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Extension of geometric series to hypergeometric function in Hindu mathematics

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Abstract
The eastern society is rich in terms of science and technology. Mathematics is considered as the base of science. The eastern history shows that the Hindu society is rich in mathematics. The evolution of mathematics can be studied from the time of 'patiganita' to the latest form of science and technology. Geometric series is an important tool in arithmetic which is now developed to hypergeometric function. Hypergeometric function is an advance function used to solve differential equations of second order. The purpose of this paper is to find the linkage between the ancient geometric series (Gunanka sreni) to the modern Hypergeometric function and to expose the work of ancient Hindu mathematicians which is believed to be narrow among the mathematics researchers of the present period. In this paper, some forms of geometric series that were used in Hindu mathematics are interpreted in terms of hypergeometric series.

Keywords: Hindu Mathematics, Bijaganita, geometric series, hypergeometric function

1. Introduction

1.1 History of Hindu mathematics

The Hindu society has a great history in mathematics. The history of mathematics in this region is as old as human civilization. It reflects the noblest thoughts of countless generations [21]. The mathematics did not had the independent existence at the time of its origination in this region. Mathematics was developed as a Vedanga, the wing of the Vedas, and an art to support the religious rituals in the Sindhu valley. Veda is the basis for development of arts and science and consequently the foundation of modern civilization. Thus India can be considered as the birthplace of modern Algebra and Arithmetic and he branches of mathematics; Algebra, Arithmetic and Geometry are believed to be started from Veda itself.

The advancement of technology in the world can be considered from the time of Ramayan about 1,000 BC [14]. Later Mahavira and Buddha taught the unique system of religious and moral philosophy of life. Mathematics was a key aspect in religious affairs and calculations for wealth, love, music and medicine. It was considered significantly important in the study of the moving heavenly bodies relating to ellipse and the planets. Mathematics was used to count the numbers, measure the diameter, position and perimeter of the islands oceans and Planets.

Ganita is the Hindu name for the mathematics and it means the science of calculations [18]. Ganita is found abundantly in vedic literature. The Vedanga Jyotish (1200 BC) gives the highest values to mathematics among all science which forms the Vedanga. Hindu manuscript describes the importance of Ganita as "As the crest on the heard of peacocks, as the germs in the hood of the snakes, so is ganita on the top of the sciences."

The Ganita consists of *Parikrama* (Fundamental operations), *Vyabahara* (Determinations), *Rajjyu*, (Geometry), *Rasi* (Rule of three), *Kalasavarna* (Operation with fractions), *Yarat tavat* (Simple equations), *Varga* (Square or Quadratic equations), *Ghana* (Cube or Cubic equation), *Varga-varga* (Biquadratic equation), and Vikalpa (Permutation and Combination) in the early renaissance period [1]. The work dealt with algebra was named as *Bijaganita* and the chapter of Bijaganita was called *Kuttaka* (628 A. D.) in his work *Brahma-Sput Siddhant, Sridharacarya.*

1.1.1 Word numerals without Place Value

Arya Bhatta I (499 A. D.) developed the idea of place value in decimal system. After the invention of zero and the place value system the same numerical symbols from 1 to 9 continued to be employed with the zero to denote the numbers. The invention of number system may be assigned to the period of 1000 BC to 6000 BC.

1.1.2 The Decimal Value system

The decimal value system is an important feature of Hindu mathematics. Its existence can be found in ancient documents written by *Lekhaka* and *Lipikara* then later *Divira, Karana and Kayastha*. In this system there are 10 symbols called the *anka* for all numbers 1-9 and the zero is called Sunya.

1.1.3 Zero or the Sunya

Zero was invented in Indian region in the beginning of the Christian era to write the numbers in decimal scale. According to *Chandah-Sutra*, the zero symbol was used in matrices by Pingala before 200 BC. It is used in Panca Siddhanta (505 A. D.). Jinabhadra Ghani, used zero as a distinct numerical symbol. The zero was also included as the other numbers in Bakhshali Manuscript before the third century.

1.2 Operations in Mathematics

Hindu mathematics consists mostly Arithmetic. The work of mathematics in ancient period is called '*patiganita*'. The word *patigapita* is formed from the words '*pati*', means "board," and '*Ganita*', means the science of calculation. Hence Ganita means the science of calculation that requires the use of a writing material (the board) [17]. According to Prthudakasvami there are twenty operations, in arithmetic. They are addition, subtraction, multiplication, division, square, square-root, etc. Addition and subtraction are considered as the very basic operations. In Sara Sangraha by Mahaviracharya [24], the names of the operations are mentioned as follows;

आदिमं गुणकारोऽत्र प्रत्युत्पन्नोऽपि तद्भवेत्।
द्वितीयं भागहाराख्यं तृतीयं कृतिरुच्यते ॥ ४६ ॥
चतुर्थं वर्गमूलं हि भाष्यते पञ्चमं घनः ।
घनमूलं ततष्षष्ठं सप्तमं च चितिस्स्मृतम् ॥ ४७ ॥

The meaning of the verse is like this

"The first among these (operations) is gunakara (multiplication), and it is also (called) pratyutpanna; the second is what is known as bhagahara (division); and krti (squaring) is said to be the third. The fourth, as a matter of course, is varga-mula (square root), and the fifth is said to the ghana (cubing); then ghanamula (cube root) is the sixth, and the seventh is known as citi (summation)"

The verses relating to the other operations are listed in from verse 48 onwards.

Multiplication

Multiplication is the extension of the addition. The number of times of the addition of a certain number reflects the product of the multiple. In Hindu mathematics multiplication was well defined. It was categorically placed as a first operation in *parikarman* (operation) as mentioned by Mahaviracharya in Ganit Sarasangharha[24]. The verse in multiplication is as follows;

प्रत्युत्पन्नः ।
तत्र प्रथमे प्रत्युत्पन्नपरिकर्मणि करणसूत्रं यथा—
गुणयेद्गुणेन गुण्यं कवाटसन्धिक्रमेण संस्थाप्य ।
राश्यर्धखण्डतत्स्थैरनुलोमविलोममार्गाभ्याम ॥ १ ॥

The meaning of the verse is as follows

" After placing (the multiplicand and the multiplier one below the other) in the manner of the hinge of a door, the multiplicand should be multiplied by the multiplier, in accordance with (either of) the two methods of normal (or) reverse working, by adopting the process of (i) dividing the multiplicand and multiplying the multiplier by a factor of the multiplicand, (ii) of dividing the multiplier and multiplying the multiplicand by a factor of the multiplier, or (iii) of them, (in the multiplication) as they are (in themselves)."

Further the book consists of the series of examples of multiplication. E. g. in verse 8, the quote is as follows;

हिमगुपयोनिधिगतिशाशिवह्निव्रतनिचयमत्र संस्थाप्य ।
सैकाशीत्या त्वं मे गुणयित्वाचक्ष्व तत्सङ्ख्याम् ॥ ८

Meaning

"In this (problem), write down (the number represented by) the group (of figures) consisting of 1, 4, 4, 1, 3 and 5 (in order from the units place upwards), and multiply it by 81; and then tell me the (resulting) number."

The multiples are not just ordinary examples but are with interesting products. For example. In verse 15, the typical example is a common but an interesting one.

सप्त शून्यं द्वयं द्वन्द्वं पञ्चैकश्च प्रतिष्ठितम् ।
त्रयःसप्ततिसङ्गुण्यं कण्ठाभरणमादिशेत् ॥ १५ ॥

Meaning

"The (figures) 7, 0, 2, 2, 5 and I are put down (in order from the units' place upwards); and then this (number) which is to be multiplied by 73, should (ako) be called a necklace (when so multiplied)."

1.2.1 Arithmetic and Geometric series

Indian mathematicians studied arithmetic and geometric series in early period. In Hindu mathematics, arithmetic series is called a "*prakṛiti saṃkhya*" or a "*dharma saṃkhya*.". The term "*prakṛiti*" refers to the starting point or the first term of the series, while "dharma" refers to the common difference between consecutive terms in the series. Arithmetic and geometric progressions are found in the Vedic literature of Indians: *Taittirīya-Saṃhitā*, *Vājasaneyī Saṃhitā* and *Kalpa-sūtra* of Bhadrabāhu etc (2000 B.C.).

Arithmetic series

Arithmetic series is kept in seventh number among *parikarman* (operation) under the topic summation in *SaraSanghraha*.. *Āryabhata*– I (499 A.D.), Brahmagupta (628 A. D.) and most of Indian mathematicians have stated various formulae to evaluate sum of n terms of series, arithmetic mean of a finite sequence, number of terms in series and common difference in Arithmetic progression. In *Aryabbatiya*

[27] the method to evaluate the sum of the arithmetic series is explained as follows

इष्टं व्येकं दलितं सपूर्वमुत्तरगुणं समुखमध्यम्¹ ।
इष्टगुणितमिष्टधनं त्वथवाद्यन्तं² पदार्धहतम् ॥ १६ ॥

Meaning

Diminish the given number of terms by one then divide by two, then increase by the number of the preceding terms (if any) then multiply by the common difference, an then the first term of the (whole) series: the result is the arithmetic mean (of the given number of terms). This multiplied by the given number of terms is the sum of the given alternately multiply the sum of the first and last terms (of the series or partial series which is to be summed up) by half the number of terms. According to the verse 61, the formula to evaluate the sum of the terms in the arithmetic progression is given as follows;

सङ्कलितम् ।

सप्तमे सङ्कलितपरिकर्मणि करणसूत्रं यथा --

रूपेणोनो गच्छो दलीकृतः प्रचयताडितो मिश्रः ।
प्रभवेण पदाभ्यस्तस्सङ्कलितं भवति सर्वेषाम् ॥ ६१ ॥

Meaning

"The number of terms in the series is (first) diminished by one and (is then) halved and multiplied by the common difference; this when combined with the first term in the series and (then) multiplied by the number of terms (therein) becomes the sum of all (the terms in the series in arithmetical progression.)"

According to the eastern mathematics the terms "s*redhi"* (progression) is used for the series and *sreni* for the line row of the numbers. Nārāyaṇa Paṇḍita(1340 to 1400 A.D.) stated the general formula for evaluating the sum of a figurate numbers that can be represented in the form of a triangular grid of points where the first row contains a single element and each subsequent row contains one more element than the previous one. He sum of sums of the arithmetic progression is called the *Vārasaṅkalita*. He generalized the formula for *Vārasaṅkalita*. The first *Vārasaṅkalita* will give sum of first order triangular numbers; the second *Vārasaṅkalita* will give sum of second order triangular numbers and similarly we can obtain the result for the sum of kth order triangular numbers or k^{th} *Vārasaṅkalita* of an arithmetic progression. According to him, the general formula for the sum of arithmetic series with first term *a*, common difference *d* and number of terms *n* is:

$$S = \frac{n}{2}\left[2a + (n-1)d\right] \qquad \ldots(1)$$

Geometric series is series of numbers where each term is obtained by multiplying the previous term by a constant factor. In Hindu literature, Geometric series is called the *gunana sreni*, or *guna sreni* or *gunottra sreni*. The first term is called the *prabha*, or *mukkha* or *adidhana*, desired term is called the *ista dhana*, common ratio is called the *gunaka*, the term is called the *pada* and the terms of the geometric series are called the *dhana*. A geometric series is a series of the form:

$$a + ar + ar^2 + ar^3 + ar^4 + \ldots \qquad \ldots(2)$$

where 'a' is the first term, and each successive term is obtaned by multiplying the previous term by '*r*' Western mathematicians are very biased towards the work of Indian mathematicians [9]. Pietro Cossali, on his work in history of algebra, believed that algebra is not developed by Greeks, but is either invented or taken from the Indians. John Wallis [28] was the first westerner mathematician, to believe that the algebra may be originated from India. It is clear from the following statement.

*"*However, it is not unlikely that the Arabs, who received from the Indians the numeral figures (which the Greeks knew not), did from them also receive the use of them, and many profound speculations concerning them, which neither Latins nor Greeks know, till that now of late we have learned them from thence. From the Indians also they might learn their algebra, rather than from Diophantus.

Sanskrit mathematics was not introduced into Europe before seventeenth century. After only the seventh teeth century, western scholars were aware of the existence of Indian algebra.

2. Literature Review

2.1 Geometric Series in Ancient Hindu Texts

Dr. K. Ramasubramanian defined mathematics as *"*ganite sankayete tatganitam: tatpraati padaktyena tatsamajanam shasrtam ukhayate: i. e. the science that deals with calculation are defined by the word *ganita*. To study about the development of mathematics in the Eastern society, one has to go back to 5000 years where the ground was fertile to mathematics. Since the original work were in Sanskrit, due to the linguistic inconvenience, many of the mathematical theories and formulas were unknown to the rest of the world. [8, 26] The vedic period (2000-500 BC) can be considered as the most remarkable time since the first significant chapters of *Sulba sutra* (mathematical formulae) was written in this time. This was done by Baudhyana on 800 BC [5, 8, 26]. During (200 BC-200AD) *Bakhshali* manuscript was discovered which contained the work of *Patiganita* together with other Arithmetic and Geometric progressions and other mathematical operations [8].

The *Bashshali* manuscript shows that *patiganita* was composed nearly about 300 A.D. together with *Aryabhatiya* (499A.D.) the *Brahma Sphuta Siddhanta* (628 A.D) the *Trisatika* (750 a. D.) the *Ganit* sarasangraha (950 A. D.) The Lilavati (1150 A. D.). The *Ganita Kaumadi* (1350 A. D.) [27]. Series was treated as one of the fundamental operations (*Sreni byavahara*) and a separate section is generally devoted to the formula and problem relating to series. For the first time Mahāvīra (815 to 878 A.D.) gave the formula to evaluate the sum S_n of n terms of the geometric progression.

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + ..$$

$$S_n = \frac{a(1-r^n)}{1-r} \qquad \ldots(3)$$

He knew that the infinite geometric series converges if its common ratio is less than one. *Mahāvīra Jain* gave the formula for sum first n terms of geometric progression. The geometrical representation of convergence of summation of geometric series is beautifully explained in *Nīlakaṇṭha's Aryabhatiya bhasya* and *Jyeṣṭhadeva's Yuktibhāṣa* [9].

The formula to evaluate the sum of the terms of the geometric series (*gunadhana*) are mentioned in the form of verse (Sloke) in Ganit Sarasanghraha [27]. These formulas are accumulated in the verses 93 to 95. The verses listed in 93 is as follows;

अथ गुणधनगुणसङ्कलितधनयोस्सूत्रम् —

पदमितगुणहतिगुणितप्रभवस्स्याद्गुणधनं तदाद्यूनम् ।
एकोनगुणविभक्तं गुणसङ्कलितं विजानीयात् ॥ ९३ ॥

Meaning

The first term (of a series in geometrical progression), when multiplied by that self-multiplied product of the common ratio in which (product the frequency of the occurrence of the common ratio is) measured by the number of terms (in the series), gives rise to the gunadhana. And it has to be understood that this gunaghana, when diminished by the first term, and (then) divided by the common ratio lessened by one, becomes the sum of the series in geometrical progression.

Similarly in verse 94, there is also another rule for finding the sum of a series in geometrical progression. It is expressed as follows;

गुणसङ्कलिते अन्यदपि सूत्रम् —

समदलविषमखरूपो गुणगुणितो वर्गताडितो गच्छः ।
रूपोनः प्रभवघ्नो व्येकोत्तरभाजितस्सारम् ॥ ९४ ॥

Meaning

"The number of terms in the series is caused to be marked (in a separate column) by zero and by one (respectively) corresponding to the even (value) which is halved and to the uneven (value from which one is subtracted till by continuing these processes zero is ultimately reached); then this (representative series made up of zero and (me is used in order from the last one therein, so that this one multiplied by the common ratio is again) multiplied by the common ratio (wherever one happens to be the denoting item), and multiplied so as to obtain the square (wherever zero happens to be the denoting item). When (the result of) this (operation) is diminished by one, and (is then) multiplied by the first term, and (is then) divided by the common ratio lessened by one, it becomes the sum (of the series)."

Likewise in the verse 95, the rule for finding the last term and the sum of the terms in a geometric progression is as follows

गुणसङ्कलितान्त्यधनानयने तत्सङ्कलितधनानयने च सूत्रम् –

गुणसङ्कलितान्त्यधनं विगतैकपदस्य गुणधनं भवति ।
तद्गुणगुणं मुखोनं व्येकोत्तरभाजितं सारम् ॥ ९५ ॥

Meaning

"The antyadham or the last term of a series in geometrical progression is the gunadhana (of another aeries) wherein the number of terms is less by one. This (antyadhana) when multiplied by the common ratio, and (then diminished by the first term, and (then) divided by the common ratio lessened by one give rise to the sum (of the series)."

There are numerous practical examples on geometric series. A typical example mentioned in the verse 96 is as follows;

गुणधनस्योदाहरणम् ।

स्वर्णद्वयं गृहीत्वा त्रिगुणधनं प्रतिपुरं समार्जयति ।
यः पुरुषोऽष्टनगर्यां तस्य कियद्वित्तमाचक्ष्व ॥ ९६ ॥

Meaning

"Having (first) obtained 2 golden coins (in some city), a man goes on from city to city, earning (every where) three times (of what he earned immediately before) Say how much he will make in the eighth city". Similarly in verse 99 we can find the similar but another example.

गुणसङ्कलितोदाहरणम् ।

दीनारपञ्चकादिद्विगुणं धनमर्जयन्नरः कश्चित् ।
प्राविक्षदष्टनगरीः कति जातास्तस्य दीनाराः ॥ ९९ ॥

Meaning

"A certain man (in going from city to city) earned money (in a geometrically progressive series) having 5 dinaras for the first term (there vof) and 2 for the common ratio. He (thus) entered 8 cities, how many are the dinaras in his (possession)"

In the *Taittiriya samhita* we can find the geometric series

10 + 20 + 40 + ---------- 8000 …. (4)

Similarly in *Kalpa sutra* and *sulba sutra* of *Bhadrabahu* (350 B.C), the series (5) is included

1+ 2 + 4+ 8 +----------- … (5)

Likewise, in chapter 5, Verse-CXXXVI of Lilavati, Bhaskaracharya [20] uses geometric series in the form of the verse as follows;

विषमे गच्छे व्येके गुणकः स्थाप्यः समेऽर्धिते वर्गः ।
गच्छक्षयांतमंत्यात् व्यस्तं गुणवर्गजं लं यत्तत् ।
व्येकं व्येकगुणोद्धृतमादिगुणं स्यात् गुणोत्तरे गणितम् ॥

Meaning

"If the number of terms in a G.P. is odd, then (n - 1) is called a 'multiplier' (M) and if it is $\frac{n}{2}$ then it is called a square [Bhaskaracharya's terminology). Now beginning with n, continue this process (n - 1) for odd and $\frac{n}{2}$ even until 0 is reached. Then keeping the common ratio (r) against 0 and start writing M or S in the opposite direction. Carry out the operations and then subtract 1 from the final result and divide it by (n-1). The result is the required sum."

Table 1: This rule is illustrated through an example with first term (a) =1, common ratio (r) = 2 and for 31 terms as follows;

31	30	15	14	7	6	3	2	1	0
S	M	S	M	S	M	S	M	S	M

Beginning with 31 we write 31-1=30, $\frac{30}{2}$ =15, 15-1=14, $\frac{14}{2}$ =7, until we reach to zero. Then in the second line, we begin with S and alternately write M and S. Now we perform the operation M for multiplication and S for square as shown below;

Table 2: Sum of 31 terms in G. P.

Number	S/M	Values	Index
30	M	2147483648	31
15	S	1073741824	30
14	M	32768	15
7	S	16384	14
6	M	128	7
3	S	64	6
2	M	8	3
1	S	4	2
0	M	2	1

To reach to the sum up to the 31st power, Bhaskaracharya has given a shorter method by squaring at even numbers and multiplying by r at odd ones. In practice multiplication is easier than squaring. Therefore the sum is obtained to be 2147483648.

There are numerous examples in Lilavati, in geometric progression. Another example solved by the same method mentioned in verse CXXXVII of Lilavati is as follows;

पूर्वं वराटकयुगं येन द्विगुणोत्तरं प्रतिज्ञातम्।
प्रत्यहमर्थिजनाय स मासे निष्कान् ददाति कति।।

Meaning

"A person gave a number the first day; the second day he added that number multiplied by itself and so on for several days. Adding every product multiplied by the fifth number."

The method for solving both of the problems is the same. It is described as follows;

"First observe whether the number of times is odd or even. If it is odd, subtract 1 from it and write it somewhere placing a mark of multiplication above it. If it is even, place a mark of square above it and half it till the number of the times is finished; after that, beginning at the bottom with the number of increase, multiply where there is a mark of multiplication and take the square. Where there is a mark of square; subtract 1 from the product and divide what remains by the number of increase lefts 1. And multiply the quotient by the number of the fifth day. The product will be the sum of the progression."

The verse CXXXVII of Lilavati, mentions the example of the geometric series as follows;

आदिर्द्वयं सखे वृद्धिः प्रत्यहं त्रिगुणोत्तरा।
गच्छ सप्तदिनं यत्र गणितं तत्र किं वद।।

Meaning

"A gentleman gave 2 cowries in charity the first day and thereafter he gave everyday twice of what he gave the previous day. How many cowries did he give away in one month?"

Bhaskaracharya gave many examples to arithmetic progressions and Geometric progressions. He gave the formula for finding the sum of the series, the last term and constant difference. Mahavira defines common ratio as "That by which the term divided by the first term is divisible again and again, subtracting unity every time in the common ratio." He considered simple equations of higher degrees in connection with the the geometric series. They are of the type;

$$ax^n = q \qquad \dots (6)$$

and

$$\frac{a(x^n - 1)}{(x-1)} = p \qquad \dots (7)$$

Where a is the first term of a Geometric progression. q is the *gunadhana*, $(n+1)^{th}$ term of a G. P. p is the sum and x is the unknown common ratio. The counter example presented by Mahavira is, the series whose first term is 3, number of terms is 6 and the sum is 4095, the common ratio is x

$$\frac{3(x^6 - 1)}{x-1} = 4095 \qquad \dots(8)$$

In India, geometric series were used extensively in the context of infinite series. One of the most well-known examples of a geometric series in in this context is the series for the sum of the infinite sequence of numbers

$$1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64} \dots,$$

where each term is obtained by dividing the previous term by 4. This series is given by:

$$1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots = \frac{4}{3} \qquad \dots(9)$$

This result is obtained by the Indian mathematician Madhava in the 14th century. Geometric series were used in India to study infinite products, closely related to infinite series. One example is the formula for the product of the infinite sequence of numbers given below. The sum of this series is obtained by Bhaskara to be $\frac{3}{4}$.

$$\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{27}\right)\dots \qquad \dots (10)$$

2.2 Geometric ratio in Scale notation

The study of Yajurveda Samhita (*Vajasareyi*) [2] shows that ten has been the basis of numeration in Hindu mathematics. In Ganit sarasangharha, the denominations of the numbers are in the form of verses(sloke) These verses numbered from 63 to 68 are as follows;

एकं तु प्रथमस्थानं द्वितीयं दशसंज्ञिकम् ।
तृतीयं शतमित्याहुः चतुर्थं तु सहस्रकम् ॥ ६३ ॥
पञ्चमं दशसाहस्रं षष्ठं स्याल्लक्षमेव च ।
सप्तमं दशलक्षं तु अष्टमं कोटिरुच्यते ॥ ६४ ॥

नवमं दशकोट्यस्तु दशमं शतकोटयः ।
अर्बुदं रुद्रसंयुक्तं न्यर्बुदं द्वादशं भवेत् ॥ ६५ ॥
खर्वं त्रयोदशस्थानं महाखर्वं चतुर्दशम् ।
पद्मं पञ्चदशं चैव महापद्मं तु षोडशम् ॥ ६६ ॥
क्षोणी सप्तदशं चैव महाक्षोणी दशाष्टकम् ।
शङ्खं नवदशं स्थानं महाशङ्खं तु विंशकम् ॥ ६७ ॥
क्षित्यैकविंशतिस्थानं महाक्षित्या द्विविंशकम् ।
त्रिविंशकमथ क्षोभं महाक्षोभं चतुर्नयम् ॥ ६८ ॥

Meaning of the verses 63 to 68

"The first place is what is known as eka (unit); the second place is named dasa (ten); the third they call as sata (hundred), while the fourth is sahasra (thousand). The fifth is dasa-sahastra (ten-thousand) and the sixth is no other than laksa (lakh). The seventh is dasa-laksa (ten-lakh) and the eighth is said to be koti (crore). The ninth dasa-koti (ten-crore) and the tenth is sata-koti (hundred-crore). The eleventh place is arbuda and the twelfth is place nyarbuda. The thirteenth place is kharva and the fourteenth is maha-kharva. Similarly the fifteenth is padma and the sixteenth maha-padma. Again the seventeenth is ksona the eighteenth maha-ksona, The nineteenth place is sankha and the twentieth is maha-sankha. The twenty first place is ksitya the twenty-second maha-ksitya. Then the twenty-third is ksobha and the twenty-fourth maha-ksobha. The first of these denomination is eka which is the unit place, second is called the dasa, and third is called the sata whose place value is hundred. The place value is placed accordingly in the multiples of ten. In other words the notational place value in the decimal system forms a geometric progression with the common ratio of ten. The summary of the notational value is shown in the table given below.

Table 3: Denomination of numbers

Position of place value	Numerical values	Values	Hindu Terminology
first	1	One	eka
Second	10	Ten	dasa
Third	100	Hundred	sata
Fourth	1, 000	Thousand	sahasra
Fifth	10, 000	Ten thousand	Dasa-sahasra
Sixth	100, 000	Hundred Thousand	Laksa –Lakh
Seventh	1, 000, 000	million	Dasa-laksa
Eighth	10, 000, 000	Ten million	Koti Crore
Ninth	100, 000, 000	Hundred million	Dasa koti
Tenth	1, 000, 000, 000	Billion	Sata Koti
Eleventh	10, 000, 000, 000	Ten Billion	Arbuda
Twelfth	100, 000, 000, 000	Hundred Billion	Nyarbuda
Thirteenth	1, 000, 000, 000, 000	Trillion	Kharva
Fourteenth	10, 000, 000, 000, 000	Ten trillion	Maha kharva
Fifteen	100, 000, 000, 000,000	Hundred trillion	padma
Sixteenth	1, 000, 000, 000, 000, 000	Quadrillion	Maha padma
Seventeenth	10, 000, 000, 000, 000, 000	Ten Quadrillion	ksona
Eighteenth	100, 000, 000, 000, 000, 000	Hundred Quadrillion	Maha-ksona
Nineteenth	1, 000, 000, 000, 000, 000, 000	Quintillion	sankha
Twentieth	10, 000, 000, 000, 000, 000, 000	Ten Quintillion	Maha sankha
Twenty first	100, 000, 000, 000, 000, 000, 000	Hundred Quintillion	ksitya
Twenty second	1,000, 000, 000, 000, 000, 000, 000	Sextillion	Maha ksitya
Twenty third	10, 000, 000, 000, 000, 000, 000, 000	Ten Sextillion	ksobha
Twenty fourth	100, 000, 000, 000, 000, 000, 000, 000	Hundred Sextillion	Maha- ksobha

Similarly in *Aryabhatita* [27] the first ten notational places in terms of decimal places is explained as follow;

THE FIRST TEN NOTATIONAL PLACES

एकं दश च शतं च सहस्रं त्वयुतं नियुते तथा प्रयुतम् ।
कोट्यर्बुदं च वृन्दं स्थानात् स्थानं दशगुणं स्यात् ॥ २ ॥

Meaning

Eka (units place), Data (tens place), Sata (hundreds place), Sahasra (thousands place), Ayuta (ten thousands place), Niyuta (hundred thousand place), Prayuta (millions place), Koti (ten millions place), Arbuda (hundred millions place), and Vrnda (thousand millions place) are, respectively, from place to place, each ten times the preceding.

2.3 Unit fraction

Mahavira (550 BC) has given the number of examples to express any fractions as the sum of numbers of unit fractions. These examples shows the existence and development of the geometric series.

1) Expressing 1 as the sum of numbers (n) of unit fractions

$$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+--------+\frac{1}{3^{n-2}}+\frac{1}{2.3^{n-2}} \quad \ldots (11)$$

The rule is: When the number of different quantities having one for their numerator is 1, the required denominator such as beginning with 1 are in the order multiplied by 3, the first and the last being multiplied again by 2 and

$$\frac{2}{3}$$

Ignoring the first and the last term of the series all the rest terms of the middle are expressed in the geometric series with the common ratio.

$\frac{1}{3}$

Apart from the geometric series, there are advance works in infinite series. Rajagopal *et al.* [23] describes Govindaavamin's second order difference interpolation formula of the form

$$f(x+nh) = f(x) + n\Delta f(x) + \frac{n(n-1)}{2}\{\Delta f(x) - \Delta f(x-h)\} \quad \dots(12)$$

Which leads to Madhava's Taylors series for sine and cosine in the form

$$f(x+nh) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \dots + \quad \dots(13)$$

These series are found in the *Aryabhatiya bhasya* of Nilakhanths Somayaji. The power series for sine, and cosine functions as mentioned above are mentioned in the form;

$$R\sin\theta = R\theta - \frac{(R\theta)^3}{3!R^2} + \frac{(R\theta)^5}{5!R^4} + \dots \quad (14)$$

$$R\cos\theta = (R\theta)^2 - \frac{(R\theta)^2}{2!R} + \frac{(R\theta)^4}{4!R^4} + \dots \quad (15)$$

Further, *Paramesvara* third order interpolation formula in *Siddhantadipika* gives the series approximation for sine and cosine function.
Likewise, in the *Tantra-sangharya* of Nilakantha and the commentary of *Yutibhasa* of Jayadev [30] the expansion for *$\tan^{-1}s$* is as follows.

$$\tan^{-1}s = s - \frac{s^3}{3} + \frac{s^5}{5} - \dots \quad (16)$$

2.4 Hypergeometric Functions

Hypergeometric functions are one of the oldest transcendental functions. Normally exponential functions are generalized in terms of hypergeometric functions and they undergo several transforms [21]. They can be manipulated analytically and plays a significant role in the number system, partition theory, graph theory, Lie algebra, etc. [17]. According to Horn [13] the power series in m variables is called hypergeometric if all its m ratios are of subsequent coefficients. The first person to use the term "hypergeometric series" was John Wallis in 1655 in his book Arithemetica Infinitorum. [12] Wallis's *Treatise on Algebra* (1685) can be considered the first historical investigation of the history of algebra. He was highly influenced by the work of Indian mathematics [6]. Then the further studies on Hypergeometric series was done by Euler and was systematically developed by Guass. In 1707-83 Leonhard Euler introduced the power series expansion of the form

$$1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \dots\dots\dots \quad (17)$$

The series (6) can be represented in the Hommer's form as

$$1 + \frac{ab}{c}\frac{z}{1}\left[1 + \frac{(a+1)(b+1)z}{(c+1)2}\left[1 + \frac{(a+2)(b+2)z}{(c+2)3}\dots\dots\dots\dots\dots\right]\right] \quad (18)$$

Or F(a, b; c; z) = $${}_2F_1\left[\begin{matrix} a & b; & \\ & c; & z \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \quad (19)$$

The equation (19) is called the hypergeometric function with two numerator parameters a, and b and the denominator parameter c. The equation (19) is equivalently written as.

$${}_2F_1\left[\begin{matrix} a,b & \\ c & ;1 \end{matrix}\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \dots(20)$$

The Pochhamer symbol [17] in (19) can be written in terms of gamma function as

$$(a)_n = a(a+1)(a+2)\dots.\{a+(n-1)\} = \frac{\Gamma(a+n)}{\Gamma(a)_n}; (a)_0 = 1 \quad (21)$$

Hypergeometric functions are not only expressed as the Euler Hypergeometric functions but are also the solutions of the second order differential equation.

$$z(1-z)\frac{d^2y}{dz^2} + [c-(a+b+1)z]\frac{dy}{dz} - aby = 0 \quad \dots(22)$$

In term of operator the above equation can be expressed as

$$[\theta(\theta+c-1) - z(\theta+a)(\theta+b)]w = 0 \quad (23)$$

Where

$$\theta = z\frac{d}{dz}$$

The series is convergent if $|z|<1$ and divergent for $|z|>1$ and z =1 for *R*(c-a-b)>0

3. Results and Discussions

Let a geometric series with the common ratio x be

$$1 + x^2 + x^3 + x^4 + x^5 + x^6 + \dots. \quad (24)$$

John Wallis extended the ordinary geometric series (1) to the hypergeometric series to the form

$$1 + a + a(a+b) + a(a+b)(a+2b) + a(a+b)(a+2b)(a+3b) + \dots. \quad (25)$$

whose nth term is given by

$$(a)_n = a(a+b)(a+2b)(a+3b)\dots.\{a+(n-1)b\} \quad (26)$$

3.1 Some primitive Hindu Relations in terms Hypergeometric Function

(a) **Mahavira's Geometric series from Hypergeometric series**

From the hypergeometric function (19), If a = b = c = 1 and z is replaced by r in (19) then,

$$a.[{}_2F_1(1,1;1;r)] = a\left[{}_2F_1\begin{bmatrix}1 & 1; & \\ & 1; & r\end{bmatrix}\right] = a\left[\sum_{n=0}^{\infty}\frac{(1)_n(1)_n}{(1)_n}\frac{r^n}{n!}\right]$$

$$= a\left[1+\frac{1.1}{1}\frac{r}{1}\left[1+\frac{(1+1)(1+1)r}{(1+1)2}\left[1+\frac{(1+2)(1+2)r}{(1+2)3}.....................\right]\right]\right]$$

$$= a\left[1+r\left[1+\frac{(2)(2)r}{(2)2}\left[1+\frac{(3)(3)r}{(3)3}.....................\right]\right]\right]$$

$$= a[1+r[1+r[1+r.....................]$$

Therefore,

$$a.[{}_2F_1(1,1;1;r)] = a + ar + ar^2 + ar^3 + ar^4 + ... \quad ...(25)$$

The equation (25) is the geometric series expressed by Mahavira in (3) whose first term is 'a' and common ratio 'r'.

i) If first term (a) = 10 and common ratio (r) =2, in (25), we obtain the geometric series (4) as mentioned in *Taittriya Samhita*. i. e.

$$10.\left({}_2F_1(1,1;1;2)\right) = 10 + 20 + 40 + ----------$$

ii) If first term (a) = 1 and common ratio (r) =2, in (25), we obtain the geometric series (5) as mentioned in *Sulba sutra*. i. e. ${}_2F_1(1,1;1;2) = 1 + 2 + 4 + ----------$

iii) If first term $a=1,$ and $r=\frac{1}{4}$ then the series (9) of *Mabhavira*, is obtained i.e.

$${}_2F_1(1,1;1;\frac{1}{4}) = 1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+...=\frac{4}{3}$$

b) **Sine series in *Aryabhatiya bhasya* of Nilakhanths Somayaji in terms of Hypergeometric function**

Replacing $b=a$ and $c=\frac{3}{2},$ and $z=-\frac{\theta^2}{4a^2}$ in equation (19) we get,

$${}_2F_1\left(a,a;\frac{3}{2};-\frac{\theta^2}{4a^2}\right) = \sum_n \frac{(a)_n(a)_n}{\left[\frac{3}{2}\right]_n}\left[-\frac{\theta^2}{4a^2}\right]^n.\frac{1}{n!}$$

Or,
$${}_2F_1\left(a,a;\frac{3}{2};-\frac{\theta^2}{4a^2}\right) = \sum_n \frac{(a)_n(a)_n(-1)^n\theta^{2n}}{a_n.a_n\left[\frac{3}{2}\right]_n 4^n n!} \quad ...(26)$$

Now, using Limit in equation (26)

$$\lim_{a\to\infty} {}_2F_1\left(a,a;\frac{3}{2};\frac{\theta^2}{4a^2}\right)$$

$$= \lim_{a\to\infty}\sum_n \frac{(a)_n(a)_n(-1)^n\theta^{2n}}{a_n.a_n\left[\frac{3}{2}\right]_n 4^n n!}$$

Or,
$$\lim_{a\to\infty} {}_2F_1\left(a,a;\frac{3}{2};\frac{\theta^2}{4a^2}\right) = \sum_n \frac{(-1)^n\theta^{2n+1}}{\left[\frac{3}{2}\right]_n 2^{2n} n!} \quad ...(27)$$

Since

$$\left[\frac{3}{2}\right]_n 2^{2n} n! = \frac{3}{2}\left(\frac{3}{2}+1\right)\left(\frac{3}{2}+2\right)\left(\frac{3}{2}+3\right).......\left(\frac{3}{2}+n-1\right)2^{2n} n!$$

$$= 3.5.7.....(3+2n-2)\frac{2^{2n}}{2^n}n!$$

$$= 3.5.7.....(2n+1)(1.2.3.4.....)2^n n!$$

Or,
$$\left[\frac{3}{2}\right]_n 2^{2n} n! = (2n+1)! \quad ...(28)$$

∵ From equations (27) and (28) we get,

$$\lim_{a\to\infty} {}_2F_1\left(a,a;\frac{3}{2};-\frac{\theta^2}{4a^2}\right) = \sum_n \frac{(-1)^n\theta^{2n+1}}{\left[\frac{3}{2}\right]_n 2^{2n} n!}$$

$$= \sum_n \frac{(-1)^n\theta^{2n+1}}{(2n+1)!}$$

Or,
$$\lim_{a\to\infty} {}_2F_1\left(a,a;\frac{3}{2};-\frac{\theta^2}{4a^2}\right)$$

$$= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!}...$$

Or,

$$\lim_{a\to\infty} {}_2F_1\left(a,a;\frac{3}{2};-\frac{\theta^2}{4a^2}\right) = \mathrm{Sin}\ \theta$$

$$\therefore\ \mathrm{Sin}\,\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!}... = \lim_{a\to\infty} {}_2F_1\left(a,a;\frac{3}{2};-\frac{\theta^2}{4a^2}\right) \quad ...(29)$$

Or.
$$R\sin\theta = R\theta - \frac{(R\theta)^3}{3!R^2} + \frac{(R\theta)^5}{5!R^4} + ... = R.\lim_{a\to\infty} {}_2F_1\left(a,a;\frac{3}{2};-\frac{\theta^2}{4a^2}\right) \quad (30)$$

Thus the Sine series in equation (14) established in terms of Hypergeometric function

c) Cosine series in *Aryabhatiya bhasya* of Nilakhanths Somayaji in terms of Hypergeometric function

Replacing $b = a$ and $c = \frac{1}{2}$, and $z = -\frac{\theta^2}{4a^2}$ in equation (19), we get,

$$ {}_2F_1\left(a, a; \frac{1}{2}; -\frac{\theta^2}{4a^2}\right) = \sum_n \frac{(a)_n (a)_n}{\left[\frac{1}{2}\right]_n} \left[-\frac{\theta^2}{4a^2}\right]^n \cdot \frac{1}{n!} $$

Or, $$ {}_2F_1\left(a, a; \frac{1}{2}; -\frac{\theta^2}{4a^2}\right) = \sum_n \frac{(a)_n (a)_n (-1)^n \theta^{2n}}{a_n . a_n \left[\frac{1}{2}\right]_n 4^n n!} $$

Now,

$$ \lim_{a \to \infty} {}_2F_1\left(a, a; \frac{1}{2}; \frac{\theta^2}{4a^2}\right) $$

$$ = \lim_{a \to \infty} \sum_n \frac{(a)_n (a)_n (-1)^n \theta^{2n}}{a_n . a_n \left[\frac{1}{2}\right]_n 4^n n!} $$

Or, $$ \lim_{a \to \infty} {}_2F_1\left(a, a; \frac{1}{2}; \frac{\theta^2}{4a^2}\right) = \sum_n \frac{(-1)^n \theta^{2n+1}}{\left[\frac{1}{2}\right]_n 2^{2n} n!} \quad \ldots(31) $$

Since

$$ \left[\frac{1}{2}\right]_n 2^{2n} n! = \frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\left(\frac{1}{2}+3\right) \ldots\ldots \left(\frac{1}{2}+n-1\right) 2^{2n} n! $$

$$ = 1.3.5.7 \ldots\ldots (1+2n-2) \frac{2^{2n}}{2^n} n! $$

$$ = 1.3.5.7 \ldots\ldots (2n-1)(1.2.3.4 \ldots\ldots) 2^n n! $$

Therefore, $\left[\frac{1}{2}\right]_n 2^{2n} n!$ =(2n)! (32)

Hence from (31) and (32)

$$ \lim_{a \to \infty} {}_2F_1\left(a, a; \frac{1}{2}; -\frac{x^2}{4a^2}\right) = \sum_n \frac{(-1)^n x^{2n}}{\left[\frac{1}{2}\right]_n 2^{2n} n!} $$

$$ = \sum_n \frac{(-1)^n x^{2n}}{(2n)!} $$ Or, $$ \lim_{a \to \infty} {}_2F_1\left(a, a; \frac{1}{2}; -\frac{\theta^2}{4a^2}\right) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} \ldots $$

Or $$ \lim_{a \to \infty} {}_2F_1\left(a, a; \frac{1}{2}; -\frac{\theta^2}{4a^2}\right) = \cos\theta $$ Or, $$ R\cos\theta = R \lim_{a \to \infty} {}_2F_1\left(a, a; \frac{1}{2}; -\frac{\theta^2}{4a^2}\right) $$

$$ = R\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} \ldots\right]. $$

$$ \therefore R\cos\theta = (R\theta)^2 - \frac{(R\theta)^2}{2!R} + \frac{(R\theta)^4}{4!R^4} + \ldots = R \lim_{a \to \infty} {}_2F_1\left(a, a; \frac{1}{2}; -\frac{\theta^2}{4a^2}\right) \quad (33) $$

Thus the Sine series in equation (15) established in terms of Hypergeometric function

(d) Arcsin series in terms of Hypergeometric function

Replacing

$a = \frac{1}{2}; b = \frac{1}{2}$ and $c = \frac{3}{2}$, and $z = x^2$

in equation (19) we get,

$$ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right) = \sum_n \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{\left[\frac{3}{2}\right]_n} (x^2)^n \frac{1}{n!} $$

Or,

$$ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right) = 1 + \frac{x^2}{2.3} + \frac{1.3.x^4}{2.4.5} + \frac{1.3.5.x^6}{2.4.6.7} + \ldots $$

Now $$ x\left[{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right)\right] $$

$$ = x\left(1 + \frac{x^2}{2.3} + \frac{1.3.x^4}{2.4.5} + \frac{1.3.5.x^6}{2.4.6.7} + \ldots.\right) $$

$$ x\left[{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right)\right] $$

$$ = x + \frac{x^3}{2.3} + \frac{1.3.x^5}{2.4.5} + \frac{1.3.5.x^7}{2.4.6.7} + \ldots $$

Or, $$ x\left[{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right)\right] = \sin^{-1} x $$

$$\therefore \operatorname{Sin}^{-1} x = x.\left[{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};x^2\right)\right] \qquad (34)$$

(e) The expansion *tan⁻¹s* as mentioned in *Tantra-sangharya* of Nilakantha and the commentary of *Yutibhasa* of Jayadev in terms of hypergeometric function. Replacing

$$a=\frac{1}{2}; b=1 \text{ and } c=\frac{3}{2}, \text{ and } z=-s^2$$

in equation (19) we get,

$${}_2F_1\left(\frac{1}{2},1;\frac{3}{2};-s^2\right)=\sum_n \frac{\left(\frac{1}{2}\right)_n (1)_n}{\left[\frac{3}{2}\right]_n}(-s^2)^n \frac{1}{n!}$$

$$=1-\frac{s^2}{.3}+\frac{.s^4}{5}-\frac{s^6}{7}+\ldots \text{ Now, } s._2F_1\left(\frac{1}{2},1;\frac{3}{2};-s^2\right)$$

$$=s\left(1-\frac{s^2}{.3}+\frac{.s^4}{5}-\frac{s^6}{7}+\ldots\right)$$

$$=s-\frac{s^3}{.3}+\frac{.s^5}{5}-\frac{s^7}{7}+\ldots=\sum_{k=0}^{\infty}\frac{(-1)^n}{(2n+1)!}s^{2n+1}$$

$= \tan^{-1} s$

$$\therefore \tan^{-1} s = s-\frac{s^3}{.3}+\frac{.s^5}{5}-\frac{s^7}{7}+\ldots \; s.\left[{}_2F_1\left(\frac{1}{2},1;\frac{3}{2};-s^2\right)\right] \qquad (35)$$

3.2 Diophantine Equations and Brahma Gupta's Theorem and Hypergeometric function

The Diophatine equation is defined by

$$Ny^2+1=x^2 \qquad (36)$$

for N>1, seeks the integer solution for (x, y). Since N is free from square factors it is known as square nature (*Varga Prakriti*) in Indian mathematics. The Brahmagupta's theorem states that if (36) has a non-trivial solution, then it has infinitely many solutions [3, 28] depending upon α,β and k through the relation,

$$x=\frac{2\alpha\beta}{k} \text{ and } y=\frac{\beta^2+N\alpha^2}{k} \qquad (37)$$

The Diophantine was rediscovered by Euler and Lagranges in 1768 and was solved by using Saalaschutz theorem, summation theorem formulas, Whipple transformation of the hypergeometric functions to form the useful identities.

4. Conclusion

Geometric series and hypergeometric series are both types of mathematical series, which are sums of an infinite number of terms. Geometric series can be expressed as the infinite series whose existence was proved in the 600 B.C. It was first used by Bhaskaracharya to ask the mathematical riddles to his daughter. Then it was used to make the interesting mathematics problems. It consisted of two variables; gunanka (common ratio) and the first term (pratham pad). AP and GP are found in vedic literature in 2000 B. C.

The hypergeometric functions are one of the most important and special functions in mathematics. They are the generalization of the exponential functions particularly the ordinary hypergeometric function

$${}_2F_{1(}(a,b;c;z)$$

is represented by hypergeometric series and is a solution to a second order differential equation. John Wallis extended the geometric series to the hypergeometric series with multiple common ratio. Thus hypergeometric function, equivalently the hypergeometric series are the standard form of geometric series with more number of parameters.

In this paper we have shown the derivation of geometric series mentioned by Mahavira in (25), expansion of sine and Cosine series as mentioned in *Aryabhatiya bhasya* of Nilakhanths Somayaji, in (30) and (33) $\sin^{-1}$x, and $\tan^{-1}$x as mentioned in *Tantra-sangharya* of Nilakantha in (35) in terms of the Hypergeometric function. The relation (37) proves the work of Brahma Gupta in Diophatine equation of the ancient time can be verified by using hypergeometric function in the present time.

Though the work of Hindu mathematicians was not widely abundant among the western mathematicians, the eastern society was highly rich in complex mathematical derivations that are relevant and are the issues of research even in the present time. This paper will add a dimension to expose the work of Hindu mathematicians of the primitive period and to relate it with the present studies.

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