International Journal of Statistics and Applied Mathematics 2023; 8(5): 220-223

ISSN: 2456-1452
Maths 2023; 8(5): 220-223
© 2023 Stats & Maths
https://www.mathsjournal.com
Received: 01-06-2023
Accepted: 03-07-2023

A Anil Kumar
ICAR-Indian Agricultural Research Institute, Pusa, New Delhi, India

B Manjunatha
ICAR-Indian Agricultural Research Institute, Pusa, New Delhi, India

Corresponding Author:
A Anil Kumar
ICAR-Indian Agricultural Research Institute, Pusa, New Delhi, India

Non-proper PBIB Designs using Star-polygon

A Anil Kumar and B Manjunatha

Abstract
This article provides a new series of non-proper partially balanced incomplete block (PBIB) designs using star-polygon graph with minimal replications. Constructed PBIB designs are of resolvable design category which can be successfully employed in sequential experimentations. A catalogue of PBIB designs for v (number of treatments) ≤ 100 along with computed efficiencies and average variances is also presented.

Keywords: Canonical efficiency factors; Star-polygon association scheme; Non-proper PBIB design; Resolvable design

1. Introduction
The concept of PBIB designs was initially developed by Bose and Nair (1951) [21]. Later, 2-associate class PBIB designs have been extensively studied in the literature [see e.g., Clatworthy (1973)] [4]. Further, 3- or higher-class PBIB designs such as rectangular designs are an important class of block designs with factorial structure for experiments with two factors [see e.g., Vartak (1955) [41], Sharma and Das (1985) [32], Suen (1989) [36], Srivastava *et al.* (2000) [35], Parsad *et al.* (2007a, 2007b) [22-23] and references cited therein]. The nested group divisible designs, a class of PBIB (3) designs, useful for 3-factor experiments was studied by Roy (1953) [30]; Raghavarao (1960) [27]; Miao *et al.* (1996) [18]; and Kageyama and Singh (2002) [9]. More generalized association scheme called extended group divisible (EGD) association scheme and related designs are known as extended group divisible (EGD) designs [Hinkelmann (1964)] [13]. Many practical applications of these designs were fostered by Parsad *et al.* (2007a, 2007b) [22-23]. circular lattice designs was introduced by Rao (1956) which were essentially PBIB(3) designs for $v = 2n^2$ treatments, where $n \geq 2$ and these were further generalized by Varghese and Sharma (2004) [39] to accommodate $2sn^2$ treatments; $n, s \geq 2$. Varghese *et al.* (2004) [39] gave a list of PBIB (3) designs and their applications to partial diallel crosses. Sharma *et al.* (2010) [32] introduced 3-associate-class tetrahedral and cubical association schemes and related PBIB(3) designs. Some light on investigations of 4-associate class PBIB designs was thrown by several authors such as Nair (1951) [21]; Tharthare (1963, 1965) [37-38]; Garg *et al.* (2011) [9]; Vinayaka and Vinaykumar (2021) [42], *etc.* Further, investigations on 2-replicate PBIB designs are limited to only Varghese and Sharma (2004) [39]; Sharma *et al.* (2010) [33]; Kipkemoi *et al.* (2013, 2015) [16-17] gave concept of affine resolvability including relationships between affine resolvability and variance balance using parametric relationships. Subsequently, described two methods for constructing affine resolvable block designs with different block sizes. Affine α-resolvable PBIB(2) designs has been investigated by Kadowaki and Kageyama (2092) [2]. For distinct block sizes, obtained some construction methods for affine resolvable rectangular type PBIB designs. Recently, a series of affine resolvable 2-replicate PBIB(4) designs with unequal blocks have been obtained by Jha *et al.* (2011) [10] and Vinaykumar *et al.* (2023) [43]. Apart from being used as block designs for multi-environmental trials, PBIB designs can find profound application in designing breeding trials *i.e.*, in obtaining mating-environmental designs involving a representative sample of crosses and partially replicated (p-rep) designs.
Here, we extend the work on 3-associate class non-proper PBIB designs further by proposing construction of PBIB (3) designs using star-polygon graph.

These designs fall under the category of resolvable designs which is used in information theory *i.e.*, constructing A^2-codes and low-density parity-check (LDPC) codes [Pei (2006); and Xu *et al*. (2015, 2020)] [46-47, 26] and in sequential experimentation over space and time [John and Williams (1995) [15]; and Morgan and Reck (2007)] [20]. However, many authors, among others, Harshbarger (1949) [12]; Bose and Nair (1962) [3]; David (1967) [5]; Patterson and Williams (1976) [45]; Williams *et al*. (1976, 1977) [44-45]; Jarrett and Hall (1978) [14]; Varghese and Sharma (2004) [39]; and Sharma *et al.* (2010) [33] were discussed on problems of construction and analysis of resolvable incomplete block designs.

The article is organized as follows: In Section 2, the star-polygon association scheme is defined along with numerical illustration. Section 3 deals with the construction of non-proper PBIB(3) designs using star polygon graph along with example. Section 4 reveals a brief discussion. At last, a table of efficient PBIB designs for $v \leq 100$ has also been obtained.

3. Star Polygon Association scheme

Let $v = 10s$ $(s \geq 2)$ be the number of treatments. Arrange these treatments on the vertices of a Star Polygon graph such that each vertex contains exactly m distinct treatments. Now we define the association scheme on these v treatments as follows: Treatment β is the first associate of α, if β lies on the same vertex of α; the second associate, if β lies on any of the quadruplets (where each quadruplet has precisely four equi-distant vertices) that intersect the vertex of α and third associates, otherwise [Vinayaka and Vinaykumar (2021)] [42]. The parameters of first kind and association matrices (called as parameters of second of kind) of the association scheme are given respectively: $v = 10s$, $n_1 = s - 1$, $n_2 = 6s$, $n_3 = 3s$, and

$$P_1 = \begin{bmatrix} s-2 & 0 & 0 \\ 0 & 6s & 0 \\ 0 & 0 & 3s \end{bmatrix},\ P_2 = \begin{bmatrix} 0 & s-1 & 0 \\ s-1 & 3s & 2s \\ 0 & 2s & s \end{bmatrix} \text{ and } P_3 = \begin{bmatrix} 0 & 0 & s-1 \\ 0 & 4s & 2s \\ s-1 & 2s & 0 \end{bmatrix}.$$

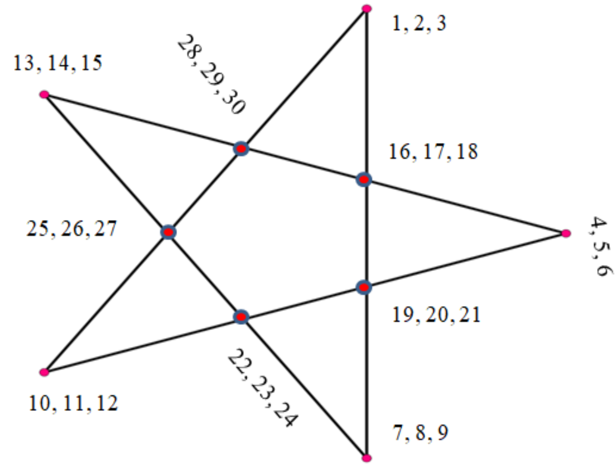

Fig 1: Arrangement of 30 treatments on vertices of a Star Polygon

Illustration 1: Let $v = 30\ (= 10 \times 3)$ treatments are arranged on the vertices of a Star Polygon graph such that each vertex contains exactly three distinct treatments as shown in Figure 1. Here, $n_1 = 2$, $n_2 = 18$, $n_3 = 9$ and the three associates of treatments, say, 1, 2, 7 and 22 are as given in Table 1. Association matrices of this illustration are as follows:

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 9 \end{bmatrix},\ P_2 = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 9 & 6 \\ 0 & 6 & 3 \end{bmatrix} \text{ and } P_3 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 12 & 6 \\ 2 & 6 & 0 \end{bmatrix}.$$

Table 1: Different associates of some treatments 1, 2, 7 and 13

Treatment	1st associates	2nd associates	3rd associates
1	2, 3	7, 8, 9, 10, 11, 12, 16, 17, 18, 19, 20, 21, 25, 26, 27, 28, 29, 30	4, 5, 6, 13, 14, 15, 22, 23, 24
2	1, 3	7, 8, 9, 10, 11, 12, 16, 17, 18, 19, 20, 21, 25, 26, 27, 28, 29, 30	4, 5, 6, 13, 14, 15, 22, 23, 24
7	8, 9	1, 2, 3, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27	4, 5, 6, 10, 11, 12, 28, 29, 30
13	14, 15	4, 5, 6, 7, 8, 9, 16, 17, 18, 22, 23, 24, 25, 26, 27, 28, 29, 30	1, 2, 3, 10, 11, 12, 19, 20, 21

Method of construction

Let $v = 10s$ $(s \geq 2)$ treatments are arranged on the vertices of a Star Polygon graph as indicated in the association scheme. By taking all possible triangles as blocks such that treatments situated on the three vertices of each triangle taken together as a block and also form another set of blocks of the design each one corresponding to a quadruplets by taking together the treatments that lie on the four vertices (points) of each quadruplet as a block, then combine all these blocks implies a non-proper PBIB(3) design based on Star Polygon association scheme with parameters $v = 10s$, $b_1 = 10$, $b_2 = 5$, $r = 5$, $k_1 = 3s$, $k_2 = 4s$, $\lambda_1 = 5$, $\lambda_2 = 2$, $\lambda_3 = 0$.

Illustration 2: Let $v = 30$ treatments for $s = 3$ are arranged on the vertices of a Star Polygon graph as given in Figure 1. By following the procedure of above method, we can get a PBIB(3) design based on Star Polygon association scheme with block contents as

(1, 2, 3, 16, 17, 18, 28, 29, 30)
(4, 5, 6, 16, 17, 18, 19, 20, 21)
(7, 8, 9, 19, 20, 21, 22, 23, 24)
(10, 11, 12, 22, 23, 24, 25, 26, 27)
(13, 14, 15, 25, 26, 27, 28, 29, 30)
(1, 2, 3, 7, 8, 9, 25, 26, 27)
(1, 2, 3, 10, 11, 12, 19, 20, 21)
(4, 5, 6, 13, 14, 15, 22, 23, 24)
(4, 5, 6, 10, 11, 12, 28, 29, 30)
(7, 8, 9, 13, 14, 15, 16, 17, 18)
(1, 2, 3, 10, 11, 12, 25, 26, 27, 28, 29, 30)
(1, 2, 3, 7, 8, 9, 16, 17, 18, 19, 20, 21)
(4, 5, 6, 10, 11, 12, 19, 20, 21, 22, 23, 24)
(4, 5, 6, 13, 14, 15, 16, 17, 18, 28, 29, 30)
(7, 8, 9, 13, 14, 15, 22, 23, 24, 25, 26, 27)

The parameters of this design are: $v = 30$, $b_1 = 10$, $b_2 = 5$, $r = 5$, $k_1 = 9$, $k_2 = 12$, $\lambda_1 = 5$, $\lambda_2 = 2$, $\lambda_3 = 0$.

A total of nine PBIB(3) designs for $v \leq 100$ generated by this method are listed in Table 2 along with their average variance $(\bar{V})$ and canonical efficiency factors (CEFs) as compared to a randomized complete block design. Details on canonical efficiency factors is available in Dey (2008). All the designs have good efficiency that is within the range of 0.8000 to 0.9628.

Table 2: PBIB(3) designs based on Star-polygon association scheme for $v \leq 100$

SI. No.	s	v	b_1	b_2	r	k_1	k_2	λ_1	λ_2	λ_3	$\bar{V}$	CEFs
1	2	20	10	5	5	6	8	5	2	0	0.4677	0.8553
2	3	30	10	5	5	9	12	5	2	0	0.4443	0.9002
3	4	40	10	5	5	12	16	5	2	0	0.4330	0.9239
4	5	50	10	5	5	15	20	5	2	0	0.4262	0.9384
5	6	60	10	5	5	18	24	5	2	0	0.4218	0.9483
6	7	70	10	5	5	21	28	5	2	0	0.4186	0.9555
7	8	80	10	5	5	24	32	5	2	0	0.4163	0.9609
8	9	90	10	5	5	27	36	5	2	0	0.4144	0.9651
9	10	100	10	5	5	30	40	5	2	0	0.4130	0.9686

Remark 1: For $s = 1$, this scheme also reduced to 2-associate class G_1 association scheme of Garg and Farooq (2014) [9]. Thus obtained design is non-proper PBIB(2) design with parameters $v = 10, b_1 = 10, b_2 = 5, r = 5, k_1 = 3, k_2 = 4, \lambda_1 = 0, \lambda_2 = 2, n_1 = 3, n_2 = 6$.

Discussion

The PBIB(3) designs with unequal block sizes obtained from the star polygon association scheme fall into the resolvable group of designs with minimal replications (*i.e.*, $r = 5$). The benefit of resolvable design is that its replications can be applied over different locations or over distinct time periods. Further, these are applicable when the experimenter's facing problem of constraint of resources. Moreover, efficiencies of these designs are quite high. Hence, these designs can be used to test a large number of cultivars in agricultural varietal trials. Also, the association schemes of these designs can be utilized in plant/animal breeding experiments to construct efficient partial diallel cross plans.

Acknowledgements

The author is thankful to Post Graduate School, ICAR-IARI, New Delhi, and ICAR-IASRI, New Delhi, India for providing the necessary facilities and the financial assistance.

References

1. Adhikari A, Bose M, Kumar D, Roy B. Applications of partially balanced incomplete block designs in developing (2, n) visual cryptographic schemes. IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences. 2007;90(5):949-951.
2. Bhagwandas Sinha K, Kageyama S. Constructions of PBIB designs based on nested group divisible association scheme. Utilitas Mathematica. 1992;41:169-174.
3. Bose RC, Nair KR. Resolvable incomplete block designs with two replications. Sankhya: The Indian Journal of Statistics, Series A. 1962, 9-24.
4. Clatworthy WH. Tables of two-associate-class partially balanced designs. National Bureau of Standards, Applied Mathematics, Series No. 63, Washington D.C; c1973.
5. David HA. Resolvable cyclic designs. Sankhya: The Indian Journal of Statistics, Series A; c1967. p. 191-198.
6. Dey A. Construction of regular group divisible designs. Biometrika. 1977;64:647-649.
7. Dey A. Canonical efficiency factors and related issues revisited. Journal of Indian Society of Agricultural Statistics. 2008;62(2):169-173.
8. Duan X, Kageyama S. Constructions of nested group divisible designs. Statistics and Probability Letters. 1993;18(1):41-48.
9. Garg DK, Farooq SA. Construction of PBIB designs through chosen lines and triangles of graphs. International Journal of Mathematics Trends and Technology. 2014;8(1):25-32.
10. Garg DK, Jhaji HS, Mishra G. Construction of Some New Triangular and Four Associate Class PBIB Designs with Two Replicates. International Journal of Mathematical Sciences and Applications. 2011;1(2):808-821.
11. Ghosh DK, Divecha J. Some new semi-regular group divisible designs. Sankhya. 1995;B57(3):453-455.
12. Harshbarger B. Triple rectangular lattices. Biometrics. 1949;5(1):1-13.
13. Hinkelmann K. Extended group divisible partially balanced incomplete block designs. The Annals of Mathematical Statistics. 1964;35:681-695.
14. Jarrett RG, Hall WB. Generalized cyclic incomplete block designs. Biometrika. 1978;65(2):397-401.
15. John JA, Williams ER. *Cyclic and computer-generated designs*. 2nd Edition. Chapman and Hall London; c1995.
16. Kipkemoi EC, Koske JK, Mutiso JM. Construction of three-associate class partially balanced incomplete block designs in two replicates. American Journal of Mathematical Science and Applications. 2013;1(1):61-65.
17. Kipkemoi EC, Mutiso JM, Bekolle D. Construction of some new three associate class partially balanced incomplete block designs in two replicates. International Journal of Academic Research in Progressive Education and Development. 2015;1:188-192.
18. Miao Y, Kageyama S, Duan X. Further constructions of nested group divisible designs. Journal of the Japan Statistical Society. 1996;26(2):231-239.
19. Mitra RK, Sinha K, Kageyama S, Singh MK. Constructions of group divisible and nested group divisible designs. Utilitas Mathematica. 2002;61:167-174.
20. Morgan JP, Reck BH. Resolvable designs with large blocks. The Annals of Statistics. 2007;35(2):747-771.
21. Nair KR. Rectangular lattices and partially balanced incomplete block designs. Biometrics. 1951;7(2):145-154.
22. Parsad R, Gupta VK, Srivastava R. Designs for cropping systems research. Journal of Statistical Planning and Inference. 2007a;137:1687-1703.
23. Parsad R, Kageyama S, Gupta VK. Use of complementary property of block designs in PBIB designs. Ars Combinatoria. 2007b;85:173-182.
24. Patterson HD, Silvey V. Statutory and recommended list trials of crop varities in the United Kingdom. Journal of the Royal Statistical Society: Series A. 1980;143(3):219-252.
25. Patterson HD, Williams E. A new class of resolvable incomplete block designs. Biometrika. 1976;63(1):83-92.
26. Pei D. A survey of construction for A2-codes. *In* Authentication codes and combinatorial designs. Discrete mathematics and its applications, ed. K. H. Rosen. Chapman and Hall/CRC, New York; c2006. p. 215-30.
27. Raghavarao D. A generalization of group divisible designs. The Annals of Mathematical Statistics; c1960. p. 756-771.

28. Rao CR. A general class of quasifactorial and related designs. Sankhya: The Indian Journal of Statistics. 1956;17(2):165-174.
29. Richeson DS. Euler's gem: The polyhedron formula and the birth of topology. Princeton University Press, Princeton, New Jersey; c2019.
30. Roy PM. Hierarchical group divisible incomplete block designs with m-associate classes. Science and Culture. 1953;19:210–211.
31. Saurabh S, Sinha K. A list of new partially balanced designs. Communications in Statistics-Theory and Methods; c2022. p. 1-4. doi: https://doi.org/10.1080/03610926.2022.2059685.
32. Sharma VK, Das MN. On resolvable incomplete block designs 1. Australian Journal of Statistics. 1985;27(3):298-302.
33. Sharma VK, Varghese C, Jaggi S. Tetrahedral and cubical association schemes with related PBIB(3) designs. Model Assisted Statistics and Applications. 2010;5(2):93-99.
34. Sinha K. A list of new group divisible designs. Journal of Research of the National Institute of Standards and Technology. 1991;96:613-615.
35. Srivastava R, Parsad R, Gupta VK. Structure resistant factorial designs. Sankhya: The Indian Journal of Statistics, Series B. 2000;62(2):257-265.
36. Suen CY. Some rectangular designs constructed by the method of differences. Journal of Statistical Planning and Inference. 1989;21(2):273-276.
37. Tharthare SK. Right angular designs. The Annals of Mathematical Statistics; c1963. p. 1057-1067.
38. Tharthare SK. Generalized right angular designs. The Annals of Mathematical Statistics; c1965. p. 1535-1553.
39. Varghese C, Sharma VK. A series of resolvable PBIB(3) designs with two replicates. Metrika. 2004;60(3):251–254.
40. Varghese C, Sharma VK, Jaggi S, Sharma A. Three-associate class partially balanced incomplete block designs and their application to partial diallel crosses. Project report, IASRI publication, New Delhi; c2004.
41. Vartak MN. On an application of Kronecker product of matrices to statistical designs. The Annals of Mathematical Statistics. 1955;26(3):420-438.
42. Vinayaka, Vinaykumar LN. Higher associate class partially balanced incomplete block designs using graphs for agricultural experiments. The Pharma Innovation Journal. 2021;10(12S):967-972. DOI: 10.22271/tpi.2021.v10.i12Sn.11252.
43. Vinaykumar LN, Varghese C, Harun M, Karmakar S. Minimally replicated PBIB designs for multi-environmental trials, Communications in Statistics - Theory and Methods, Advance online publication; c2023. DOI: 10.1080/03610926.2023.2185753.
44. Williams ER, Patterson HD, John JA. Resolvable designs with two replications. Journal of the Royal Statistical Society. Series B (Methodological); c1976. p. 296-301.
45. Williams ER, Patterson HD, John JA. Efficient two-replicate resolvable designs. Biometrics; c1977. p. 713-717.
46. Xu H, Feng D, Sun C, Bai B. Construction of LDPC codes based on resolvable group divisible designs. In 2015 International Workshop on High Mobility Wireless Communications (HMWC); c2015. p. 111-115.
47. Xu H, Yu Z, Feng D, Zhu H. New construction of partial geometries based on group divisible designs and their associated LDPC codes. Physical Communication. 2020;39:1-38. DOI: https://doi.org/10.1016/j.phycom.2019.100970