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Magero Eric Nyaanga
School of Biological Physical Mathematics and Actuarial Sciences, Jaramogi Oginga Odinga University of Science and Technology PO Box 210-40601, Bondo, Kenya

Oduor D Brian
School of Biological Physical Mathematics and Actuarial Sciences, Jaramogi Oginga Odinga University of Science and Technology, PO Box 210 – 40601, Bondo, Kenya

Muga M Zablon
School of Biological Physical Mathematics and Actuarial Sciences, Jaramogi Oginga Odinga University of Science and Technology, PO Box 210 – 40601, Bondo, Kenya

Corresponding Author:
Magero Eric Nyaanga
School of Biological Physical Mathematics and Actuarial Sciences, Jaramogi Oginga Odinga University of Science and Technology PO Box 210 – 40601, Bondo, Kenya

Derivation of the asset pricing equation using Ornstein-Uhlenbeck model with transaction cost

Magero Eric Nyaanga, Oduor D Brian and Muga M Zablon

Abstract
The Ornstein-Uhlenbeck process has gained attention in the field of finance as a means of modelling the volatility of the underlying asset price. It is a stochastic process that models the behaviour of a variable over time. It is often used to model the mean-reverting process where the variable tends to return to a long-term average over time. It can model interest rates, volatilities and many more financial variables. The Ornstein-Uhlenbeck model offers a valuable framework for analysing and predicting the volatility of financial assets, allowing researchers to gain deeper insights into the dynamics of market fluctuations and make more informed decisions. It is notable that option pricing models assumes that there is no transaction cost in trading. To better reflect the dynamic nature of the market and accommodate real-world conditions, it is necessary to develop a model that considers this factor. Based on the findings of this study, we have applied a similar approach by incorporating transaction costs into the Ornstein-Uhlenbeck model and developed an asset pricing model using the Ornstein-Uhlenbeck model with transaction costs. This model allows for a comprehensive analysis of the factors influencing asset prices, including volatility and transaction costs. It also helps investors to gain insights into the impact of transaction costs on option prices within the context of the Ornstein-Uhlenbeck model, providing a more comprehensive understanding of the dynamics involved.

Keywords: Ornstein-Uhlenbeck, business operations and management, transaction cost

1. Introduction
Transaction costs, being a vital factor in business operations and management, should be integrated into option pricing models. By incorporating transaction costs, these models can better capture the evolving dynamics of market capital prices. The study conducted has demonstrated how transaction costs can be incorporated into the Ornstein-Uhlenbeck model and has further highlighted the effects of transaction costs and changes in volatility on option pricing models. This model contributes to a more comprehensive understanding of market dynamics by accounting for transaction costs and volatility, allowing for more accurate pricing and assessment of options in the changing landscape of financial markets. By incorporating these factors, the model enables investors and decision-makers to make more informed choices in their investment strategies.

2. Preliminaries
2.1. Ornstein-Uhlenbeck
The Ornstein-Uhlenbeck model is a widely utilized mean reversion model in finance, particularly for modelling interest rates and commodities markets. It is employed for simulation and estimation purposes, allowing researchers and practitioners to analyse the behaviour and dynamics of these markets. The process serves as a valuable tool in understanding the patterns of mean reversion and capturing important characteristics of interest rates and commodities in financial modelling [7]. Ornstaein-Uhlenbeck model, also referred to as the process is the most known model for such work [8]. In recent times, the Ornstein-Uhlenbeck model has emerged in the field of finance as a viable model for capturing the volatility inherent in asset processes. This process serves as an extension of the Vasicek model and is particularly well-suited for implementing maximum likelihood estimation techniques.

The Ornstein-Uhlenbeck model is governed by a stochastic differential equation, which describes its dynamic behaviour and allows for accurate modelling and estimation of volatility in financial contexts [7].

$$dx_t = \lambda(\mu - X_t)dt + \sigma dw_t \quad (1)$$

2.2. Transaction Cost

In the field of economics and related disciplines, the concept of transaction cost refers to the expenses incurred when engaging in economic transactions within a market. It represents the costs involved in conducting trade and participating in market activities. These costs encompass various factors such as fees, charges, commissions, and other expenditures associated with executing and facilitating economic exchanges. Transaction costs play a significant role in the overall efficiency and functioning of markets. The notion that transactions are fundamental to economic analysis was first proposed by John R. Commons, an institutional economist, in 1931. Later, in 2008, Oliver E. Williamson expanded on this concept in his influential article on Transaction Cost Economics [11], made the concept of transaction costs popular [5]. Douglass C. North contends that institutions, which refer to the framework of rules within a society, play a crucial role in shaping transaction costs. According to North, institutions that effectively reduce transaction costs can contribute to economic growth and development [4]. According to Williamson, transaction costs encompass the expenses associated with operating a company within an economic system. Unlike production costs, decision-makers assess both transaction costs and production costs when formulating strategies for their organizations. Transaction costs encompass the overall expenses incurred throughout the transaction process, including planning, decision-making, plan adjustments, dispute resolution, and post-sales activities [2]. Therefore, the transaction cost is one of the key significant factors in business operation and management [9]. To accurately reflect the dynamics of market capital price changes, it is necessary to develop a model that aligns more closely with market realities. The original Black-Scholes option pricing model did not incorporate transaction expenses as a variable, despite their significance to traders. However, in actual transactions, different customers and exchanges may impose varying transaction fee structures for different derivative products or quantities of the same product. Notably, when investors engage in frequent trading to manage risks and generate modest profits, transaction costs gradually become significant as the number of transactions increases. As a result, the initial model gradually deviates from the actual scenario. Unlike the Black-Scholes formula, which assumes no transaction costs on the underlying asset, a more realistic model should account for these costs [10]. In their study, [33] focused on the optimal investment and proportional reinsurance strategy using the mean reverting Ornstein-Uhlenbeck model and net profit condition. They incorporated transaction costs into a geometric Brownian motion framework and found that insurance companies can maximize long-term returns on investment and reinsurance while reducing operational risks through effective trading. Building on their approach, we aim to incorporate transaction costs into the Ornstein-Uhlenbeck model and derived an option pricing model.

2.3 Stochastic processes

A variable whose value changes over time in an uncertain manner is referred to as following a stochastic process. Stochastic processes are mathematical models that describe the random evolution of variables or systems. They are widely used in various fields, including finance, physics, economics, and engineering, to capture and analyse the uncertainty and randomness inherent in many real-world phenomena. Hence it obeys laws of probability. Mathematically, a stochastic is process $X = [X(t); t\epsilon(0, \alpha)]$ a collection of random variables such that for each t in the index set $(0, \alpha), X(t)$ is a random variable where $X(t)$ is the state of the process at time t. A discrete time stochastic process is characterized by changes in the variable occurring only at specific, fixed points in time.

2.4 Markov process

A Markov process is characterized by its inherent unpredictability in terms of future behaviour, as it cannot be accurately predicted solely based on past observations except for the current state. It involves elements of randomness or probability. Various phenomena, such as the behaviour of businesses or economies (e.g., stock prices), traffic flow, or the progression of an epidemic, can be described as Markov processes. The term" Markov" originates from the name of the Russian mathematician Andrei Andreevich Markov (1856-1922), who developed Markov analysis. In essence, a Markov stochastic process is a specific type of stochastic process in which only the current value of a variable holds relevance for predicting future movements. Consequently, the past does not exert a deterministic influence on the future.

2.5 Geometric Brownian motion

Geometric Brownian Motion (GBM) is a stochastic process commonly used to model the behaviour of asset prices, particularly in the field of quantitative finance. It is a continuous-time process that incorporates both drift and randomness.

In the context of financial markets, the GBM model assumes that the logarithm of the asset price follows a Brownian motion with drift. Mathematically, the GBM can be represented as: Samuelson [6] and Black [1].

$$ds(t) = \mu s(t)dt + \sigma s(t)dwt \quad (2)$$

Where, $S(t)$ represents the asset price at time t, μ is the drift coefficient that determines the expected rate of return or growth rate of the asset price, σ is the volatility coefficient that measures the level of randomness or uncertainty, $dW(t)$ represents the differential of a Wiener process or Brownian motion, dt denotes the differential of time. The term $\mu S(t)dt$ accounts for the deterministic component or the expected growth rate of the asset price, while the term $\sigma S(t)dW(t)$ represents the stochastic or random component that incorporates the volatility of the asset price.

2.6 Brownian motion

Brownian motion is a concept that illustrates the inherent randomness and unpredictability observed in the behaviour of microscopic particles suspended in a fluid or gas. When it comes to financial investment, Brownian motion plays a crucial role in the development of mathematical models used to understand and predict the behaviour of asset prices. The Geometric Brownian Motion (GBM) model, which incorporates random fluctuations, is widely used to simulate and forecast asset price movements.

2.7 Generalized Wiener process

The basic Wiener process dZ that has been developed so far has a drift rate of 0 and a variance rate of 1.0. The drift rate of 0 means that the expected value of Z at any future time is equal to its current value. The variance rate of 1.0 means that the variance of the change in Z in time interval of length T equals T. A generalised Wiener process for a variable X can be defined in terms of dZ as;

$$dX = adt + bdz \quad (3)$$

Where mean rate a and variance rate b are constants, adt is the expectation of dX and bdZ is the addition of noise or variability to the path followed by X, while b is the diffusivity.

3. Main Results

3.1. Derivation of the asset pricing equation using Ornstein-Uhlenbeck model with transaction cost

Ornstein-Ulenbeck model is an extended Vasicek model suitable in applying the maximum likelihood estimation given by stochastic differential equation as;

$$dx_t = \lambda(\mu - X_t)dt + \sigma dw_t \quad (4)$$

According to Li, *et al.* (2020) [3], for a proper investment strategy, one aspect that needs to be consistent but assumed by Black- Scholes equation is the transaction cost. From equation (4) we therefore incorporate transaction cost as done by Li, *et al.* (2020) [3] to give.

$$dx_t = \lambda(\mu - X_t - \theta)dt + \sigma dw_t \quad (5)$$

Where; λ is the rate of mean reversion.
μ is the equilibrium force or value around X_t which tends to oscillate.
Ɵ is the transaction cost
σ is the volatility.
Therefore $\lambda(\mu - X - \theta)$ gives the drift term rate.

When X_t: Ɵ > μ, it results to negative value and if X_t: Ɵ < μ it results to positive value that pulls back to a higher equilibrium rate.
Solving equation (5) involves taking derivatives of $e^{\lambda t}X_t$ which yields.

$$d(e^{\lambda t}X_t) = X_t\lambda e^{\lambda t}dt + e^{\lambda t}dX_t \quad (6)$$

When we multiply both sides of equation (2) by $e^{\lambda t}$, gives

$$e^{\lambda t}dX_t = e^{\lambda t}\lambda(\mu - X_t - \theta)dt + e^{\lambda t}\sigma dw_t \quad (7)$$

Opening the brackets in equation (8) and substituting equation (6) we obtain.

$$d(e^{\lambda t}X_t) = \lambda e^{\lambda t}\mu dt - \lambda e^{\lambda t}\theta dt + e^{\lambda t}\sigma dw_t \quad (8)$$

When we take integral from time $t = 0$ to t we obtain

$$e^{\lambda t}X_t = X_0 + \int_0^t \lambda e^{\lambda s}\mu ds - \int_0^t \lambda e^{\lambda s}\theta ds + \int_0^t e^{\lambda s}\ \sigma dw_s \quad (9)$$

Which yields

$$X_t = X_0 e^{-\lambda t} + \int_0^t \lambda e^{-\lambda(t-s)}\mu ds - \int_0^t \lambda e^{-\lambda(t-s)}\theta ds + \int_0^t e^{-\lambda(t-s)}\ \sigma dw_s \quad (10)$$

$$X_t = X_0 e^{-\lambda t} + \mu(1 - e^{-\lambda t}) - \theta(1 - e^{-\lambda t}) + \int_0^t e^{-\lambda(t-s)}\ \sigma dw_s \quad (11)$$

The second component on the right-hand side of the equation (11) is $\int_0^t e^{-\lambda(t-s)}\ \sigma dw_s$ which follows a normal distribution with mean zero and its variance.

$$E[\left(\int_0^t e^{-\lambda(t-s)}\ \sigma dw_s)^2\right] = (\int_0^t e^{-\lambda(t-s)}\ \sigma)^2 ds = \frac{\sigma^2}{2\lambda}(1 - e^{-2\lambda t}) \quad (12)$$

The mean will be given by

$$E(X_t) = \mu - \theta + (X_0 - \mu_0 + \theta_0)e^{-\lambda t} \quad (13)$$

The variance will also be given by

$$\text{Var}\,(X_t) = \frac{\sigma^2}{2\lambda}(1 - e^{-2\lambda t}) \quad (14)$$

As time goes on, the Ornstein-Uhlenbeck model tends to settle around its long-term mean while its randomness becomes less prominent, resulting in a distribution that becomes increasingly Gaussian. This means that equation (13) and equation (14) can be expressed as

$$X_t = X_0 e^{-\lambda t} + \mu(1 - e^{-\lambda t}) - \theta(1 - e^{-\lambda t}) + \sigma\sqrt{\frac{1-e^{-2\lambda t}}{2\lambda}}W \quad (15)$$

This is the asset pricing equation using Ornstein-Uhlenbeck model with transaction cost.

4. Conclusion

In this paper we have derived an asset pricing equation using Orstein – Uhlenbeck model with transaction cost that leads into tractable solutions for a number of financial challenges that can help investors in making viable decisions when setting up their investment strategies.

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