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A monte Carlo comparison of normality tests based on entropy estimators

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Abstract
In this paper, seven tests for normality of datasets that are based on entropy estimators are presented. The type-I-error rates and the powers of the statistics are compared through extensive simulation studies at a sample size of *n* = 10. The alternative distributions used in the comparative study are categorized into 5 according to symmetry and support. It is discovered that all the tests considered have good control over type-I-error and that no test is universally the best among them.

Keywords: Empirical critical value of a test, entropy estimator, power of a test, test for normality, type-I-error rate

Introduction

Goodness-of-fit to the normal distribution of datasets has been tested using different techniques in the literature. These techniques have been developed using different characterizations of the normal distribution and some transforms of the normal distribution. Some of such characterizations include quantile function, distribution function, moment generating function, characteristic function, Laplace transform, entropy function, skewness and kurtosis. As a result, tests that are developed using a particular characterization are often regarded as tests belonging to a class.

One of the important classes of tests for normality is the class of tests based on the entropy estimators. Suppose a random variable *X* follows a distribution $F(x)$ with probability density function $f(x)$. The entropy of the random variable, denoted by $H(f)$, is defined by Shannon (1948) [9] as:

$$H(f) = -\int_{-\infty}^{\infty} f(x)\log\{f(x)\}\,dx \tag{1}$$

Several estimators of (1) have been proposed in the literature. Such estimators include Vasicek (1976) [11], van Es (1992) [10], Ebrahimi *et al*. (1994) [5], Correa (1995) [3], Yokota and Shiga (2004) [13], Alizadeh Noughabi and Arghami (2010) [1], Zamanzade and Arghami (2011) [14], Kohansal and Rezakhah (2016) [7], Bitaraf *et al*. (2017) [2] and Madukaife (2023) [8]. The estimators are obtained using different nonparametric methods, thereby leaving the estimators as biased. Some of the nonparametric methods include kernel density estimation, window size spacing, nearest neighbor technique and quantile density estimation of particular interest are the estimators based on window size (*m*) spacing, which have dominated this area of research, obtained by a transformation of (1) according to Vasicek (1976) [11] as:

$$H(f) = \int_0^1 \log\left\{\frac{d}{dp}F^{-1}(p)\right\}dp;\ p \in (0,1) \tag{2}$$

Suppose $x_1, x_2, \dots, x_n$ is a random sample of size *n* from a continuous distribution $F(x)$

having a probability density function $f(x)$. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the sample order statistics obtained from the sample such that $X_{(i)}, i=1, 2, , \ldots, n$ is the ith smallest observation in the sample. For a non-negative integar, m, such that $m \leq \frac{n}{2}$, the sample m spacing is defined on ith order by $X_{(i+m)} - X_{(i-m)}$. Using the spacings, researchers have obtained several estimators of (1) and (2). Some of these estimators have in turn been applied to develop goodness-of-fit statistics for testing the normality of datasets. This is because it is known that among all the statistical distributions that possess a density function $f(x)$ and having variance σ^2, the entropy is maximized by the normal distribution.

In this paper, the empirical powers of seven of these tests are compared. A review of these statistics is presented in Section 2 while the empirical power comparison of the tests is presented in Section 3 with resultant discussion. Thereafter, the paper is concluded in Section 4.

2. A review of the normality tests

There are seven tests for normality based on the spacings approach of the Shannon entropy estimators which are compared in this paper. In this section, a review of these tests for normality is presented.

2.1 Test of normality based on Vasicek estimator

Vasicek (1976) [11] obtained an estimator of the derivative of the pth quantile $F^{-1}(p)$ by the slope, given by:

$$\frac{n}{2m}\left[X_{(i+m)} - X_{(i-m)}\right]; \quad \text{for} \quad \frac{(i-1)}{n} < p < \frac{i}{n}, i = m+1, m+2, \ldots, n-m$$

Using the estimated derivative, one of the pioneer estimators of the entropy is obtained by Vasicek as:

$$HV_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{2m}\left[X_{(i+m)} - X_{(i-m)}\right]\right\} \tag{3}$$

Based on the estimator in (3), Vasicek (1976) [11] obtained a statistic for testing the normality of datasets. The statistic is given by:

$$TV_{mn} = \frac{n}{2ms}\left\{\prod_{i=1}^{n}\left[X_{(i+m)} - X_{(i-m)}\right]\right\}^{1/n} \tag{4}$$

Where

$$s = \sqrt{n^{-1}\sum_{i=1}^{n}(x_i - \overline{X})^2}$$

The statistic is established to be consistent and invariant with respect to changes in location and scale. Also, the null distribution of normality of a dataset is rejected for small values of the statistic.

2.3 Tests of normality according to Esteban *et al*. (2001) [6]

The Vasicek estimator in (3) received some criticisms as having a wrong slope measure when $i \leq m$ or $i \geq n-m+1$. Such criticisms include Dudewicz and Van der Meulen (1981) [4], van Es (1992) [10], Ebrahimi *et al*. (1994) [5], Correa (1995) [3], etc. Based on these criticisms, van Es (1992) [10] proposed an estimator of entropy, given by:

$$HE_{mn} = \frac{1}{n-m}\sum_{i=1}^{n-m}\log\left\{\frac{n+1}{m}\left[X_{(i+m)} - X_{(i)}\right]\right\} + \sum_{k=m}^{n}\frac{1}{k} + \log\{m\} - \log\{n+1\}, \tag{5}$$

where $1 \leq k \leq n$. Also, Wieczorkowski and Grzegorzewski (1999) [12] modified the Vasicek estimator by adding a bias correction to have a new entropy estimator given by:

$$HW_{mn} = HV_{mn} - \log(n) + \log(2m) - \left(1 - \frac{2m}{n}\right)\psi(2m) + \psi(n+1) - \frac{2}{n}\sum_{i=1}^{m}\psi(i+m-1), \tag{6}$$

where HV_{mn} is the Vasicek estimator and $\psi(x)$ is the digamma function defined by $\psi(x) = \frac{d\log\Gamma(x)}{dx} = \frac{\Gamma'(x)}{\Gamma(x)}$.

Esteban *et al*. (2001) [6], among other things, proposed tests of normality based on these estimators in (5) and (6). The tests are:

$$TE_{mn} = \frac{\exp\{HE_{mn}\}}{\hat{\sigma}} = \frac{1}{\hat{\sigma}} \exp\left\{\sum_{i=1}^{n} \frac{1}{k}\right\} \left[\prod_{i=1}^{n-m} \left(X_{(i+m)} - X_{(i)}\right)\right]^{\frac{1}{(n-m)}} \tag{7}$$

And

$$TW_{mn} = \frac{\exp\{HW_{mn}\}}{\hat{\sigma}} = \frac{1}{\hat{\sigma}} \left[\prod_{i=1}^{n} \left(X_{(i+m)} - X_{(i-m)}\right)\right]^{\frac{1}{n}} \exp(c) \tag{8}$$

where HE_{mn} and HW_{mn} are the van Es and the Wieczorkowski and Grzegorzewski estimators respectively,

$$c = -\left(1 - \frac{2m}{n}\right)\psi(2m) + \psi(n+1) - \frac{2}{n}\sum_{i=1}^{m} \psi(i+m-1)$$

and $\hat{\sigma}$ is the estimated standard deviation. Esteban *et al*. (2001) [6] stated that the statistics are invariant with respect to transformations of location and scale and concluded that they are both powerful tools for testing normality.

Test of normality based on Alizadeh Noughabi and Arghami estimator

Alizadeh Noughabi and Arghami (2010) [1] proposed to estimate entropy $H(f)$ of an unknown continuous probability density f by:

$$HA_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log\left\{\frac{n}{a_i m}\left[X_{(i+m)} - X_{(i-m)}\right]\right\}, \tag{9}$$

Where

$$a_i = \begin{cases} 1, & 1 \le i \le m \\ 2, & m+1 \le i \le n-m \\ 1, & n-m+1 \le i \le n, \end{cases}$$

$X_{(i-m)} = X_{(1)}$ for $i \le m$ and $X_{(i+m)} = X_{(n)}$ for $i \ge n-m$. They established that the estimator in (9) is consistent with smaller mean square error than the previous estimators in the literature. Based on the Kullback – Leibler divergence between two distributions f and f_0, usually denoted by $D(f, f_0)$, they obtained a statistic for testing the normality of a dataset. The statistic is denoted by TA_{mn} and defined as:

$$TA_{mn} = \log\sqrt{2\pi\hat{\sigma}^2} + 0.5 - HA_{mn}, \tag{10}$$

where $\hat{\sigma}^2$ is the estimated variance and HA_{mn} is the entropy estimator in (9). They stated that the test which rejects the null distribution of normality for large values of the statistic is invariant with respect to changes in location and scale.

Test of normality based on the Zamanzade and Arghami estimator

Zamanzade and Arghami (2011) [14] proposed a bias and root mean square error (RMSE) reduced estimator of the Shannon entropy. It is given by:

$$HZ_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log\left\{\frac{n}{a_i m}\left[X_{(i+m)} - X_{(i-m)}\right]\right\}, \tag{11}$$

Where

$$a_i = \begin{cases} \dfrac{i}{m}, & 1 \le i \le m \\ 2, & m+1 \le i \le n-m \\ \dfrac{n-i+1}{m}, & n-m+1 \le i \le n, \end{cases}$$

Based on the estimator in (11), they proposed a test for assessing the normality of datasets using the same concept of Kullback – Leibler distance measure between two continuous distributions f and f_0. The statistic is given by:

$$TZ_{mn} = \log\sqrt{2\pi\hat{\sigma}^2} + 0.5 - HZ_{mn}, \tag{12}$$

where HZ_{mn} is as defined in (11). The test rejects normality of datasets for large values of the statistic and is said to be affine invariant.

Test of normality based on the Bitaraf *et al.* estimator

Bitaraf *et al*. (2017) [2] introduced an internal jth spacing to the Vasicek (1976) [11] modified estimator to obtain a new estimator given by:

$$HB_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log\{T_{i.}\}, \tag{13}$$

where $T_{i.} = \frac{1}{2}\sum_{j=0}^{1} T_{ij}$; $T_{ij} = \frac{n}{w_{ij}(m-j)}\left[X_{(i+m-j)} - X_{(i-m+j)}\right]$ and $w_{ij} = \begin{cases} 1, & 1 \le i \le m-j \\ 2, & m-j+1 \le i \le n-m+j \\ 1, & n-m+j+1 \le i \le n, \end{cases}$.

They established the consistency of the estimator and obtained a statistic for assessing normality of datasets based on the estimator. The statistic is given by:

$$TB_{mn} = \frac{\exp\{HB_{mn}\}}{\hat{\sigma}} = \frac{1}{\hat{\sigma}}\prod_{i=1}^{n}\left\{\frac{n\left(X_{(i+m)} - X_{(i-m)}\right)}{2w_{i0}m} + \frac{n\left(X_{(i+m-1)} - X_{(i-m+1)}\right)}{2w_{i1}(m-1)}\right\}^{\frac{1}{n}} \tag{14}$$

They equally stated that the statistic is affine invariant and rejects the null distribution of normality for small values of TB_{mn}.

Test of normality based on Madukaife estimator

Madukaife (2023) [8] modified the Bitaraf *et al*. (2017) [2] doubly spacing approach to obtain a new estimator of the Shannon entropy. The estimator is given by:

$$HM_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{1}{2}\left(T_{i0} + T_{i1}\right)\right\}, \tag{15}$$

where $T_{ij} = \frac{n}{w_{ij}(m-2j)}\left[X_{(i+m-j)} - X_{(i-m+j)}\right]$ and $w_{ij} = \begin{cases} 1+\frac{1}{3}, & 1 \le i \le m-j \\ 2, & m-j+1 \le i \le n-m+j \\ 1+\frac{1}{3}, & n-m+j+1 \le i \le n, \end{cases}$.

Based on the statistic in (15), Madukaife (2023) [8] obtained a statistic for testing the normality of datasets. The statistic is given by:

$$TM_{mn} = \frac{\exp\{HB_{mn}\}}{\hat{\sigma}} = \frac{1}{s}\prod_{i=1}^{n}\left\{\frac{n\left(X_{(i+m)} - X_{(i-m)}\right)}{2w_{i0}m} + \frac{n\left(X_{(i+m-1)} - X_{(i-m+1)}\right)}{2w_{i1}(m-2)}\right\}^{\frac{1}{n}} \tag{16}$$

where $s = \sqrt{n^{-1}\sum_{i=1}^{n}(x_i - \overline{X})^2}$. The test is said to be consistent and affine invariant, and rejects the null distribution of normality for small values of TM_{mn}.

Simulation study

This section is of two aspects. The first is the empirical critical values of the seven tests under comparison. The second is the empirical power comparison of the tests under several classes of distributions.

Empirical critical values of the tests

No attempt was made to obtain the theoretical distributions of the various tests for normality under investigation in this study. As a result, empirical critical values of the tests are obtained through Monte Carlo simulation and computation. Precisely, 100,000 samples of the standard normal distribution are simulated at each sample size, *n* = 10, 20 and 50. Under a combination of each sample size (*n*) and window size (*m*), each of the seven competing statistics is computed in each of the 100,000 samples. The α -level critical value (c_α) of each of the tests in each combination of n and m is the 100α percentile of the 100,000 computed values of the statistic, for tests that reject null distribution of normality for small values of the statistics ($P(T_{mn} < c_\alpha) = \alpha$), and/or the $100(1-\alpha)$ percentile of the 100,000 computed values of the statistics, for tests that reject normality for large values of the statistics ($P(T_{mn} > c_\alpha) = \alpha$). The results are presented in Table 1.

Table 1: Empirical critical values of the tests, α = 0.05

n	m	TV_{mn}	TE_{mn}	TW_{mn}	TA_{mn}	TZ_{mn}	TB_{mn}	TM_{mn}
10	1	1.6644	2.9269	0.8288	2.6823	2.6787	-	0.9040
	2	2.0323	2.6871	3.6615	2.4304	2.2814	2.4992	-
	3	2.0912	2.5119	2.9919	2.3070	2.0068	3.0060	3.8172
	4	2.0837	2.3338	1.2106	2.2156	1.7401	3.4657	3.4840
	5	2.0230	2.1993	0.3182	2.1280	1.4767	3.8997	3.4750
20	1	2.1908	3.2010	0.7301	2.3925	2.3953	-	1.1360
	2	2.6260	3.2010	9.5400	2.1950	2.1208	2.8651	-
	3	2.7040	3.2010	20.7432	2.1072	1.9534	3.2403	4.4617
	4	2.6969	3.2010	22.0894	2.0446	1.8142	3.4830	3.9168
	5	2.6478	3.2010	15.7484	2.0016	1.6718	3.6858	3.7546
	6	2.5900	3.2010	8.6518	1.9584	1.5370	3.8661	3.6891
	7	2.5253	3.2010	3.9300	1.9130	1.4074	4.0402	3.6688
	8	2.4606	3.2010	1.5411	1.8759	1.2685	4.2182	3.6713
	9	2.3932	3.2010	0.5368	1.8399	1.1399	4.3993	3.6781
	10	2.3287	3.2010	0.1698	1.7974	1.0049	4.5827	3.6930
30	1	2.4156	3.3331	0.6445	2.2785	2.2781	-	1.2409
	2	2.8799	3.3331	14.4857	2.0906	2.0448	3.0169	-
	3	2.9962	3.3331	49.7084	2.0173	1.9175	3.3636	4.7418
	4	3.0172	3.3331	82.4870	1.9684	1.8134	3.5715	4.1524
	5	3.0003	3.3331	91.4103	1.9316	1.7147	3.7291	3.9774
	6	2.9615	3.3331	77.9328	1.8960	1.6198	3.8678	3.9067
	7	2.9131	3.3331	55.2207	1.8680	1.5271	3.9853	3.8619
	8	2.8573	3.3331	34.0204	1.8357	1.4312	4.1015	3.8391
	9	2.8043	3.3331	18.7477	1.8063	1.3434	4.2086	3.8238
	10	2.7481	3.3331	9.4337	1.7808	1.2523	4.3255	3.8205
	11	2.6928	3.3331	4.4060	1.7548	1.1631	4.4377	3.8210
	12	2.6367	3.3331	1.9253	1.7318	1.0731	4.5499	3.8252
	13	2.5832	3.3331	0.7947	1.7040	0.9858	4.6658	3.8307
	14	2.5316	3.3331	0.3128	1.6810	0.8942	4.7881	3.8384
	15	2.4824	3.3331	0.1179	1.6506	0.8038	4.9155	3.8546

Empirical power study

The powers of the seven statistics that are being compared are computed from 12 different continuous statistical distributions at different sample sizes through Monte Carlo simulation. The distributions are grouped into 5, namely: symmetric distributions with support $(-\infty, \infty)$ as *Group I*; asymmetric distributions with support $(-\infty, \infty)$ as *Group II*; asymmetric distributions with support $(0, \infty)$ as *Group III*; symmetric distributions with support $(0,1)$ as *Group IV* and asymmetric distributions with support $(0,1)$ as *Group V*. The distributions include:

Group I: Support $(-\infty, \infty)$, symmetric
Standard normal
Standard Laplace
Student's t with 2 degrees of freedom

Group II: Support $(-\infty, \infty)$, asymmetric
Skew standard Laplace
Gumbel with α = 0, β = 1
Gumbel with α = 1, β = 3

Group III: Support $(0, \infty)$, asymmetric
Standard exponential
Gamma with β = 1, α = 2
Standard lognormal
Weibull with β = 1, α = 2
Chi-square with 2 degrees of freedom

Group IV: Support $(0,1)$, symmetric
Uniform with (0, 1)
Beta with α = 2, β = 2
Beta with α = 0.5, β = 0.5 (arcsine)
Group V: Support $(0,1)$, asymmetric
Beta with α = 1, β = 3
Beta with α = 3, β = 0.5

Under each combination of *n* and *m*, 10,000 samples are simulated from each distribution and each of the competing statistics is computed from each sample. Normality of each sample is tested at 0.05 level of significance. The power of each statistic is defined as the proportion of the 10,000 samples that are rejected by the statistic. It is expressed in this work in percentage. The results are presented in Table 2 for sample sizes *n* = 10 with the optimal window size in parenthesis and the highest power under each alternative distribution in bold font.

Table 2: Empirical power comparison of the tests, *n* = 10, α = 0.05

Distributions	TV_{mn}	TE_{mn}	TW_{mn}	TA_{mn}	TZ_{mn}	TB_{mn}	TM_{mn}
Normal (0,1)	5.3 (1)	5.0 (1)	5.2 (2)	5.2 (1)	5.4 (1)	4.8 (2)	5.0 (3)
Laplace (0,1)	9.8 (5)	11.3 (4)	9.0 (5)	42.5 (5)	42.1 (4)	9.9 (2)	9.2 (5)
Students' t (2)	19.9 (5)	24.4 (5)	18.8 (5)	65.8 (4)	65.9 (2)	18.6 (5)	18.4 (5)
Skew Laplace (0,1,1)	9.3 (1)	9.9 (1)	9.2 (5)	42.2 (4)	42.8 (2)	9.6 (5)	9.3 (5)
Gumbel (0,1)	12.6 (5)	12.2 (5)	13.2 (4)	31.6 (2)	31.8 (2)	12.9 (5)	12.9 (5)
Gumbel (1,3)	13.1 (5)	15.9 (1)	13.5 (4)	99.7 (5)	99.7 (2)	13.8 (4)	12.6 (5)
Exponential (1)	45.0 (3)	32.8 (4)	48.2 (4)	21.8 (1)	21.7 (1)	49.1 (4)	46.6 (4)
Gamma (1,3)	44.9 (3)	32.0 (4)	47.3 (4)	2.2 (1)	2.1 (1)	46.8 (4)	46.3 (4)
Lognormal (0,1)	59.6 (3)	49.6 (3)	61.7 (4)	57.3 (2)	57.2 (1)	61.3 (4)	60.9 (5)
Chi-square (2)	46.9 (4)	42.9 (2)	47.0 (4)	74.3 (2)	75.1 (2)	48.1 (4)	48.3 (4)
Weibull (1,2)	45.1 (3)	44.8 (4)	47.5 (4)	74.7 (2)	74.4 (2)	48.3 (4)	48.6 (4)
Uniform (0,1)	16.6 (2)	16.2 (2)	16.7 (2)	0.1 (1)	0.1 (1)	16.9 (2)	16.8 (2)
Beta (2,2)	7.5 (3)	5.9 (2)	7.9 (2)	0.0	0.1 (1)	31.5 (3)	30.8 (2)
Beta (0.5,0.5)	50.9 (2)	50.5 (2)	50.4 (2)	2.6 (1)	2.9 (1)	80.1 (3)	50.5 (1)
Beta (1,3)	24.1 (3)	28.2 (4)	24.6 (4)	0.4 (1)	0.8 (1)	58.3 (3)	58.5 (3)
Beta (3,0.5)	67.1 (2)	68.9 (2)	67.9 (3)	18.5 (1)	18.4 (1)	89.8 (2)	88.9 (4)

From the results in Table 2, all the seven tests recorded approximately 5.0 power performance under the normal distribution, which is the null distribution. This shows that they all have good control over the type-I-error since the level of significance at which the tests are carried out is the same 5.0% level. Under the remaining alternative distributions in groups I and II with support from $(-\infty,\infty)$, the TA_{mn} and TZ_{mn} statistics showed dominant superiority in terms of power performance among all the tests considered, with the TZ_{mn} outperforming the TA_{mn} to appear as the most powerful statistics in those categories of alternative distributions.

Under the distributions with support from the $(0,\infty)$ considered in this work, the TW_{mn}, TA_{mn}, TZ_{mn}, and the TB_{mn} all recorded high power performances with the TW_{mn}, and TB_{mn} as the most powerful among all the tests considered under this category of alternative distributions. Finally for distributions with support from the $(0,1)$, both symmetric and asymmetric which are considered, the TB_{mn} and TM_{mn} showed dominant superiority in terms of power performance among all the statistics considered, with the TB_{mn} being more powerful than the TM_{mn} to be the most powerful statistic under this category of alternative distributions.

Conclusion

Quite a good number of tests for normality that are based on the Shannon entropy estimators exist. Previous studies have shown that these seven considered in this study have potential of having good power performances. The result of the comparison carried out in this study have shown that none of the statistics is universally the most powerful under different categories of alternative distributions. As a result, for alternative distributions with support from $(-\infty,\infty)$, the TA_{mn} and TZ_{mn} can be regarded as good statistics for testing normality. Also for alternative distributions with support from $(0,\infty)$, the TW_{mn}, and TB_{mn} can be regarded as good statistics for testing normality. Finally for alternative distributions with support from $(0,1)$, the TB_{mn} and TM_{mn} can be regarded as good statistics for testing normality of datasets.

It is expected from statistical theory that as the sample sizes increase, the powers of the tests considered in this study will increase. However, it is recommended to carry out this comparison at different sample sizes to ascertain if the behaviour of the various statistics will maintain the same pattern as seen in this study.

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