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The inverse proposition of Taylor-type mean value theorem

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Abstract
Through the study of the Taylor-type mean value theorem, we prove correctness of the inverse proposition of the Taylor-type mean value theorem under some certain conditions. Furthermore, the inverse theorem of the Cauchy mean value is extended to the case of higher order.

Keywords: The theorem of Taylor-type mean value, inverse question, monotonicity, the theorem of intermediate value

1. Introduction

G. Polyá and G. Szegö [1] first introduced the inverse question of Lagrange mean value theorem, i.e., if a function *f*(*x*) is continuously differential in the interval (a,b), for $\xi \in (a,b)$, can we find $x_1 < \xi < x_2$ such that $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(\xi)$. And at the same time they also gave a counterexample to explain that this inverse proposition is usually incorrect. From then on, there are many specialists who are interested in this field and did a lot of work. For example, L. Zuo [2] studied sufficient conditions on the rightness of the inverse proposition of the Lagrange mean value theorem. X. You [3] researched the correctness of the inverse proposition of high-order difffferential mean value theorem. Y. Chen and J. Cai [4] explored the inverse proposition on the Cauchy mean value theorem and Y. Feng [5] investigated the inverse proposition of the differential mean theorem under the setting of sub-differentiation.

Since the Taylor-type mean value [6] holds, it is natural to take into account its inverse proposition. In this paper we are devoted to finding sufficient conditions to make the inverse proposition established. The main result of this paper is Theorem 3.1 and Theorem 3.2. Theorem 3.1 shows that the inverse proposition of the Taylor mean value holds under some certain conditions. Theorem 3.2 generalizes the inverse theorem of the Cauchy mean value theorem to high-order form. Our proof is fairly different from [7].

2. Preliminaries

We first give the Taylor-type mean value theorem and the inverse theorem of Cauchy mean

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Value theorem.

Theorem 2.1 [6] Let $f(x), g(x)$ have $n-1$-order continuous derivatives at $[a,b]$ and $f^{(n)}(x), g^{(n)}(x)$ exist in (a,b), $n \in \mathbf{N}$, and $g^{(n)}(x) \neq 0$ for any $x \in (a,b)$. Then there exists $\xi \in (a,b)$ such that.

$$\frac{f(b)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(b-a)^k}{g(b)-\sum_{k=0}^{n-1}\frac{g^{(k)}(a)}{k!}(b-a)^k} = \frac{f^{(n)}(\xi)}{g^{(n)}(\xi)}$$

Theorem 2.2 [7, Theorem 1] Let $f(x), g(x)$ be differential on $[a,b]$, $g'(x) \neq 0$ and $f'(x_1)g'(x_2) - f'(x_2)g'(x_1) < 0$ (or $f'(x_1)g'(x_2) - f'(x_2)g'(x_1) > 0$) if $x_1 < x_2$, $x_1, x_2 \in [a,b]$. Then there exists $x_0 \in (a,b)$, such that

(1) if $\xi \in (a, x_0)$, there exists $p \in (a,b)$ such that

$$\frac{f(p) - f(a)}{g(p) - g(a)} = \frac{f'(\xi)}{g'(\xi)};$$

(2) if $\xi \in (x_0, b)$, there exists $q \in (a,b)$ such that

$$\frac{f(b) - f(q)}{g(b) - g(q)} = \frac{f'(\xi)}{g'(\xi)}.$$

3. Main results

According to Taylor-type mean value theorem, we can get the inverse proposition of it: Let $f(x), g(x)$ have $n-1$-order continous and derivatives at $[a,b]$ and $f^{(n)}(x), g^{(n)}(x)$ exist in (a,b), $n \in \mathbf{N}$. Then for any $\xi \in (a,b)$, there exist $a \leq x_1 < \xi < x_2 \leq b$ such that

$$\frac{f(x_2) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_1)}{k!}(x_2 - x_1)^k}{g(x_2) - \sum_{k=0}^{n-1} \frac{g^{(k)}(x_1)}{k!}(x_2 - x_1)^k} = \frac{f^{(n)}(\xi)}{g^{(n)}(\xi)}. \tag{3.1}$$

But it is wrong. Indeed, take two functions at $[-1,1]$: $f(x) = x^{n+2}$ and $g(x) = x^n$. When $-1 \leq x_1 < 0 < x_2 \leq 1$,

$$f(x_2) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_1)}{k!}(x_2 - x_1)^k$$

$$= x_2^{\,n+2} - \sum_{k=0}^{n-1} C_{n+2}^k x_1^{\,n+2-k}(x_2 - x_1)^k$$

$$= C_{n+2}^n x_1^{\,2}(x_2 - x_1)^n + C_{n+2}^{n+1} x_1 (x_2 - x_1)^{n+1} + C_{n+2}^{n+2}(x_2 - x_1)^{n+2}$$

$$= \left[\frac{n^2 + n}{2} x_1^{\,2} + n x_1 x_2 + x_2^{\,2}\right](x_2 - x_1)^n$$

$$> \left(\frac{n x_1}{2} + x_2\right)^2 (x_2 - x_1)^n \geq 0.$$

Let $\xi = 0$. Then right hand side of (3.1) is equal to zero, but the left hand side of (3.1) is not the case. Thus the inverse proposition of the Taylor-type mean value theorem does not hold unless some certain conditions are added.

To prove the main results, we need some important lemmas.

Lemma 3.1 Let $f(x), g(x)$ have $n-1$-order continuous derivatives at $[a,b]$ and $f^{(n)}(x), g^{(n)}(x)$ exist in (a,b), $n\in\mathbf{N}$, $g^{(n)}(x)\neq 0$ and $f^{(n)}(x_1)g^{(n)}(x_2)-f^{(n)}(x_2)g^{(n)}(x_1)<0$ (or $f^{(n)}(x_1)g^{(n)}(x_2)-f^{(n)}(x_2)g^{(n)}(x_1)>0$) for $a\le x_1<x_2\le b$. If for any $\xi_1\in(a,b)$, there exists $p_1\in(a,b)$ such that

$$\frac{f(p_1)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(p_1-a)^k}{g(p_1)-\sum_{k=0}^{n-1}\frac{g^{(k)}(a)}{k!}(p_1-a)^k}=\frac{f^{(n)}(\xi_1)}{g^{(n)}(\xi_1)},$$

Then for any $\xi_2\in(a,\xi_1)$, there exists $p_2\in(\xi_2,p_1)$ such that

$$\frac{f(p_2)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(p_2-a)^k}{g(p_2)-\sum_{k=0}^{n-1}\frac{g^{(k)}(a)}{k!}(p_2-a)^k}=\frac{f^{(n)}(\xi_2)}{g^{(n)}(\xi_2)}.$$

Proof. Construct a function

$$G_1(x)=\frac{f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k}{g(x)-\sum_{k=0}^{n-1}\frac{g^{(k)}(a)}{k!}(x-a)^k}-\frac{f^{(n)}(\xi_2)}{g^{(n)}(\xi_2)},\ x\in[a,b].$$

Then

$$G_1(\xi_2)=\frac{f(\xi_2)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(\xi_2-a)^k}{g(\xi_2)-\sum_{k=0}^{n-1}\frac{g^{(k)}(a)}{k!}(\xi_2-a)^k}-\frac{f^{(n)}(\xi_2)}{g^{(n)}(\xi_2)},$$

And

$$G_1(p_1)=\frac{f(p_1)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(p_1-a)^k}{g(p_1)-\sum_{k=0}^{n-1}\frac{g^{(k)}(a)}{k!}(p_1-a)^k}-\frac{f^{(n)}(\xi_2)}{g^{(n)}(\xi_2)}.$$

In the light of Theorem 2.1, there exists $\xi'\in(a,\xi_2)$ such that

$$G_1(\xi_2)=\frac{f^{(n)}(\xi')}{g^{(n)}(\xi')}-\frac{f^{(n)}(\xi_2)}{g^{(n)}(\xi_2)}$$

But

$$G_1(p_1)=\frac{f^{(n)}(\xi_1)}{g^{(n)}(\xi_1)}-\frac{f^{(n)}(\xi_2)}{g^{(n)}(\xi_2)}$$

And $\frac{f^{(n)}(x)}{g^{(n)}(x)}$ is a strictly monotone function. Hence $G_1(\xi_2)G_1(p_1)<0$.

According to the intermediate value theorem of one-variable continuous function, there exists $p_2 \in (\xi_2, p_1)$ such that

$G_1(p_2)=0$, Which implies that Lemma 3.1 holds.

Lemma 3.2 Let two functions f, g satisfy the conditions of Lemma 3.1. If for any $\xi_3 \in (a,b)$, there exists $q_1 \in (a,\xi_3)$ such that

$$\frac{f(b)-\sum_{k=0}^{n-1}\frac{f^{(k)}(q_1)}{k!}(b-q_1)^k}{g(b)-\sum_{k=0}^{n-1}\frac{g^{(k)}(q_1)}{k!}(b-q_1)^k}=\frac{f^{(n)}(\xi_3)}{g^{(n)}(\xi_3)}$$

Then for any $\xi_4 \in (\xi_3, b)$, there exists $q_2 \in (q_1,\xi_4)$ such that

$$\frac{f(b)-\sum_{k=0}^{n-1}\frac{f^{(k)}(q_2)}{k!}(b-q_2)^k}{g(b)-\sum_{k=0}^{n-1}\frac{g^{(k)}(q_2)}{k!}(b-q_2)^k}=\frac{f^{(n)}(\xi_4)}{g^{(n)}(\xi_4)}.$$

Proof. Construct

$$G_2(x)=\frac{f(b)-\sum_{k=0}^{n-1}\frac{f^{(k)}(x)}{k!}(b-x)^k}{g(b)-\sum_{k=0}^{n-1}\frac{g^{(k)}(x)}{k!}(b-x)^k}-\frac{f^{(n)}(\xi_4)}{g^{(n)}(\xi_4)},\ x\in[a,b].$$

Then

$$G_2(\xi_4)=\frac{f(b)-\sum_{k=0}^{n-1}\frac{f^{(k)}(\xi_4)}{k!}(b-\xi_4)^k}{g(b)-\sum_{k=0}^{n-1}\frac{g^{(k)}(\xi_4)}{k!}(b-\xi_4)^k}-\frac{f^{(n)}(\xi_4)}{g^{(n)}(\xi_4)},$$

$$G_2(q_1)=\frac{f(b)-\sum_{k=0}^{n-1}\frac{f^{(k)}(q_1)}{k!}(b-q_1)^k}{g(b)-\sum_{k=0}^{n-1}\frac{g^{(k)}(q_1)}{k!}(b-q_1)^k}-\frac{f^{(n)}(\xi_4)}{g^{(n)}(\xi_4)}.$$

According to Theorem 2.1, there exists $\xi'' \in (\xi_4, b)$ such that

$$G_2(\xi_4)=\frac{f^{(n)}(\xi'')}{g^{(n)}(\xi'')}-\frac{f^{(n)}(\xi_4)}{g^{(n)}(\xi_4)}.$$

But

$$G_2(q_1)=\frac{f^{(n)}(\xi_3)}{g^{(n)}(\xi_3)}-\frac{f^{(n)}(\xi_4)}{g^{(n)}(\xi_4)}$$

And $\frac{f^{(n)}(x)}{g^{(n)}(x)}$ is a strictly monotone function on $[a,b]$ from assumptions. Thus

$G_2(\xi_4)G_2(q_1)<0$.

According to the intermediate value theorem of one-variable continuous function, there exists $q_2 \in (q_1, \xi_4)$ such that $G_2(q_2)=0$ which yields that Lemma 3.2 holds.

Next we give the main results of this paper.

Theorem 3.1 Let $f(x), g(x)$ have $n-1$-order continuous derivatives at $[a,b]$ and $f^{(n)}(x), g^{(n)}(x)$ exist in (a,b), $n \in \mathbf{N}$, $g^{(n)}(x) \neq 0$ and $f^{(n)}(x_1)g^{(n)}(x_2) - f^{(n)}(x_2)g^{(n)}(x_1) < 0$ (or $f^{(n)}(x_1)g^{(n)}(x_2) - f^{(n)}(x_2)g^{(n)}(x_1) > 0$) for $a \le x_1 < x_2 \le b$. Then for any $\xi \in [s,r] \subset [a,b]$, there exist $x_1, x_2 \in [s,r]$, $x_1 < \xi < x_2$ such that

$$\frac{f(x_2) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_1)}{k!}(x_2 - x_1)^k}{g(x_2) - \sum_{k=0}^{n-1} \frac{g^{(k)}(x_1)}{k!}(x_2 - x_1)^k} = \frac{f^{(n)}(\xi)}{g^{(n)}(\xi)}.$$

Proof. In view of Theorem 2.1, there exist $\xi_1 \in [s, \xi]$ and $\xi_2 \in [\xi, r]$ such that

$$\frac{f(\xi) - \sum_{k=0}^{n-1} \frac{f^{(k)}(s)}{k!}(\xi - s)^k}{g(\xi) - \sum_{k=0}^{n-1} \frac{g^{(k)}(s)}{k!}(\xi - s)^k} = \frac{f^{(n)}(\xi_1)}{g^{(n)}(\xi_1)},$$

$$\frac{f(r) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\xi)}{k!}(r - \xi)^k}{g(r) - \sum_{k=0}^{n-1} \frac{g^{(k)}(\xi)}{k!}(r - \xi)^k} = \frac{f^{(n)}(\xi_2)}{g^{(n)}(\xi_2)}.$$

Construct a function on $D = [s, \xi] \times [\xi, r]$

$$H(\alpha, \beta) = \frac{f^{(n)}(\xi)}{g^{(n)}(\xi)} - \frac{f(\beta) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k}{g(\beta) - \sum_{k=0}^{n-1} \frac{g^{(k)}(\alpha)}{k!}(\beta - \alpha)^k}.$$

Then

$$H(s, \xi) = \frac{f^{(n)}(\xi)}{g^{(n)}(\xi)} - \frac{f^{(n)}(\xi_1)}{g^{(n)}(\xi_1)},$$

$$H(\xi, r) = \frac{f^{(n)}(\xi)}{g^{(n)}(\xi)} - \frac{f^{(n)}(\xi_2)}{g^{(n)}(\xi_2)}.$$

Since $\frac{f^{(n)}(x)}{g^{(n)}(x)}$ is a strictly monotone function on [a,b] and $s<\xi_1<\xi<\xi_2<r$, we have that $H(s,\xi)\cdot H(\xi,r)<0$.

Note that $H(\alpha,\beta)$ is continuous on rectangle D. Thus there exists a point (x_1,x_2) on the segment AB where A$=(s,\xi)$, B=(ξ,r), such that $H(x_1,x_2)=0$ by the intermediate value theorem of multi-variable functions. Obviously $s<x_1<\xi<x_2<r$ and

$$H(x_1,x_2)=\frac{f^{(n)}(\xi)}{g^{(n)}(\xi)}-\frac{f(x_2)-\sum_{k=0}^{n-1}\frac{f^{(k)}(x_1)}{k!}(x_2-x_1)^k}{g(x_2)-\sum_{k=0}^{n-1}\frac{g^{(k)}(x_1)}{k!}(x_2-x_1)^k}=0,$$

i.e.

$$\frac{f(x_2)-\sum_{k=0}^{n-1}\frac{f^{(k)}(x_1)}{k!}(x_2-x_1)^k}{g(x_2)-\sum_{k=0}^{n-1}\frac{g^{(k)}(x_1)}{k!}(x_2-x_1)^k}=\frac{f^{(n)}(\xi)}{g^{(n)}(\xi)}.$$

We complete the proof.

Remark 3.1 Theorem 3.1 indicates that the inverse proposition of the Taylor-type mean value sets up under the certain assumptions. Thus Theorem 3.1 extends [7, Lemma 1] from first derivative to higher derivative situation.

Theorem 3.2 Let two functions $f(x), g(x)$ on $[a,b]$ satisfy the conditions of Theorem 3.1. Then there exist $x_0\in(a,b)$ such that

(1) When $\xi\in(a,x_0)$, there exists $p\in(a,b)$,such that

$$\frac{f(p)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(p-a)^k}{g(p)-\sum_{k=0}^{n-1}\frac{g^{(k)}(a)}{k!}(p-a)^k}=\frac{f^{(n)}(\xi)}{g^{(n)}(\xi)};$$

(2) When $\xi\in(x_0,b)$, there exist $q\in(a,b)$ such that

$$\frac{f(b)-\sum_{k=0}^{n-1}\frac{f^{(k)}(q)}{k!}(b-q)^k}{g(b)-\sum_{k=0}^{n-1}\frac{g^{(k)}(q)}{k!}(b-q)^k}=\frac{f^{(n)}(\xi)}{g^{(n)}(\xi)}.$$

Proof. Construct sets

$$A=\left\{\xi\in(a,b)\mid\exists p\in(a,b],\frac{f(p)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(p-a)^k}{g(p)-\sum_{k=0}^{n-1}\frac{g^{(k)}(a)}{k!}(p-a)^k}=\frac{f^{(n)}(\xi)}{g^{(n)}(\xi)}\right\},$$

$$B=\left\{\xi\in(a,b)\,|\,\exists q\in[a,b),\frac{f(b)-\sum_{k=0}^{n-1}\frac{f^{(k)}(q)}{k!}(b-q)^k}{g(b)-\sum_{k=0}^{n-1}\frac{g^{(k)}(q)}{k!}(b-q)^k}=\frac{f^{(n)}(\xi)}{g^{(n)}(\xi)}\right\}.$$

According to Theorem 2.1, we know that A,B are empty. Define $\alpha=\sup_{\xi\in A}\xi$, $\beta=\inf_{\xi\in B}\xi$.

We prove that $\alpha\geq\beta$ by contradiction. If $\alpha<\beta$, suppose $\delta\in(\alpha,\beta)$. There exists $\xi\in(a,b)$ such that

$$\frac{f(b)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(b-a)^k}{g(b)-\sum_{k=0}^{n-1}\frac{g^{(k)}(a)}{k!}(b-a)^k}=\frac{f^{(n)}(\xi)}{g^{(n)}(\xi)}.$$

Clearly $a<\delta,\xi<b$. If $a<\delta\leq\xi$, by Lemma 3.1 there exists $p''\in(a,b)$ such that

$$\frac{f(p'')-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(p''-a)^k}{g(p'')-\sum_{k=0}^{n-1}\frac{g^{(k)}(a)}{k!}(p''-a)^k}=\frac{f^{(n)}(\delta)}{g^{(n)}(\delta)}.$$

Then $\delta\leq\alpha$, Which contradicts $\delta\in(\alpha,\beta)$ in view of the definition of α. If $\xi<\delta<b$, from Lemma 3.2 there exists $q''\in(a,b)$ such that

$$\frac{f(b)-\sum_{k=0}^{n-1}\frac{f^{(k)}(q'')}{k!}(b-q'')^k}{g(b)-\sum_{k=0}^{n-1}\frac{g^{(k)}(q'')}{k!}(b-q'')^k}=\frac{f^{(n)}(\delta)}{g^{(n)}(\delta)}.$$

Thus $\delta\geq\beta$. Which contradicts $\delta\in(\alpha,\beta)$ due to the definition of β. Therefore $\alpha<\beta$ is false. So $\alpha\geq\beta$. Take $x_0\in[\beta,\alpha]\subset(a,b)$. Combining Lemma 3.1 and Lemma 3.2 induces that x_0 satisfies the requirement of Theorem 3.2. The proof is complete.

Remark 3.2 If $n=1$, then Theorem3.2 reduces to Theorem 2.2. Hence Theorem 3.2 generalizes the main result of [7].

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