third century.

1.2 Operations in Mathematics
Hindu mathematics consists mostly Arithmetic. The work of mathematics in ancient period is called 'patigapita'. The word patigapita is formed from the words 'patti', means "board," and 'Ganita', means the science of calculation. Hence Ganita means the science of calculation that requires the use of a writing material (the board) [17]. According to Prthudakasvami there are twenty operations, in arithmetic. They are addition, subtraction, multiplication, division, square, square-root, etc. Addition and subtraction are considered as the very basic operations. In Sara Sangrha by Mahaviracharya [24], the names of the operations are mentioned as follows;

Meaning
“After placing (the multiplicand and the multiplier one below the other) in the manner of the hinge of a door, the multiplicand should be multiplied by the multiplier, in accordance with (either of) the two methods of normal (or) reverse working, by adopting the process of (i) dividing the multiplicand and multiplying the multiplier by a factor of the multiplicand, (ii) of dividing the multiplier and multiplying the multiplicand by a factor of the multiplier, or (iii) of them, (in the multiplication) as they are (in themselves).”

Further the book consists of the series of examples of multiplication. E. g. in verse 8, the quote is as follows;

The meaning of the verse is as follows

1.2.1 Arithmetic and Geometric series
Indian mathematicians studied arithmetic and geometric series in early period. In Hindu mathematics, arithmetic series is called a "prakriti samkhya" or a "dharm samkhy.". The term "prakriti" refers to the starting point or the first term of the series, while "dharma" refers to the common difference between consecutive terms in the series. Arithmetic and geometric progressions are found in the Vedic literature of Indians: Taittiriya-Samhitā, Vājasaneyī Saṁhitā and Kalpa-sūtra of Bhadrabahu etc (2000 B.C.).

Arithmetic series
Arithmetic series is kept in seventh number among parikarman (operation) under the topic summation in SaraSangrha. Aryabhata - I (499 A.D.), Brahmagupta (628 A. D.) and most of Indian mathematicians have stated various formulae to evaluate sum of n terms of series, arithmetic mean of a finite sequence, number of terms in series and common difference in Arithmetic progression. In Aryabhatiya
The method to evaluate the sum of the arithmetic series is explained as follows:

\[ S_n = \frac{n}{2} [2a + (n - 1)d] \quad \text{...(1)} \]

Geometric series is series of numbers where each term is obtained by multiplying the previous term by a constant factor. In Hindu literature, Geometric series is called the śr̥ddhi (progression) for the series and sreni for the line row of the numbers. Nārāyaṇa Pāṇḍita (1340 to 1400 A.D.) stated the general formula for evaluating the sum of a figurate numbers that can be represented in the form of a triangular grid of points where the first row contains a single element and each subsequent row contains one more element than the previous one. He sum of two arithmetic progressions is called the Vārasaṅkalita. He generalized the formula for Vārasaṅkalita. The first Vārasaṅkalita will give sum of first order triangular numbers; the second Vārasaṅkalita will give sum of second order triangular numbers and similarly we can obtain the result for the sum of kth order triangular numbers or kth Vārasaṅkalita of an arithmetic progression. According to him, the general formula for the sum of arithmetic series with first term a, common difference d and number of terms n is:

\[ S = \frac{n}{2} [2a + (n - 1)d] \]

2. Literature Review
2.1 Geometric Series in Ancient Hindu Texts
Dr. K. Ramasubramanian defined mathematics as “ganite sankayete tattaganitam: tatpraditi padadyatena tatsamanam sharstam ukhayate: i.e. the science that deals with calculation are defined by the word ganita. To study about the development of mathematics in the Eastern society, one has to go back to 5000 BC where the ground was fertile to mathematics. Since the original work were in Sanskrit, due to the linguistic inconvenience, many of the mathematical theories and formulas were unknown to the rest of the world. The vedic period (2000-500 BC) can be considered as the most remarkable since the first significant chapters of Sulba sutra (mathematical formulae) was written in this time. This was done by Baudhāyaṇa on 800 BC. During (200 BC-200AD) Bakhshali manuscript was discovered which contained the work of Patiganita together with other Arithmetical and Geometric progressions and other mathematical operations.

The Bakhshali manuscript shows that patiganita was composed nearly about 300 A.D. together with Aryabhatiya (499A.D.) the Bṛhama Sphuta Siddhānta (628 A.D) the Trisati (750 a. d.) the Gāṇit sarasangraha (950 A. D.) The Lilavati (1150 A. D.) The Ganita Kaumudi (1350 A. D.) The Ganita Kaumudi (1350 A. D.) [27]. Series was treated as one of the fundamental operations (Sreni byavahara) and a separate section is generally devoted to the formula and problem relating to series. For the first time Mahāvīra (815 to 878 A.D.) gave the formula to evaluate the sum \( S_n \) of n terms of the geometric progression.

\[ S_n = a + ar + ar^2 + ar^3 + ar^4 + ... \]

\[ S_n = \frac{a(1-r^n)}{1-r} \quad \text{...(3)} \]

He knew that the infinite geometric series converges if its common ratio is less than one. Mahāvīra Jain gave the formula for sum first n terms of geometric progression. The geometrical representation of convergence of summation of geometric series is beautifully explained in Nilakantha’s Aryabhatiya bhāṣya and Jyeṣṭhadeva’s Yuktibhāṣa [9]. The formula to evaluate the sum of the terms of the geometric series (gunadhana) are mentioned in the form of verse (Sloke)
in Ganit Sarasanghraha [27]. These formulas are accumulated in the verses 93 to 95. The verses listed in 93 is as follows;

![Equation](image)

**Meaning**
The first term (of a series in geometrical progression), when multiplied by that self-multiplied product of the common ratio in which (product the frequency of the occurrence of the common ratio is) measured by the number of terms (in the series), gives rise to the gunadhana. And it has to be understood that this gunaghana, when diminished by the first term, and (then) divided by the common ratio lessened by one, becomes the sum of the series in geometrical progression.

Similarly in verse 94, there is also another rule for finding the sum of a series in geometrical progression. It is expressed as follows;

![Equation](image)

**Meaning**
“The number of terms in the series is caused to be marked (in a separate column) by zero and by one (respectively) corresponding to the even (value) which is halved and to the uneven (value from which one is subtracted till by continuing these processes zero is ultimately reached); then this (representative series made up of zero and one is used in order from the last one therein, so that this one multiplied by the common ratio is again) multiplied by the common ratio (wherever one happens to be the denoting item), and multiplied so as to obtain the square (wherever zero happens to be the denoting item). When (the result of) this (operation) is diminished by one, and (is then) multiplied by the first term, and (is then) divided by the common ratio lessened by one, it becomes the sum (of the series).”

Likewise in the verse 95, the rule for finding the last term and the sum of the terms in a geometric progression is as follows;

![Equation](image)

**Meaning**
“The antyadham or the last term of a series in geometrical progression is the gunadhana (of another aeries) wherein the number of terms is less by one. This (antyadhana) when multiplied by the common ratio, and (then diminished by the first term, and (then) divided by the common ratio lessened by one give rise to the sum (of the series).”

There are numerous practical examples on geometric series. A typical example mentioned in the verse 96 is as follows;

![Equation](image)

**Meaning**
“Having (first) obtained 2 golden coins (in some city), a man goes on from city to city, earning (every where) three times (of what he earned immediately before) Say how much he will make in the eighth city”. Similarly in verse 99 we can find the similar but another example.

![Equation](image)

**Meaning**
“A certain man (in going from city to city) earned money (in a geometrically progressive series) having 5 dinaras for the first term (there of) and 2 for the common ratio. He (thus) entered 8 cities, how many are the dinaras in his (possession)”

In the Taittiriya samhita we can find the geometric series

10 + 20 + 40 + -------- 8000 .... (4)

Similarly in Kalpa sutra and sulba sutra of Bhadrabahu (350 B.C), the series (5) is included

1 + 2 + 4+ 8 +---------- ... (5)

Likewise, in chapter 5, Verse-CXXXVI of Lilavati, Bhaskaracharya [20] uses geometric series in the form of the verse as follows;

![Table](image)

**Table 1:** This rule is illustrated through an example with first term (a) =1, common ratio (r) = 2 and for 31 terms as follows;

<table>
<thead>
<tr>
<th>n</th>
<th>S</th>
<th>M</th>
<th>S</th>
<th>M</th>
<th>S</th>
<th>M</th>
<th>S</th>
<th>M</th>
<th>S</th>
<th>M</th>
<th>S</th>
<th>M</th>
</tr>
</thead>
</table>
| 31 | 30 | 15 | 14 | 7 | 6 | 3 | 2 | 1 | 0

"498"
Beginning with 31 we write 31-1=30, \( \frac{30}{2} = 15 \), 15-1=14, \( \frac{14}{2} = 7 \), until we reach to zero. Then in the second line, we begin with S and alternately write M and S. Now we perform the operation M for multiplication and S for square as shown below;

<table>
<thead>
<tr>
<th>Number</th>
<th>S/M</th>
<th>Values</th>
<th>Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>M</td>
<td>2147483648</td>
<td>31</td>
</tr>
<tr>
<td>15</td>
<td>S</td>
<td>1073741824</td>
<td>30</td>
</tr>
<tr>
<td>14</td>
<td>M</td>
<td>32768</td>
<td>15</td>
</tr>
<tr>
<td>7</td>
<td>S</td>
<td>16384</td>
<td>14</td>
</tr>
<tr>
<td>6</td>
<td>M</td>
<td>128</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>S</td>
<td>64</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>S/M</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>S/M</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>S/M</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

To reach to the sum up to the 31st power, Bhaskaracharya has given a shorter method by squaring at even numbers and multiplying by \( r \) at odd ones. In practice multiplication is easier than squaring. Therefore the sum is obtained to be 2147483648.

There are numerous examples in Lilavati, in geometric progression. Another example solved by the same method mentioned in verse CXXXVII of Lilavati is as follows;

\[ \frac{3x^n - 1}{x - 1} = 4095 \]

In India, geometric series were used extensively in the context of infinite series. One of the most well-known examples of a geometric series in in this context is the series for the sum of the infinite sequence of numbers

\[ 1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots, \]

where each term is obtained by dividing the previous term by 4. This series is given by:

\[ 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \frac{4}{3} \]

This result is obtained by the Indian mathematician Madhava in the 14th century. Geometric series were used in India to study infinite products, closely related to infinite series. One example is the formula for the product of the infinite sequence of numbers given below. The sum of this series is obtained by Bhaskara to be \( \frac{3}{4} \).

\[ \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{9}\right) \left(1 - \frac{1}{27}\right) \ldots \]

2.2 Geometric ratio in Scale notation

The study of Yajurveda Samhita (Vajasareyi) \([2]\) shows that ten has been the basis of numeration in Hindu mathematics. In Ganit sarasangharha, the denominations of the numbers are in the form of verses(sloke) These verses numbered from 63 to 68 are as follows:
express any fractions as the sum of numbers of unit fractions. Mahavira (550 BC) has given the number of examples to place to place, each ten times the preceding.

Twenty fourth \(\frac{1}{24}\)

Twenty third \(\frac{1}{23}\)

Twenty first \(\frac{1}{21}\)

Seventeenth \(\frac{1}{17}\)

Nineteenth \(\frac{1}{19}\)

Eighteenth \(\frac{1}{18}\)

Fourteenth \(\frac{1}{14}\)

Tenth \(\frac{1}{10}\)

Hundred \(\frac{1}{100}\)

One thousand \(\frac{1}{1000}\)

Ten thousand \(\frac{1}{10000}\)

Hundred thousand \(\frac{1}{100000}\)

Million \(\frac{1}{1000000}\)

Hundred million \(\frac{1}{10000000}\)

Billion \(\frac{1}{100000000}\)

Trillion \(\frac{1}{1000000000}\)

Quadrillion \(\frac{1}{10000000000}\)

Quintillion \(\frac{1}{100000000000}\)

Sextillion \(\frac{1}{1000000000000}\)

Septillion \(\frac{1}{10000000000000}\)

Octillion \(\frac{1}{100000000000000}\)

Nonillion \(\frac{1}{1000000000000000}\)

Tricillion \(\frac{1}{10000000000000000}\)

The first place is known as eka (unit); the second they call as dasa (hundred); the third is sata (thousand). The fifth is dasa (ten); the fourth is sahasra (thousand). The sixth is sahasra (ten-thousand) and the seventh is laksha (lakh). The seventh is dasa-laksa (ten-lakhs) and the eighth is said to be koti (crore). The ninth dasa-koti (ten-crore) and the tenth is sata-koti (hundred-crore). The eleventh place is arbuda and the twelfth is place nyarbuda. The thirteenth place is kharva and the fourteenth is maha-kharva. Similarly the fifteenth is padma and the sixteenth maha-padma. Again the seventeenth is ksona the eighteenth maha-ksona. The nineteenth place is sankha and the twentieth is maha-sankha. The twenty first place is ksona the twenty-second maha-ksobha. Then the twenty-third is ksona and the twenty-fourth maha-ksobha. The first of the first ten notational places in the decimal system forms a geometric progression with the common ratio of ten. The summary of the notational value is shown in the table given below.

Table 3: Denomination of numbers

<table>
<thead>
<tr>
<th>Position of place value</th>
<th>Numerical values</th>
<th>Values</th>
<th>Hindu Terminology</th>
</tr>
</thead>
<tbody>
<tr>
<td>first</td>
<td>1</td>
<td>One</td>
<td>eka</td>
</tr>
<tr>
<td>Second</td>
<td>10</td>
<td>Ten</td>
<td>dasa</td>
</tr>
<tr>
<td>Third</td>
<td>100</td>
<td>Hundred</td>
<td>sata</td>
</tr>
<tr>
<td>Fourth</td>
<td>1,000</td>
<td>Thousand</td>
<td>sahasra</td>
</tr>
<tr>
<td>Fifth</td>
<td>10,000</td>
<td>Ten thousand</td>
<td>Dasa-sahasra</td>
</tr>
<tr>
<td>Sixth</td>
<td>100,000</td>
<td>Hundred thousand</td>
<td>Laks–laksh</td>
</tr>
<tr>
<td>Seventh</td>
<td>1,000,000</td>
<td>Million</td>
<td>Dasa-laksa</td>
</tr>
<tr>
<td>Eighth</td>
<td>10,000,000</td>
<td>Ten million</td>
<td>Koti Cre</td>
</tr>
<tr>
<td>Ninth</td>
<td>100,000,000</td>
<td>Hundred million</td>
<td>Dasa Koti</td>
</tr>
<tr>
<td>Tenth</td>
<td>1,000,000,000</td>
<td>Billion</td>
<td>Sata Koti</td>
</tr>
<tr>
<td>Eleventh</td>
<td>10,000,000,000</td>
<td>Ten Billion</td>
<td>Arbuda</td>
</tr>
<tr>
<td>Twelfth</td>
<td>100,000,000,000</td>
<td>Hundred Billion</td>
<td>Nyarbuda</td>
</tr>
<tr>
<td>Thirteenth</td>
<td>1,000,000,000,000</td>
<td>Trillion</td>
<td>Kharva</td>
</tr>
<tr>
<td>Fourteenth</td>
<td>10,000,000,000,000</td>
<td>Ten Trillion</td>
<td>Maha kharva</td>
</tr>
<tr>
<td>Fifteenth</td>
<td>100,000,000,000,000</td>
<td>Hundred Trillion</td>
<td>padma</td>
</tr>
<tr>
<td>Sixteenth</td>
<td>1,000,000,000,000,000</td>
<td>Quadrillion</td>
<td>Maha padma</td>
</tr>
<tr>
<td>Seventeenth</td>
<td>10,000,000,000,000,000</td>
<td>Ten Quadrillion</td>
<td>ksona</td>
</tr>
<tr>
<td>Eighteenth</td>
<td>100,000,000,000,000,000</td>
<td>Hundred Quadrillion</td>
<td>Maha-ksona</td>
</tr>
<tr>
<td>Nineteenth</td>
<td>1,000,000,000,000,000,000</td>
<td>Quintillion</td>
<td>sankha</td>
</tr>
<tr>
<td>Twentieth</td>
<td>10,000,000,000,000,000,000</td>
<td>Ten Quintillion</td>
<td>Maha sankha</td>
</tr>
<tr>
<td>Twenty first</td>
<td>100,000,000,000,000,000,000</td>
<td>Hundred Quintillion</td>
<td>ksonya</td>
</tr>
<tr>
<td>Twenty second</td>
<td>1,000,000,000,000,000,000,000</td>
<td>Sextillion</td>
<td>Maha ksonya</td>
</tr>
<tr>
<td>Twenty third</td>
<td>10,000,000,000,000,000,000,000</td>
<td>Ten Sextillion</td>
<td>ksobha</td>
</tr>
<tr>
<td>Twenty fourth</td>
<td>100,000,000,000,000,000,000,000,000</td>
<td>Hundred Sextillion</td>
<td>Maha-ksobha</td>
</tr>
</tbody>
</table>

Similarly in Aryabhatita the first ten notational places in terms of decimal places is explained as follow;

Meaning of the verses 63 to 68

“The first place is what is known as eka (unit); the second place is named dasa (ten); the third they call as sata (hundred), while the fourth is sahasra (thousand). The fifth is dasa-sahastra (ten-thousand) and the sixth is no other than laksha (lakh). The seventh is dasa-laksha (ten-lakhs) and the eighth is said to be koti (crore). The ninth dasa-koti (ten-crore) and the tenth is sata-koti (hundred-crore). The eleventh place is arbuda and the twelfth is place nyarbuda. The thirteenth place is kharva and the fourteenth is maha-kharva. Similarly the fifteenth is padma and the sixteenth maha-padma. Again the seventeenth is ksona the eighteenth maha-ksona. The nineteenth place is sankha and the twentieth is maha-sankha. The twenty first place is ksona the twenty-second maha-ksobha. Then the twenty-third is ksona and the twenty-fourth maha-ksobha. The first of the first ten notational places in the decimal system forms a geometric progression with the common ratio of ten. The summary of the notational value is shown in the table given below.

These examples shows the existence and development of the geometric series.

1) Expressing 1 as the sum of numbers (n) of unit fractions

\[
1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots = \frac{1}{2} + \frac{1}{2^2} \sum_{n=2}^{\infty} \frac{1}{3^n} = \frac{2}{3} 
\]

The rule is: When the number of different quantities having one for their numerator is 1, the required denominator such as beginning with 1 are in the order multiplied by 3, the first and the last being multiplied again by 2 and

\[
\frac{2}{3} \quad \text{and} \quad \frac{2}{3} 
\]

Ignoring the first and the last term of the series all the rest terms of the middle are expressed in the geometric series with the common ratio.
Apart from the geometric series, there are advance works in infinite series. Rajagopal et al. [23] describes Govindaavamin’s second order difference interpolation formula of the form
\[ f(x + nh) = f(x) + n\Delta f(x) + \frac{n(n - 1)}{2} \left[ \Delta^2 f(x) - \Delta f(x - h) \right] \]  
...(12)
Which leads to Madhava’s Taylors series for sine and cosine in the form
\[ f(x + nh) = f(x) + hf^{(1)}(x) + \frac{h^2}{2!} f^{(2)}(x) + \ldots \]  
...(13)
These series are found in the Aryabhatiya bhasya of Nilakanths Somayaji. The power series for sine, and cosine functions as mentioned in the form;

\[
R \sin \theta = R \theta - \frac{(R \theta)^3}{3! R^2} + \frac{(R \theta)^5}{5! R^4} + \ldots 
\]
\[
R \cos \theta = (R \theta)^2 - \frac{(R \theta)^3}{2! R^2} + \frac{(R \theta)^4}{4! R^4} + \ldots 
\]

Further, Paramesvara third order interpolation formula in Siddhantadipika gives the series approximation for sine and cosine function.

Likewise, in the Tantra-sangharya of Nilakantha and the commentary of Yuktibhasa of Jayadev [30] the expansion for tan's is as follows.

\[
\tan^{-1} s = s - \frac{s^3}{3} + \frac{s^5}{5} - \ldots 
\]

### 2.4 Hypergeometric Functions

Hypergeometric functions are one of the oldest transcendental functions. Normally exponential functions are generalized in terms of hypergeometric functions and they undergo several transforms [23]. They can be manipulated analytically and plays a significant role in the number system, partition theory, graph theory, Lie algebra, etc. [17]. According to Horn [13] the power series in m variables is called hypergeometric if all its m ratios are of subsequent coefficients. The first person to use the term "hypergeometric series" was John Wallis in 1655 in his book Arithmetica Infinitorum. [12] Wallis’s Treatise on Algebra (1685) can be considered the first historical investigation of the history of algebra. He was highly influenced by the work of Indian mathematics [6]. Then the further studies on Hypergeometric series was done by Euler and was systematically developed by Guass. In 1707-83 Leonard Euler introduced the power series expansion of the form

\[
1 + \frac{ab z^1}{1! c} + \frac{a(a+1)(b+1) z^2}{2! c(c+1)} + \frac{a(a+1)(a+2)(b+1)(b+2) z^3}{3! c(c+1)(c+2)} + \ldots ...
\]

The series (6) can be represented in the Hommer’s form as

\[
1 + \frac{ab z}{c} + \frac{a(a+1)(b+1) z^2}{c(c+1) 2!} + \frac{a(a+1)(a+2)(b+1)(b+2) z^3}{c(c+1)(c+2) 3!} + \ldots ...
\]

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\]

\[
1 + \frac{ab z}{c} + \frac{a(a+1)(b+1) z^2}{c(c+1) 2!} + \frac{a(a+1)(a+2)(b+1)(b+2) z^3}{c(c+1)(c+2) 3!} + \ldots ...
\]

\[
1 + \frac{ab z}{c} + \frac{a(a+1)(b+1) z^2}{c(c+1) 2!} + \frac{a(a+1)(a+2)(b+1)(b+2) z^3}{c(c+1)(c+2) 3!} + \ldots ...
\]

The equation (19) is called the hypergeometric function with two numerator parameters a, and b and the denominator parameter c. The equation (19) is equivalently written as.

\[
_{2}F_{1} \left[ \begin{array}{c}
a, b; \\
c; \\
z 
\end{array} \mid 1 \right] = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \ldots (20)
\]

The Pochhammer symbol [17] in (19) can be written in terms of gamma function as

\[
(a)_n = a(a+1)(a+2)\ldots[a+(n-1)] = \frac{\Gamma(a+n)}{\Gamma(a)} (a)_0 = 1
\]

Hypergeometric functions are not only expressed as the Euler Hypergeometric functions but are also the solutions of the second order differential equation.

\[
z(1-z) \frac{d^2 y}{dz^2} + [c - (a + b + 1)z] \frac{dy}{dz} - aby = 0 \ldots (22)
\]

In term of operator the above equation can be expressed as

\[
[\theta(\theta + c - 1) - z(\theta + a)(\theta + b)]w = 0 \ldots (23)
\]

Where

\[
\theta = z \frac{d}{dz}
\]

The series is convergent if |z|<1 and divergent for |z|>1 and z = 1 for R(c-a-b)>0

### 3. Results and Discussions

Let a geometric series with the common ratio x be

\[
1 + x^2 + x^3 + x^4 + x^5 + x^6 + \ldots \ldots \ldots (24)
\]

John Wallis extended the ordinary geometric series (1) to the hypergeometric series to the form

\[
1+a+a(a+b)+a(a+b)(a+2b)+a(a+b)(a+2b)(a+3b)+\ldots \ldots (25)
\]

whose nth term is given by

\[
(a)_n = a(a+b)(a+2b)(a+3b)\ldots[a+(n-1)b]
\]

### 3.1 Some primitive Hindu Relations in terms Hypergeometric Function

(a) Mahavira’s Geometric series from Hypergeometric series

From the hypergeometric function (19), If a = b = c = 1 and z is replaced by r in (19) then,
\[ a \cdot \left[F_{1}(1,1; l; r)\right] = a \left(\sum_{n=0}^{\infty} (a)_n (l)_n r^n\right) \]

\[ = \left[1 + \frac{1 + r}{1 + l}\right] \left(\sum_{n=0}^{\infty} \frac{(1 + l)(1 + r)r^n}{(1 + l)^2}\right) \]

\[ = \left[1 + r \left(1 + \frac{(2)(2)r}{(2)(3)}\right)\right] \left[1 + \frac{1}{n!} (3)3^r\right] \]

\[ = a \left[1 + r \left[1 + r \left[1 + r \ldots\right]\right]\right] \]

Therefore,

\[ a \cdot \left[F_{1}(1,1; l; r)\right] = a + ar + ar^2 + ar^3 + ar^4 + \ldots \ldots (25) \]

The equation (25) is the geometric series expressed by Mahavira in (3) whose first term is ‘a’ and common ratio ‘r’.

i) If first term \((a) = 10\) and common ratio \((r) = 2\), in (25), we obtain the geometric series (4) as mentioned in *Taittriya Samhita*. i.e.

\[ 10 \left(F_{1}(1,1; l; r)\right) = 10 + 20 + 40 + \ldots \ldots \ldots \]

ii) If first term \((a) = 1\) and common ratio \((r) = 2\), in (25), we obtain the geometric series (5) as mentioned in *Sulba sutra*. i.e.

\[ 2 F_{1}(1,1; l; r) = 1 + 2 + 4 + \ldots \ldots \ldots \]

\[ r = \frac{1}{4} \text{ then the series (9) of } \text{Mahavira}, \text{is obtained i.e.} \]

\[ 2 F_{1}(1,1; l; r) = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \frac{4}{3} \]

b) Sine series in *Aryabhatiya bhasya* of Nilakanths Somayaji in terms of Hypergeometric function

Replacing \(b = a\) and \(c = \frac{3}{2}\), and \(z = -\frac{\theta^2}{4a^2}\) in equation (19) we get,

\[ 2 F_{1}(a, a; \frac{3}{2}; \frac{\theta^2}{4a^2}) = \sum_{n} \frac{(a)_n (a)_n \left(-\frac{\theta^2}{4a^2}\right)^n}{n!} \]

\[ = \sum_{n} \frac{(a)_n (a)_n \left(-\frac{\theta^2}{4a^2}\right)^n}{n!} \cdot \frac{1}{n!} \]

\[ 2 F_{1}(a, a; \frac{3}{2}; \frac{\theta^2}{4a^2}) = \sum_{n} \frac{(a)_n (a)_n \left(-\frac{\theta^2}{4a^2}\right)^n}{n!} \cdot \frac{1}{n!} \]

\[ \ldots (26) \]

Or,

Now, using Limit in equation (26)

\[ \lim_{a \to \infty} \frac{2 F_{1}(a, a; \frac{3}{2}; \frac{\theta^2}{4a^2})}{a \to \infty} \]

\[ \frac{2 F_{1}(a, a; \frac{3}{2}; \frac{\theta^2}{4a^2})}{a \to \infty} \]

Thus the Sine series in equation (14) established in terms of Hypergeometric function
c) Cosine series in Aryanbhatya bhasya of Nilakhanths Somayaji in terms of Hypergeometric function

Replacing $b = a$ and $c = \frac{1}{2}$ and $z = -\theta^2$ in equation (19), we get,

$$ _2F_1 \left( a, a; \frac{1}{2}; -\frac{\theta^2}{4a^2} \right) = \sum_n \left( \frac{a_n}{n} \right) \left( -\theta^2 \right)^n \cdot \frac{1}{n!} $$

$$ _2F_1 \left( a, a; \frac{1}{2}; -\frac{\theta^2}{4a^2} \right) = \sum_n \frac{(-1)^n \theta^{2n}}{a_n a_n n!} $$

Or,

$$ \lim_{a \to \infty} _2F_1 \left( a, a; \frac{1}{2}; -\frac{\theta^2}{4a^2} \right) = \sum_n \frac{(-1)^n \theta^{2n}}{a_n a_n n!} $$

Thus the Sine series in equation (15) established in terms of Hypergeometric function

(d) Arccos in terms of Hypergeometric function

Replacing

$$ a = \frac{1}{2}, b = \frac{1}{2}, c = \frac{3}{2} \quad \text{and} \quad z = x^2 $$

in equation (19) we get,

$$ _2F_1 \left( \frac{1}{2}, \frac{3}{2}; \frac{1}{2}; x^2 \right) = \sum_n \frac{\left( \frac{1}{2} \right)^n}{\left( \frac{3}{2} \right)^n} (x^2)^n \frac{1}{n!} $$

$$_2F_1 \left( \frac{1}{2}, \frac{3}{2}; \frac{1}{2}; x^2 \right) = \sum_n \frac{\left( \frac{1}{2} \right)^n}{\left( \frac{3}{2} \right)^n} (x^2)^n \frac{1}{n!} \quad \text{Or,}$$

$$ x \left[ _2F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2 \right) \right] = x + \frac{x^3}{2.3} + \frac{1.3x^4}{2.4.5} + \frac{1.3.5x^6}{2.4.6.7} + ... $$

$$ x \left[ _2F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2 \right) \right] = x + \frac{x^3}{2.3} + \frac{1.3x^4}{2.4.5} + \frac{1.3.5x^6}{2.4.6.7} + ... $$

Or,

$$ x \left[ _2F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2 \right) \right] = \frac{\sin^{-1} x}{2} \quad \text{and} \quad z = x^2 $$

"503"
\[ x \left[ \sum_{n=0}^{\infty} \frac{1}{(2)_n (1)_n} (-s^2)^n \right] \]
\[ \therefore \sin^{-1} x = \frac{1}{2} F_1 \left( \frac{1}{2}; \frac{3}{2}; -x^2 \right) \]  \hspace{1cm} (34)

(e) The expansion \( \tan^{-1}'s \) as mentioned in \textit{Tantra-sangharya} of Nilakantha and the commentary of \textit{Yutibhasa} of Jaydev in terms of hypergeometric function. Replacing

\[ a = \frac{1}{2}; b = 1; \hspace{0.5cm} c = \frac{3}{2}, \hspace{0.5cm} \text{and} \hspace{0.5cm} z = -s^2 \]

in equation (19) we get,

\[ \sum_{n=0}^{\infty} \frac{1}{(2)_n (1)_n} (-s^2)^n \frac{1}{n!} \]

\[ = 1 + \frac{s^2}{3} + \frac{s^4}{5} - \frac{s^6}{7} + \ldots \]

Now,

\[ s \left[ 1 + \frac{s^2}{3} + \frac{s^4}{5} - \frac{s^6}{7} + \ldots \right] \]

\[ = s - \frac{s^3}{3} + \frac{s^5}{5} - \frac{s^7}{7} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{(2n+1)!}{s^{2n+1}} \]

\[ = \tan^{-1} s \]

\[ \therefore \tan^{-1} s = s - \frac{s^3}{3} + \frac{s^5}{5} - \frac{s^7}{7} + \ldots \left[ \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} s^{2n+1} \right] \]  \hspace{1cm} (35)

3.2 Diophantine Equations and Brahma Gupta’s Theorem and Hypergeometric function

The Diophantine equation is defined by

\[ NY^2 + 1 = x^2 \]  \hspace{1cm} (36)

for \( N \neq 1 \), seeks the integer solution for \( (x, y) \). Since \( N \) is free from square factors it is known as square nature (\textit{Varga Prakriti}) in Indian mathematics. The Brahma Gupta’s theorem states that if (36) has a non-trivial solution, then it has infinitely many solutions \(^{[3, 28]} \) depending upon \( \alpha, \beta \) and \( k \) through the relation,

\[ x = \frac{2\alpha \beta}{k} \hspace{0.5cm} \text{and} \hspace{0.5cm} y = \frac{\beta^2 + N\alpha^2}{k} \]  \hspace{1cm} (37)

The Diophantine was rediscovered by Euler and Lagranges in 1768 and was solved by using Saalaschutz theorem, summation theorem formulas, Whipple transformation of the hypergeometric functions to form the useful identities.

4. Conclusion

Geometric series and hypergeometric series are both types of mathematical series, which are sums of an infinite number of terms. Geometric series can be expressed as the infinite series whose existence was proved in the 600 B.C. It was first used by Bhaskaracharya to ask the mathematical riddles to his daughter. Then it was used to make the interesting mathematics problems. It consisted of two variables; gunanka (common ratio) and the first term (pratham pad). AP and GP are found in vedic literature in 2000 B.C. The hypergeometric functions are one of the most important and special functions in mathematics. They are the generalization of the exponential functions particularly the ordinary hypergeometric function

\[ _2 F_1 \left( a, b; c; z \right) \]

is represented by hypergeometric series and is a solution to a second order differential equation. John Wallis extended the geometric series to the hypergeometric series with multiple common ratio. Thus hypergeometric function, equivalently the hypergeometric series are the standard form of geometric series with more number of parameters.

In this paper we have shown the derivation of geometric series mentioned by Mahavira in (25), expansion of sine and Cosine series as mentioned in \textit{Aryabhatiya bhasya} of Nilakantahs Somayaji, in (30) and (33) \( \sin^x \), and \( \tan^x \) as mentioned in \textit{Tantra-sangharya} of Nilakantha in (35) in terms of the Hypergeometric function. The relation (37) proves the work of Brahma Gupta in Diophantine equation of the ancient time can be verified by using hypergeometric function in the present time.

Though the work of Hindu mathematicians was not widely abundant among the western mathematicians, the eastern society was highly rich in complex mathematical derivations that are relevant and are the issues of research even in the present time. This paper will add a dimension to expose the work of Hindu mathematicians of the primitive period and to relate it with the present studies.

5. References

11. Hutton C. Mathematical and Philosophical Dictionary.; c1795.