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Rhotrix and its application in construction of balance incomplete block design

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Abstract
Balanced Incomplete Block Designs (BIBD) play a pivotal role in experimental design, facilitating efficient and structured data collection. This study explores a novel approach to constructing BIBD designs through the innovative mathematical framework of Rhotrices. Rhotrices, situated between consecutive square matrices, introduce a unique algebraic system reminiscent of matrix algebra. In this research, we delve into the fundamental concepts of Rhotrices and their application in BIBD construction.

Keywords: Rhotrix, coupled matrix, heart of rhotrix, Hadamard rhotrix, BIBD

1. Introduction
Rhotrix theory, a relatively recent scientific paradigm, particularly within the realm of mathematics, was initially introduced by Ajibade (2003) [1]. Since its inception, numerous researchers have demonstrated a keen interest in advancing both the theories and practical applications of rhotrices over the past decade. It is important to note that the concept of rhotrices is still in its early stages of development, yet researchers have been actively engaged in its expansion. This expansion often draws parallels with matrix concepts, frequently involving transformations that convert matrices into rhotrices and vice versa. Notable references in this regard include the works of Sani (2004, 2007, 2008) [7-9], Tudunkaya and Makanjuola (2010) [14], Sharma and Kanwar (2011, 2012a, 2012b) [10-12], Sharma *et al.* (2013) and Isere (2016), among others. In contemporary mathematical and statistical research, rhotrices have gained significant momentum, finding applications across various domains. One noteworthy application is in the field of experimental design, specifically for constructing Balanced Incomplete Block Designs (BIBD), as exemplified in the work of Sharma and Kumar (2014) [13]. In this paper, we introduce the fundamental concepts of Rhotrix and its core components, followed by a brief overview of preliminary Rhotrix algebra. Subsequently, we define BIBD and highlight the application of Rhotrix in the construction of BIBD, drawing inspiration from the approach outlined by Sharma and Kumar (2014) [13].

2. Preliminaries
2.1 Rhotrix
Mathematical arrays that are in some way between 2-dimensinal and 3-dimensional square matrix are termed as rhotrix [Ajibade (2003)] [1]. The word rhotrix is due to rhomboidal arrangements of matrix elements. The smallest rhotrix R of order 3 can be defined as.

$$R = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle : \text{a, b, c, d and e} \in \Re$$

2.1.1 Heart of Rhotrix
The element at perpendicular intersection of the two diagonals of any Rhotrix is known as the heart of the Rhotrix. Therefore, the heart of the above Rhotrix R which is denoted by $h(R)$ is c i.e., $h(R) = c$. It is to be mentioned that a Rhotrix is always of odd order i.e., if there exist a

Rhotrix of order n, then n must be odd. The cardinality of the Rhotrix R of order n will be $\frac{(n^2+1)}{2}$. Therefore, the cardinality of the above Rhotrix R of order 3 is 5.
Let following is a n dimensional Rhotrix (R_n)

$$R_n = \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & & a_{31} & c_{21} & a_{22} & c_{21} & a_{13} & & \\ & & & & \vdots & & & & \\ & a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} & \\ & & & & \vdots & & & & \\ a_{t(t-2)} & & c_{(t-1)(t-2)} & & a_{(t-1)(t-1)} & & c_{(t-2)(t-1)} & & a_{(t-2)t} \\ & & & a_{t(t-2)} & c_{(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle$$

Here
t = $\frac{(n+1)}{2}$. In the above Rhotrix so $n = 3$ then $t = 2$ similarly if $n = 5$ then $t = 3$ and so on.

Example: Following are two examples of Rhotrix of order 3 (R_3) and 5 (R_5) respectively

$$R_3 = \left\langle \begin{array}{ccc} & 1 & \\ 4 & 3 & 6 \\ & 5 & \end{array} \right\rangle$$

$$R_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 4 & 3 & 6 & \\ 7 & 7 & 5 & 0 & 3 \\ & 1 & 2 & 3 & \\ & & 9 & & \end{array} \right\rangle$$

2.1.2 Coupled matrices of a Rhotrix

It is to be noted that, the transpose of a matrix is an operator which flips a matrix over its diagonal, that is equivalent to rotating columns of the matrix by 90° in anti-clock wise direction. The coupled matrices of Rhotrix can be obtained by rotating its columns by 45° in anti-clock wise direction instead of 90°. By doing so, first one can, we get a special matrix with missing value of order n.
Let's a rhotrix of order 5 i.e., R_5 is

$$R_5 = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & a_{22} & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle$$

$$R_5^{\frac{T}{2}} = \begin{bmatrix} a_{11} & & a_{12} & & a_{13} \\ & c_{11} & & c_{12} & \\ a_{21} & & a_{22} & & a_{23} \\ & c_{21} & & c_{22} & \\ a_{31} & & a_{32} & & a_{33} \end{bmatrix}$$

Here, $\frac{\boldsymbol{T}}{\boldsymbol{2}}$ indicates half rotation in compare to transpose but direction is same (anti-clock wise). We observe two coupled matrices, higher order matrix **A** is known as major matrix and lower order matrix **C** is known as minor matrix,

$$\boldsymbol{A}_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } \boldsymbol{C}_{2\times 2} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.$$

In general, for a Rhotrix of order n (R_n), one can have

$R_n^{\frac{T}{2}} = \langle a_{ij}, c_{lk} \rangle^{\frac{T}{2}} = [\boldsymbol{A}_{ij}\ \boldsymbol{C}_{lk}] = [\boldsymbol{A}\ \boldsymbol{C}]$ (say), where $i, j = 1,2,3 \ldots, t$ and $l, k = 1,2,3 \ldots (t-1)$.

So, **A** and **C** are coupled square matrices of order t and (t-1), where $t = \frac{(n+1)}{2}$.

Example: let's take a rhotrix of order 3 (R_3)

$$R_3 = \left\langle \begin{array}{ccc} & 1 & \\ 4 & 3 & 6 \\ & 5 & \end{array} \right\rangle$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 1 & 6 \\ 4 & 5 \end{bmatrix} \; and \; \boldsymbol{C}_1 = [3]$$

as here minor matrix is of order 1 which can be consider as a scalar.
Now consider an order 5 (R_5) Rhotrix

$$R_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 4 & 3 & 6 & \\ 7 & 7 & 5 & 0 & 3 \\ & 1 & 2 & 3 & \\ & & 9 & & \end{array} \right\rangle$$

$$\boldsymbol{A}_5 = \begin{bmatrix} 1 & 6 & 3 \\ 4 & 5 & 3 \\ 7 & 1 & 9 \end{bmatrix} \; and \; \boldsymbol{C}_2 = \begin{bmatrix} 3 & 0 \\ 7 & 3 \end{bmatrix}$$

2.1.3 Transpose of Rhotrix

The transpose of a Rhotrix can be obtained by exchanging row with corresponding column of a Rhotrix. Let, R be a rhotrix of order 3 as

$$R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \quad \text{then } R^{T} = \left\langle \begin{array}{ccc} & a & \\ d & h(R) & b \\ & e & \end{array} \right\rangle$$

The transpose of two Rhotrices R and Q of same order hold the property $(RQ)^T = Q^T R^T$.

Example: $R_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 4 & 3 & 6 & \\ 7 & 7 & 5 & 0 & 3 \\ & 1 & 2 & 3 & \\ & & 9 & & \end{array} \right\rangle$ so, its transpose as

$$(R)^T = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 6 & 3 & 4 & \\ 3 & 0 & 5 & 7 & 7 \\ & 3 & 2 & 1 & \\ & & 9 & & \end{array} \right\rangle$$

2.1.4 Addition of Rhotrix

The addition of Rhotrix is similar to matrix. For two Rhotrices of same order, add the corresponding element. Suppose, R and Q are two Rhotrices of order 3, then the addition operation can be performed as

$$R + Q = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{array} \right\rangle$$

<R +> is a commutative group same as matrix, with identity element $O = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle$

For addition of two Rhotrices, following are satisfied.

- **Closed under operation**: If R and Q are two real Rhotrices of same order n then their addition is also a real Rhotrix of same order.
- **Associative**: If there exist another Rhotrix P of same order n, then $R + (Q + P) = (R + Q) + P$.

- **Existence of Inverse**: $R + Q = O$, where O is identity element then $Q = -R$.
  Example: Let the two rhotrices R and Q of order 5 as

$$R_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 4 & 3 & 6 & \\ 7 & 7 & 5 & 0 & 3 \\ & 1 & 2 & 3 & \\ & & 9 & & \end{array} \right\rangle \text{ and } \quad Q_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 5 & 9 & 8 & \\ 2 & 3 & 5 & 0 & 1 \\ & 2 & 8 & 7 & \\ & & 5 & & \end{array} \right\rangle$$

$$R_5 + Q_5 = \left\langle \begin{array}{ccccc} & & 2 & & \\ & 9 & 12 & 14 & \\ 9 & 10 & 10 & 0 & 4 \\ & 3 & 10 & 10 & \\ & & 14 & & \end{array} \right\rangle$$

2.1.5 Scalar multiplication

Multiplication of a scalar □ $\in \Re$ and a real Rhotrix R, results in multiplication of each element with the scalar □.
Let

$$\text{R} = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \text{ then } □\text{R} = \left\langle \begin{array}{ccc} & \alpha a & \\ \alpha b & \alpha h(R) & \alpha d \\ & \alpha e & \end{array} \right\rangle$$

Example: Let take a real □□as 3 and a Rhotrix R of order 3 as

$$R_3 = \left\langle \begin{array}{ccc} & 1 & \\ 4 & 3 & 6 \\ & 5 & \end{array} \right\rangle$$

$$3R_3 = \left\langle 3 \quad \begin{array}{ccc} & 1 & \\ 4 & 3 & 6 \\ & 5 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 3 & \\ 12 & 9 & 18 \\ & 15 & \end{array} \right\rangle$$

2.1.6 Multiplication of Rhotrix

To establish similarity between rhotrix and matrix first we have to define row and column of rhotrix. Let, R be a Rhotrix of order 3 which is given as

$$\text{R} = \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array}$$

Then, rows of R can be viewed as (a d), (b e) whereas the columns can be viewed as (a b), (d e). Rhotrix multiplication is then same as matrix row-column multiplication applied in both the coupled matrices. For example,

$$\text{R} \circ \text{Q} = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & af + dg & \\ bf + eg & h(R)h(Q) & aj + dk \\ & bj + ek & \end{array} \right\rangle$$

However, multiplication of rhotrix is non-commutative but associative as like matrix.

The generalization of Rhotrix multiplication by taking two rhotrices of n order as

$$R_n = \langle a_{ij} \; c_{lk} \rangle \text{ and } \quad Q_n = \langle b_{pq} \; d_{rs} \rangle$$

where $i, j = 1,2,3 \ldots t$ and $l, k = 1,2,3 \ldots (t-1)$ can be expressed as

$$R_n Q_n = \langle a_{ij} \; c_{lk} \rangle \times \langle b_{pq} \; d_{rs} \rangle = \left\langle \sum_{p,j}^{t} (a_{ij} \cdot b_{pq}), \sum_{r,k}^{t-1} (c_{lk} \cdot d_{rs}) \right\rangle$$

Example

$$R_3 = \left\langle \begin{array}{ccc} & 1 & \\ 4 & 3 & 6 \\ & 5 & \end{array} \right\rangle \; and \quad Q_3 = \left\langle \begin{array}{ccc} & 1 & \\ 2 & 2 & 3 \\ & 1 & \end{array} \right\rangle$$

$$R_3 \circ Q_3 = \left\langle \begin{array}{ccc} & 1 & \\ 4 & 3 & 6 \\ & 5 & \end{array} \right\rangle \times \left\langle \begin{array}{ccc} & 1 & \\ 2 & 2 & 3 \\ & 1 & \end{array} \right\rangle$$

$$R_3 \circ Q_3 = \left\langle \begin{array}{ccc} & 13 & \\ 14 & 6 & 9 \\ & 17 & \end{array} \right\rangle$$

Multiplicative identity

The identity property of multiplication states that when any Rhotrix I is multiplied by any other rhotrix R, the rhotrix R does not change.

R∘I = I∘R = R

Let R = $\left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ and I = $\left\langle \begin{array}{ccc} & f & \\ g & h(I) & j \\ & k & \end{array} \right\rangle$ be the multiplicative identity then

R∘I = $\left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & af+dg & \\ bf+eg & h(R)h(Q) & aj+dk \\ & bj+ek & \end{array} \right\rangle$

Comparing LHS and RHS, one can get

$af + dg = a$

$h(R)h(I) = h(R)$

$bj + ek = e$

$bf + eg = b$

$aj + dk = d$

by solving these equations, we get $h(I) = 1, f = 1, k = 1 \; and \; g = j = 0$ so,

I = $\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle$

Rhotrix I is known as **identity rhotrix**, the coupled matrices of identity rhotrix are identity matrix.

$$I_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 & \\ & & 1 & & \end{array} \right\rangle \Rightarrow (I_5)^{\frac{T}{2}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; and \; \boldsymbol{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiplicative Inverse

A multiplicative inverse or reciprocal for a rhotrix R, denoted by R^{-1}, is a rhotrix which when multiplied by R yields the multiplicative identity i.e., an identity Rhotrix I. Therefore,

$$R \circ R^{-1} = I \Rightarrow \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & af+dg & \\ bf+eg & h(R)h(Q) & aj+dk \\ & bj+ek & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle$$

On comparing both sides we get,

$af + dg = 1$

$bj + ek = 1$

$bf + eg = 0$

$aj + dk = 0$

$and\ h(R)h(I) = 1$

Solving- From $h(R)h(I) = 1 \Rightarrow h(I) = 1/h(R)$

Now, from the above equations, one can write

$f = \frac{1-dg}{a} = \frac{e+dbf}{ae} \Rightarrow f = \frac{e}{ae-bd}$,

Similarly,

$k = \frac{a}{ae-bd}$, $g = \frac{-b}{ae-bd}$, $j = \frac{-d}{ae-bd}$ provided ae-bd ≠ 0 and h(R) ≠0

Therefore,

$$R^{-1} = \left\langle \begin{array}{ccc} & \frac{e}{ae-bd} & \\ \frac{-b}{ae-bd} & \frac{1}{h(R)} & \frac{-d}{ae-bd} \\ & \frac{a}{ae-bd} & \end{array} \right\rangle$$

A Rhotrix R is invertible if both the coupled matrices of R i.e. $\mathbf{A}_t$ and $\mathbf{C}_{t-1}$ are non-singular.

If, $\boldsymbol{A}^{-1} = [q_{ij}]_{t\times t}$ and $\boldsymbol{C}^{-1} = [r_{lk}]_{t-1\times t-1}$, then

$R^{-1} = \langle q_{ij}\ c_{lk} \rangle$

2.1.7 Determinant of rhotrix

Let $R_3 = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ with the coupled matrices of R_3 as $\boldsymbol{A} = \begin{bmatrix} a & d \\ b & e \end{bmatrix}$ and $\boldsymbol{C} = [h(R)]$. Then the determinant of R_3 is

$|R_3| = |\boldsymbol{A}||\boldsymbol{C}| = h(R)(ae - bd)$

Thus, product of determinant of the coupled matrix is the determinant of the rhotrix itself.

Example: Consider a Rhotrix of order 3, $R_3 = \left\langle \begin{array}{ccc} & 1 & \\ 4 & 3 & 6 \\ & 5 & \end{array} \right\rangle$

$\boldsymbol{A}_2 = \begin{bmatrix} 1 & 6 \\ 4 & 5 \end{bmatrix}$ and $\boldsymbol{C}_1 = [3] |\boldsymbol{A}| = [5 - 24] = -19$ and $|\boldsymbol{C}| = 3$

$|R_3| = |\boldsymbol{A}||\boldsymbol{C}| = -19 \times 3 = -57$

Theorem: For two n dimensional Rhotrices R_n and Q_n, $|R_n \circ Q_n| = |R_n||Q_n|$

Proof

Let $R_n = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ and $Q_n = \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$

$$R_n \circ Q_n = \left\langle \begin{array}{ccc} & af + dg & \\ bf + eg & h(R)h(Q) & aj + dk \\ & bj + ek & \end{array} \right\rangle$$

Then,

$$|R_n \circ Q_n| = h(R)h(Q)[(af + dg)(bj + ek) - (bf + ge)(aj + dk)]$$

$$= h(R)h(Q)[(abfj + degk + bdgj + aefk) - (abfj + degk + bdfk + aegi)]$$

$$= h(R)h(Q)[(aefk + aegi) - (bdfk + bdgi)]$$

$$= h(R)h(Q)[(ae - bd)(fk - gj)]$$

$$= h(R)(ae - bd)h(Q)(fk - gj)$$

$|R_n \circ Q_n| = |R_n||Q_n|$ Hence proved.

2.2 Hadamard Rhotrix in Finite Field A rhotrix H_n is defined as Hadamard rhotrix over GF(2^q) if it can be represented as follow

$$H_n = \left\langle \begin{array}{ccc} & U & \\ V & U & V \\ & U & \end{array} \right\rangle$$

where and U and V are two Hadamard sub-rhotrices of the n-dimensional rhotrix H_n.

For example: Hadamard rhotrix of order 7 is.

$$H_7 = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & 1 & 0 & 1 & 1 & 1 & \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 0 & 1 & 1 & 1 & 1 & \\ & & 1 & 1 & 0 & & \\ & & & 1 & & & \end{array} \right\rangle$$

2.3 Balanced Incomplete Block Designs (BIBD)

A BIB design is an arrangement of v treatments in b blocks each of size k (<v) such that

(i) Each treatment occurs at most once in a block

(ii) Each treatment occurs in exactly r blocks

(iii) Each pair of treatments occurs together in exactly λ blocks.

Example: Following is a BIB design for $v = b = 5, r = k = 4$ and $\lambda = 3$:

(1,2,3,4)
(1,2,3,5)
(1,2,4,5)
(1,3,4,5)
(2,3,4,5)

The symbols v, b, r, k and λ are the parameters of the design which satisfy the following relations.

$$vr = bk \,, b \geq v$$

and $\lambda(v - 1) = r(k - 1)$

3. Methodology

Following section deals with construction of BIBD using Rhotrix.

Case 1: Let, the order of the Hadamard rhotrix over GF(2) is $n = 8t + 1$. Now, if H_n be a Hadamard rhotrix over GF(2) having two coupled matrixes of order $4t + 1$ and $4t$, where $n = 8t + 1$, then there exists symmetric BIB design with parameter. $v = b = 4t, r = k = 3t, \lambda = 2t$ from the matrix of order $4t + 1$ and $v = b = 4t - 1, r = k = 3t - 1, \lambda = 2t - 1$ from the matrix of order $4t$. Following is an example.

Let $H_9 = \left\langle \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & 0 & 1 & & & \\ & & 0 & 1 & 1 & 1 & 1 & & \\ & 0 & 1 & 1 & 1 & 0 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ & 0 & 0 & 1 & 1 & 0 & 1 & 1 & \\ & & 1 & 1 & 1 & 0 & 1 & & \\ & & & 1 & 1 & 0 & & & \\ & & & & 1 & & & & \end{array} \right\rangle$ so, n=9= 8t+1 with t =1. The coupled matrices of H_9 are P_5 and K_4 as

$$\boldsymbol{P}_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} \text{ and } K_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Now, considering P_5, the core of P_5 can be obtained as

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

which is the incidence matrix for a BIBD with parameter v=b=4, r=k=3, λ=2 Then

$\mathbf{NN}^T = (r-\lambda) I + \lambda J$, where I is the identity matrix of order v and J is the matrix of unity, of order v. Here,

$$\mathbf{NN}^T = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$$

$$= (3-2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Thus, r=3, λ=2. Here, v=b=4, k=3

If the treatments denoted as 1, 2, 3, and 4 then block contents of the design obtained based on the incidence matrix N is

$B1$	1	2	3
$B2$	2	3	4
$B3$	1	3	4
$B4$	1	2	4

Now considering the core of minor matrix k, denoted as $\mathbf{N}^* = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ which is the incidence matrix of the BIBD such that

$$\mathbf{N}^*\mathbf{N}^{*T} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$= (2-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 1 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Thus, r=2 and λ =1. Here, v=b=3, k=2

If the treatments denoted as 1, 2, and 3 then the block content of the design obtained based on the incidence matrix **N*** is

$B1$	1	2
$B2$	2	3
$B3$	1	3

Case 2: Let, the order of the Hadamard rhotrix over GF(2) is n= 8t-1. Now, if H_n be a Hadamard rhotrix over GF(2) having two coupled matrixes of order 4t and 4t-1, where n=8t-1, then there exists symmetric BIB design with parameter. v=b=4t, r=k=3t-1, □=2t-1 from the matrix of order 4t and v=b=4t-2, r=k=3t-1, □=2t from the matrix of order 4t-1. Following is an example.

Let $H_7 = \left\langle \begin{array}{ccccccc} & & & 0 & & & \\ & & 1 & 1 & 1 & & \\ & 1 & 0 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ & 0 & 1 & 1 & 1 & 1 & \\ & & 1 & 1 & 0 & & \\ & & & 1 & & & \end{array} \right\rangle$ So, n=7= 8t-1 so t =1, Coupled matrices H_7 are P_4 and K_3 as

$$\boldsymbol{P}_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \text{ and } K_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Now, considering P_4, the core of P_4 can be obtained as

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

which is the incidence matrix for a BIBD with parameter v=b=3, r=k=2, □=1 Then

$\mathbf{NN}^T$= (r-□) I+□□J, where I is the identity matrix of order v and J is the matrix of unity, of order v.

$$\mathbf{NN}^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= (3-2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

If the treatments denoted as 1, 2, and 3 then block contents of the design obtained based on the incidence matrix N is

$B1$ 1 2
$B2$ 2 3
$B3$ 1 3

Now we take the core of minor matrix k, denoted as $\mathbf{N}^* = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ which is the incidence matrix of the BIBD with the parameter $v = b = 4t - 2, r = k = 3t - 1, \lambda = 2t,$ where t=1, but as matrix is not sufficiently large order so it is not possible to make BIBD because it doesn't satisfy the formula $\mathbf{NN}^T$= (r-□)I+□□ J.

4. Summary and conclusion remarks

The utilization of Rhotrices in the construction of Balanced Incomplete Block Designs (BIBD) marks a significant advancement in the field of experimental design. This innovative approach, which finds its roots in the relatively recent concept of Rhotrix theory, has demonstrated its potential to simplify and enhance the process of BIBD generation.

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6. Competing interests

The authors have no potential conflicts to report that are important to the article's content.

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