International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2023; 8(5): 53-57

https://www.mathsjournal.com
Received: 02-06-2023
Accepted: 05-07-2023

Veysel Kılınç
Van, Turkey

Corresponding Author:
Veysel Kılınç
Van, Turkey

On the basisness problem for a discontinious sturm-liouville operator

Veysel Kılınç

Abstract
The objective of this research paper is to explore the Riesz basis properties exhibited by the eigenfunction of the Sturm-Liouville equation, considering boundary conditions and transmission conditions that involve a spectral parameter. It is assumed that the coeffficient function of the equation is discontinuous. The fundamental purpose of this scholarly article is to examine the Riesz basis properties exhibited by the eigenfunctions of the Sturm-Liouville equation. Specifically, it examines the influence of boundary conditions and transmission conditions, which incorporate a spectral parameter.

Keywords: Business, eigenfunction, sturm-liouville operator, parameters governing the propagation circumstances

Introduction
Taking into consideration the given equation

$$-u'' + q(x)u = \lambda\rho(x)u\,,\quad x\in(-1,1) \tag{1}$$

the boundary conditions

$$U_1(u)\equiv u'(-1)=0 \tag{2}$$

$$U_2(u)\equiv(\alpha_{11}+\lambda\alpha_{12})u(1)-(\alpha_{11}+\lambda\alpha_{12})u'(1)=0 \tag{3}$$

As well as the prevailing transmission conditions at the specified locations $-1<a_1<a_2<1$

$$T_1u\equiv u(a_1-0)-u(a_1+0)=0 \tag{4}$$

$$T_2u\equiv u'(a_1-0)-\delta u'(a_1+0)=0 \tag{5}$$

$$T_3u\equiv u(a_2-0)-u(a_2+0)=0 \tag{6}$$

$$T_4u\equiv u'(a_2-0)-\gamma u'(a_2+0)=0 \tag{7}$$

Here λ is a complex parameter, $q(x)\in C(I),\, I=[-1,a_1)\cup(a_1,a_2)\cup(a_2,1]$ is a real valued function, $\alpha_{ij}\,(i,j=1,2)$ and $\delta>0$, $\gamma>0$ are real constants and $\rho(x)$ is discontinous function

$$\rho(x)=\begin{cases}\rho_1^2, -1\le x<a_1,\\ \rho_2^2, a_1<x<a_2,\\ \rho_3^2, a_2<x\le 1,\end{cases}$$

such that $\rho_1 \neq \rho_2 \neq \rho_3$.

In this paper, We conduct an analysis on the fundamental characteristics of the eigenfunctions pertaining to the boundary value problem (1.1)-(1.7), where the spectral parameter is incorporated within the boundary condition (1.3) and the transmission conditions (1.4)-(1.6). In order to accomplish this, we establish a distinct Hilbert space and proceed to construct a linear operator within it. As a result, the boundary problem (1.1)-(1.7) can be interpreted as an eigenvalue problem for this linear operator in the Hilbert space. Through our investigation, we demonstrate the self-adjointness of this operator and establish that the eigenfunctions associated with it serve as a basis within the specified Hilbert space.

We note that, the Sturm-Liouville problem with discontinuos coefficient and transmission conditions plays an important role in solving in electro-magnetics, geophysics and other fields of enginering and mathematical pyhsics [1], for example, boundary value problems with the form (1.1)-(1.7) is encountered in vibrating string problems [1]. The Sturm-Liouville problems, characterized by boundary conditions that depend on eigenparameters, have garnered increasing attention due to their relevance in physical applications and have been thoroughly examined in academic research [2-5]. And there is a vast amount of literature on this subject [6-14]. In a previous investigation conducted in 2013, Şen [14] examined the Sturm-Liouville problem, whereas Mamedov and Demirbilek [21], in their recent study conducted in 2022, investigated the same problem but with a discontinuous coefficient and included an eigenparameter in the boundary condition and obtained asymptotic formulas for the eigenvalue and eigenfunctions.

The following sections outline the organization of this paper: Section 2 provides an introduction to a unique inner product within the specialized Hilbert space *H*, where we define a self-adjoint linear operator *L*. This operator is designed in such a way that the eigenvalues of the boundary value problem (1.1)-(1.7) align with those of the operators *L*. Moving on to Section 3, we present a proof demonstrating that the eigenfunctions of *L* establish a basis in *H*. It has been shown the eigenfunctions of the spectral problem with eigenparameter- dependent boundary conditions formed defect basis in L_2 for different cases by Gulmamedov and Mamedov [5].

2. Adequate Hilbert Space can be represented using an operator-based formulation

Let us

$$\varphi(x)=(\varphi_1(x),\varphi_2(x),\varphi_3(x)),$$

where

$\varphi_1(x)=\varphi(x)_{[-1,a_1)}$, $\varphi_2(x)=\varphi(x)_{(a_1,a_2)}$, $\varphi_3(x)=\varphi(x)_{(a_2,-1]}$, $\ell\varphi(x)\equiv\dfrac{1}{\rho(x)}\left[-\varphi''+q(x)\varphi\right]$

and $\Delta=\alpha_{12}\alpha_{21}-\alpha_{11}\alpha_{22}\neq 0$.

We determine a special Hilbert space and we represent this space by $H=L_2(I)\times\mathbb{C}$. The Hilbert space of all elements $F(x)=(\varphi(x),K),\ (K\in\mathbb{C})$ is given by

$$G(x)=(\phi(x),h)=(\phi_1(x),\phi_2(x),\phi_3(x))\in H,\ h\in\mathbb{C}$$

and special inner product is defined as:

$$\langle F,G\rangle=\rho_1^2\int_{-1}^{a_1}\varphi_1(x)\overline{\phi_1(x)}dx+\rho_2^2\delta\int_{a_1}^{a_2}\varphi_2(x)\overline{\phi_2(x)}dx+\rho_3^2\delta\gamma\int_{a_2}^{1}\varphi_3(x)\overline{\phi_3(x)}dx+\frac{\delta\gamma}{\Delta}Kh.$$

We define the operator L in H as follows:

$$LF=\begin{pmatrix}\ell(\varphi)\\ \alpha_{11}\varphi(1)-\alpha_{21}\varphi''(1)\end{pmatrix} \tag{8}$$

where the domain

$$D(L)=\begin{cases}F\in H:\varphi(x),\varphi^{'}(x)\in AC(I),\ell\varphi\in L_2(I),\\U_1(\varphi)=T_1(\varphi)=T_2(\varphi)=T_3(\varphi)=0,K=\alpha_{12}\varphi(1)-\alpha_{22}\varphi^{'}(1)\end{cases}.$$

We can easily obtain that the boundary value transmission problem (1.1)-(1.7) is equivelent to operator L in H as:

$$LF=\lambda F \tag{2.2}$$

The boundary value problem $(1.1)-(1.7)$ exhibits a clear correspondence between its eigenvalues and those of the operator *L*, while its eigenfunctions correspond to the primary components of the respective eigenfunctions associated with the operator *L*. The asymptotic expressions for eigenvalues and their corresponding eigenfunctions of boundary value problems (1.1)-(1.7) are derived in T. In the case $\Delta>0$, according to research [10-12], we give the following lemma:

Lemma 2.1. The operator possesses self-adjoint properties within the Hilbert space denoted as *H*.

3. Basisness of Eigenfuctions

Theorem 3.1. The eigenfunctions corresponding to the operator L establish a complete orthonormal basis within the space H.

Proof: According to Şen [14], The operator L possesses a countably infinite set of eigenvalues $\{\lambda_k\}_0^\infty$, with each eigenvalue tending towards infinity as the argument approaches infinity. Consequently, except for the isolated eigenvalue λ_k, the operator $L-\lambda I$ exhibits an inverse within the Hilbert space H. Specially, the eigenvalue taking $\lambda=0$, In H, the operator L^{-1} is defined and it is self-adjoint. It is noteworthy that L^{-1} possesses a countable number of eigenvalues at most, and each of these eigenvalues converges to zero as the infinity is approached. Consequently, it can be concluded that L^{-1} is a compact operator. According to the Hilbert-Schmidt theorem concerning compact operators, the eigenfunctions of the operator L constitute an orthonormal basis in H. The theorem is proved.

Corollary 3.1. Let K_0 denote an arbitrary, predetermined nonnegative integer. The system of eigenfunctions $\{u_k\}_0^\infty$ $(K\neq K_0)$ of the boundary value problem $(1.1)-(1.7)$ is a complete and form a minimal system in $L_2(I)$.

Proof: By Theorem 3.1. the system eigenfunctions $u_k(x)=\{u_k(x),d_k\}$ of operator L form a basis in $H=L_2(\mathrm{I})\times\mathbb{C}$ where $u_k(x)=\{u_{1k}(x),u_{2k}(x),u_{3k}(x)\}$, $d_k\in\mathbb{C}$.

Consequently, within the context of H, the system $\{u\}_k^\infty$ can be deemed both complete and minimal, and we represent the orthogonal projection as P, an alternative way to express the given expression $Pu_k(x)=u_k(x)$ is as follows. From here it can be seen that $P=1$. The completeness and minimality of the complementary system in $P(H)=L_2(-1,1)$, denoted as $\{Pu_k(x)\}_0^\infty=\{u_k(x)\}_0^\infty$, can be established by applying Lemma 2.1 from 1983 [14], which states that upon removing a single element from $\{u_k(x)\}_0^\infty$, the resulting system retains these desirable properties. Consequently, the completeness and minimality of the eigenfunctions $\{u_k(x)\}_0^\infty$ for $\{u_k(x)\}_0^\infty$ (where k is any non-negative integer) in $L_2(-1,1)$ have been established, thus concluding the corollary.

To investigate the case $\Delta<0$, we assume that the operator L is defined by the formula (2.1) on the domain $D(L)$ and introduce the operator J.

In the case, The scalar product in $H_1=L_2(I)\times\mathbb{C}$ is determined by the equation

$$\langle F,G\rangle=\rho_1^2\int_{-1}^{a_1}\varphi_1(x)\overline{\phi_1(x)}dx+\rho_2^2\delta\int_{a_1}^{a_2}\varphi_2(x)\overline{\phi_2(x)}dx+\rho_3^2\delta\gamma\int_0^1\varphi_3(x)\overline{\phi_3(x)}dx-\frac{\delta\gamma}{\Delta}K.h \tag{3.1}$$

In this instance, the boundary value problem $(1.1)-(1.7)$ can be viewed as an analogous formulation to either the eigenvalue problem or the eigenvalue problem associated with the operators pencil within the H_1 space,

$$(B-\lambda J)Y=0 \tag{3.2}$$

where $B = JL$. Indeed, the operator $G = B - \lambda J$ exhibits the properties of being both bounded and invertible. Applying the operator J to the self-side of (2.2), we obtain that (2.2) is equivalent to (3.2).

Lemma 3.1. The operator L is J - selfadjoint in the space H_1. In the Hilbert space H_1, the operator L exhibits J - selfadjointness.

Proof: The domain $D(L)$ is dense in the space H_1. From the equalites $(3.1),(3.2),(2.1)$ and applying two times the integration by parts, we have that $\langle BF, G\rangle = \langle F, BG\rangle$ for $F, G \in D(L)$.

The operator L possesses J -symmetry in the H_1 space. The eigenvalues of the boundary problem $(1.1)-(1.7)$ are zeros of an entire function and form a bounded set [14]. In the given scenario, where the operator L possesses a discrete spectrum, it can be observed that the operator B, being symmetric, leads to the conclusion that the operator $J\mathrm{A}$ is self-adjoint.

Theorem 3.2. The eigenfunctions of the operatör L constitute a Riesz basis within the Hilbert space H.

Proof: Based on Lemma 3.1, it can be concluded that the operator $L = J^{-1}B$ is $J-$ selfadjoint in the Hilbert space H. Based on the findings presented in Theorem 3.1 and the theorem established in Azizov's study [20], it can be deduced that the eigenfunctions of the operator L constitute a Riesz basis within the Hilbert space H.

Conclusion

In this study, the Sturm Liouville problem is handled. Basic definitions and theorems are given to reach the main results of the problem. Specifically, a distinct inner product is defined within a specialized Hilbert space, leading to the formulation of a linear operator. Subsequently, an examination is conducted to determine the Riesz basis property of the eigenfunctions associated with a Sturm-Liouville operator featuring a discontinuous potential function.

References

1. Anderseen RS. The Effect of Discontinuous in Density and Shear Velocity on the Asymptotic Overtone Structure of Torsional Eigenfrequences of the Earth. Geophysical Journal Royal Astronomical Society. 1997;50:306-309.
2. Tikhonov AN, Samarskii AA. Equation of Mathematical Physics. Oxford, New York; c1963.
3. Fulton CT. Two-Point Boundary Value Problems with Eigenvalue Parameter Contained in the Boundary Conditions. Proc. Of the Royal Society of Edinburgh Section A: Mathematics. 1977;77(3-4):293-308. DOI: 10. 1017/S030821050002521X.
4. Kapustin NY, Moisseev EI. A Remark on the Convergence Problem for Spectral Expansions Corresponding to a Classiscal Problem with Spectral Parameter in the Boundary Condition. Differential Equations. 2001;37(12):1677-1683. DOI: 10. 1023/A: 1014406921176.
5. Gulmamedov VY, Mamedov Kh.R. On Basis Property for a Boundary-Value Problem with a Spectral Parameter in the Boundary Condition, Cankaya University. Journal of Arts and Sciences, 2006, 5.
6. Binding PA, Browne PJ, Siddighi K. Sturm-Liouville Problems with Eigenparameter Dependent Conditions. Proc. Edinburgh Math. Soc. 1999;37(1):57-72.
7. Kobayashi M. Eigenvalues of Discontinuous Sturm-Liouville Problem with Symmetric Potentials. Comp. Math. Appl. 1989;18:357-364.
8. Mamedov Kh. R. On a Basis Problem for a Second Order Differential Equation with a Discountinuous Coefficient and a Spectral Parameter in the Boundary Conditions. Geometry, Integrability and Quantization; c2005. p. 218-225.
9. Mamedov Kh. R. On one Boundary Value Problem with Parameter in the Boundary Conditions. Spektralnaya Teoriya Operatorov İ Prilojeniya. (Spectral Theory of Operators and its Applications. 1997; no 11, 117-121). 1997;11:117-121.
10. Ala V, Mamedov Kh. R. On Basis Property of Root Functions for a Class of the Second Order Differential Operators. Bulletin of the South Ural State University. (Ser. Math. Mech. Phys.). 2020;12(3):15-21.
11. Ala V, Mamedov Kh. R. On Basis Property of Root Functions for a Class Second Order Differantial Operator. Applied Mathematics and Nonlinear Siences. 2020;5(1):361-368.
12. Ala V, Mamedov Kh. R. On a Discontinuous Sturm- Liouville Problem with Eigenvalue Parameter in the Boundary Conditions. Dynamic Systems and Applications. 2020;29:182-191.
13. Shkalikov AA. Boundary Value Problems for Ordinary Differential Equations with a Spectral Parameter in the Boundary Conditions. Trudy Sem. Im. I. G. Petrovsogo. 1983;9:190-229.
14. Şen E. A Sturm-Liouville Problem with a Discontinuos Coefficient and Containing an Eigenparameter in the Boundary Condition. Physics Research International. Article ID 159243; c2013.
15. Azizov T, Iokhidov. Foundations of the Theory of Linear Operators in Space with Indefinite Metric, Moseow, Nauka; c1986 (in Russian)
16. Wang A, Sun J, Hao X, Jao S. Completness of Eigenfunctions of Sturm-Liouville Problems with Transmission Couditions. Methods and Applications of Analysis. 2009;16(3):299-312.
17. Şen E. A Sturm-Liouville Problem with a Discontinuos Coefficient and Containing an Eigenparameter in the Boundary Condition. Physics Research International. Article ID 159243; c2013.
18. Ala V, Mamedov Kh. R. Basisness of Eigenfunctions of a Discontinuous Sturm- Liouville Operator. J Adv. Math. Stud; c2020.

19. Azizov T, Iokhidov I. Foundations of Theory of Linear Operators in Spaces with Indefinite Metric. Moscow, Nauka. (in Russian); c1986.
20. Mamedov Kh. R, Demirbilek U. Basisness Problem for a Discontinuous Sturm –Lıouvılle Operator with a Spectral Parameter on the Boundary Conditions, International Anatolian Congress on Scientific Research. Proceedıngs Book, Volume-II, ISBN 978-625-8254-10-5. 2022 Dec;736-740:27-29.
21. Demirbilek U, Mamedov Kh. R. On The Expansion Formula for a Singular Sturm-Liouville Operator. Journal of Science and Arts, 2021; V. 21, No1, pp. 67-76.
22. Demirbilek U, Mamedov Kh. R, Suleymanov N.S. Two Fold Expansion Formula For a Non Self Adjoint Boundary Value Problem. Journal of Contemporary Applied Mathematics, 2022; V.12, No.2, pp.46-56.