International Journal of Statistics and Applied Mathematics 2023; 8(6): 71-76

ISSN: 2456-1452
Maths 2023; 8(6): 71-76

https://www.mathsjournal.com
Received: 22-07-2023
Accepted: 26-08-2023

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A mathematical modelling in two phase flow in arterioles during thalassemia

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DOI: https://dx.doi.org/10.22271/maths.2023.v8.i6a.1415

Abstract
A person is suffering from thalassemia will have too few red blood cells and too little haemoglobin, and the red blood cells may be too small. The impact of thalassemia can range from mild to severe and life threatening. We have presented a non-Newtonian two phase blood model. Blood is considered as a homogeneous mixture of blood cells and plasma, and also hold power low. We have formed equation of continuity and the equation of motion into tensorial form. We obtained a relation between haematocrit and pressure drop that predict fluctuation in blood flow. We have also collected pathological data of a thalassemia patient, and done graphical study of fluctuation in blood flow. Adopted solution techniques are analytical as well as numerical.

Keywords: Two phase blood flow, blood flow during thalassemia, non-new toni an, power law model, haematocrit, blood pressure drop

Introduction
Thalassemia is an inherited blood disorder. It causes the body to make less haemoglobin. There are two main types of thalassemia; alpha and beta, Beta thalassemia is affected by different genes of parents suffering from thalassemia [2]. There are several types of beta thalassemia; major and minor, different people will have different symptoms, based on which type of major beta thalassemia (Cooley Anemia) is reduce the production of haemoglobin. Haemoglobin is the iron-containing protein in red blood cells and plasma that carries oxygen to cells throughout the body [1, 3].

In people with major beta thalassemia (Cooley Anemia) low levels of haemoglobin lead to a lack of oxygen in many parts of the body. Affected individuals also have a shortage of red blood cells and plasma, which can cause pale skin, fussy, having a poor appetite, having many infections, belly swelling and slowed growth. People with major beta thalassemia are at an increased risk of developing abnormal blood clots [2, 3].

Structure and the function of arterioles
An arteriole is a small-diameter blood vessel in the microcirculation that extends and branches out from an artery and leads to capillaries. Arterioles have muscular walls (usually only one to two layers of smooth muscle cells) and are the primary site of vascular resistance. The greatest change in blood pressure and velocity of blood flow and plasma occurs at the transition of arterioles to capillaries. This function is extremely important because it prevents the thin, one-layer capillaries from exploding upon pressure. The arterioles achieve this decrease in pressure, as they are the site with the highest resistance [12].

Real Model
Frame of reference: - In this model we have choose orthogonal curvilinear generalized three-dimensional co-ordinate system denoted by E^3 called three-dimensional Euclidean space of the moving blood. All quantities related to blood flow written in tensorial form which is comparatively more realistic. Let P be n pin in space with co-ordinate X^i with respect to axes OX^i, O as origin where $i = 1,2,3$, At time $t, V^k = V^k(X^i, t)$ be velocity of blood, $p = p(X^i, t)$.

thermodynamically pressure and $\rho = \rho(X^i, t)$ density. Since blood vessels are cylindrical the governing equations have to transform into cylindrical co-ordinate system [6].

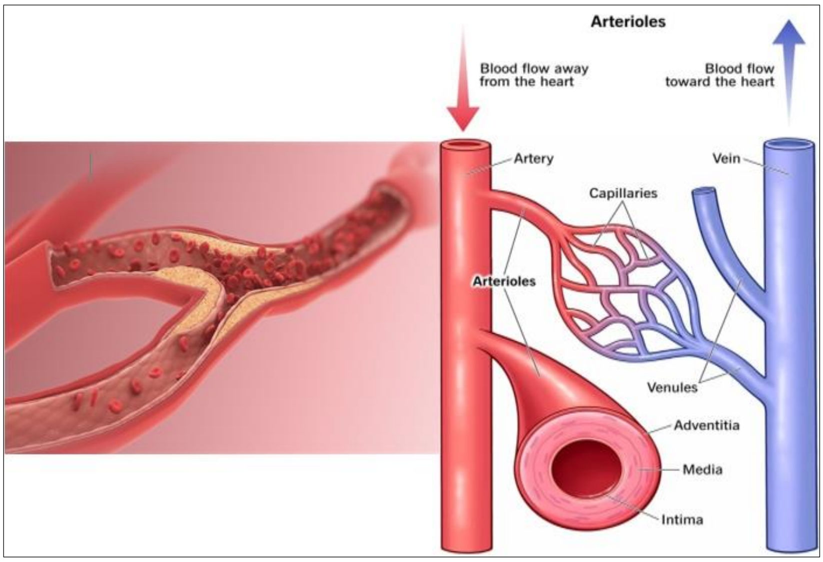

Fig 1: Blood Flow in Human Body [21]

Constitution of blood

Blood is bio fluid or fluid connective tissue. Blood consists of a suspension of cells in an aqueous solution called plasma which composed of about 90% water and 7% protein. There are about 95% are red blood cells or erythrocytes whose main function is to transport oxygen from lungs to all the cells of the body and removal of carbon dioxide formed by metabolic process in the body to lungs. About 45% of the blood volume in an average human is occupied by red blood cells. This fraction is known as the hematocrit of the remaining white blood cells or leucocytes constitute about one sixth or 1% of total and these play a impartment role in the body to infection and platelets form 5% of the total blood and they perform a function related to blood clotting [18].

Formulation

According to Sharman I.W. and Sherman V.G. blood is mixed fluid. There are two phases in the blood, one is plasma and other is blood cells. The blood cells are enclosed with a semi-permeable membrane whose density is greater than that of plasma. These blood cells are uniformly distributed in plasma.

Equation of continuity for two phase blood

According to Upadhyay V. The flow of blood is affected by the presence of blood cells. This effect is directly proportional to the volume occupied by blood cells. Let the volume portion covered by blood cells in unit volume be X, where $X = \frac{H}{100}$ and H is hematocrit the volume percentage of blood cells. Then the volume portion covered by plasma will be $1 - X$. If the mass ratio of blood cells to plasma is r then

$$r = \frac{X\rho_c}{(1-X)\rho_p}$$

Where ρ_c and ρ_p are densities of blood cells and plasma respectively. The both phase the blood cells and plasma move with common velocity. Campbell and Pitcher have presented a model for this condition. Equation of continuity for two phases according to principle of conservation of mass defined by J.N. and Gupta R. C. As follows-

$$\frac{\partial X\rho_c}{\partial t} + \left(X\rho_p v^i\right)_{,i} = 0$$

$$\frac{\partial (1-X)\rho_p}{\partial t} + \left((1-X)\rho_p v^i\right)_{,i} = 0$$

Where v is common velocity of two phase blood cells and plasma, $(X\rho_c v^i)_{,i}$ is covariant derivative of $(X\rho_c v^i)$ with respect to x^i and $((1-X)\rho_c v^i)_{,i}$ is covariant derivative of $(1-X)\rho_c v^i)$ with respect to x^i. If ρ_m be uniform density of blood then

$$\frac{1+r}{\rho_m} = \frac{r}{\rho_c} + \frac{1}{\rho_p} \text{ where } \rho_m = X\rho_c + (1-X)\,\rho_p \quad (3.4)$$

Combined equation (3.2) and (3.3) and using (3.4) we get

$$\frac{\partial \rho_m}{\partial t} + (\rho_m v^i)_I = 0 \quad (3.5)$$

Equation of motion for two phase blood flow

According to Ruch T.C. and H.D. the hydro dynamical pressure p between two phases of can be supposed to be uniform because the both phases are always in equilibrium state in blood (1973). According to principle of conservation of momentum equation of motion of two-phase blood cells and plasma

$$X\rho_c \frac{\partial v^i}{\partial t} + (X\rho_c vj)v,ij = -Xp,jgij + X\eta_c(gjkvi,k),j \quad (3.6)$$

And

$$(1-X)\,\rho_m \frac{\partial v^i}{\partial t} + \{(1-X)\,\rho_p v^j\}v^i_{,j} = -(1-X)p_{,j}g^{ij} + (1-x)\,\eta_p(g^{jk}v^i_{,k})_{,j} \quad (3.7)$$

Now adding (3.6) and (3.7) and using (3.4) then the equation of motion for blood flow will be

$$\rho_m \frac{\partial v^i}{\partial t} + (\rho_m v^j)v^i_{,j} = -p_{,j}g^{ij} + \eta_m(g^{jk}v^i_{,k})_{,j} \quad (3.8)$$

Where η_m=Xη_c +(1-X) η_p is the viscosity coefficient of blood as a mixture of two phases. As velocity of blood flow decreases, the viscosity of blood increases. Since the arterioles are remote from heart therefore velocity of blood decreases. The Herschel Bulkley law hold good on two phase blood flow through the arterioles and whose constitutive equation as follow-

$T' = \eta_m e^n + T_p$ ($T' > T_p$) and e=0 ($T' < T_p$)

where T_p is yield stress.
When strain rate e=0 ($T' < T_p$) a core region is formed which flow just like a plug. Let radius of plug be r_p and the stress action on the surface of plug will be T_p. Equation of force acting on the plug

, $P\pi r_p^2 = T_p \pi 2 r_p$ or $r_p = 2\frac{r_p}{p}$

The constitutive equation for rest part of blood vessel is
$T' = \eta_m e^n + T_p$ or $T' - T_p = \eta_m e^n = T_e$ where T_e is effective stress whose generalized form will be $T^{ij} = -pg^{ij} + T_e^{ij}$ where

$T_e^{ij} = \eta_m e^{ijn}$, $e^{ij} = g^{jk} v_k^{\ i}$

Equation of continuity

$$\frac{1}{\sqrt{g\sqrt{g v^i}},_i} = 0 \tag{3.9}$$

Equation of motion

$$\rho_m \frac{\partial v^i}{\partial t} + \rho_m v^j v^i_{,j} = -T_e{}^{ij}{}_{,j} \tag{3.10}$$

Where all the symbol has their usual meaning.

Newton Raphson Method: the general Newton Raphson Method formula is

$$X_{n+1} = x_n - \frac{f(x_n)}{f(x_n)} \tag{3.11}$$

The above formula is repeated until a sufficiently precise value is obtained.

IV. Solution

Let x^1 = r, $x^2 = \theta$ and x^3 = z be cylindrical co-ordinates and square length of element Christoffel's symbols of first and second kind are given bellow-

$$[ij,k] = \frac{1}{z}\left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{jk}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right] \text{ and } \left\{\frac{k}{ij}\right\} = g^{k\alpha}[ij,\alpha]$$

$[g_{ij}]$ be matrix of metric tensor and $[g^{ij}]$ be matrix of conjugate matrix tensor where

$$G_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r2 & 0 \\ 0 & 0 & 1 \end{pmatrix} g^{ij} = = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Metric elements

$g_{\theta\theta} = r^2$, $g_{zz} = 1$ Or $g_{11} = 1$, $g_{22} = r^2$, $g_{33} = 1$

Christoffel's symbol of second kind for cylindrical co-ordinates. Where value of r_p taken from equation of motion.

Result

The flow flux of two phased blood flow in arterioles

$$Q = \int_0^{r_p} 2\pi r v_p dr + \int_{r_p}^{R} 2\pi r v dr$$

$$= \int_0^{r_p} 2\pi r \frac{n}{n+1}\left(\frac{p}{2\eta_m}\right)^{\frac{1}{n}} (R - r_p)^{\frac{1}{n}+1} dr + \int_{r_p}^{R} 2\pi r \frac{n}{n+1}\left(\frac{p}{2\eta_m}\right)^{\frac{1}{n}} \left[(R - r_p)^{\frac{1}{n}+1} - (r - r_p)^{\frac{1}{n}+1}\right] dr$$

Using equations then

$$Q = \frac{2\pi n}{n+1}\left(\frac{P}{2\eta_m}\right)^{\frac{1}{n}} (R - r_p)^{\frac{1}{n}+1} \left[\frac{r^2}{2}\right]_0^{r_p} + \frac{2\pi n}{n+1}\left(\frac{P}{2\eta_m}\right)^{\frac{1}{n}} \left[\frac{r^2}{2}(R - r_p)^{\frac{1}{n}+1} - \frac{r(r - r_p)^{\frac{1}{n}}}{\frac{1}{n}+2} + \frac{(r - r_p)^{\frac{1}{n}+1}}{\left(\frac{1}{n}+2\right)\left(\frac{1}{n}+3\right)}\right]_{rp}^{R}$$

$$Q = \frac{\pi n}{n+1}\left(\frac{P}{2\eta_m}\right)^{\frac{1}{n}} (R)^{\frac{1}{n}+3} \left[\frac{r_p^2}{R^2}\left(1 - \frac{r_p}{R}\right)^{\frac{1}{n}+1} + \left(1 + \frac{r_p}{R}\right)\left(1 - \frac{r_p}{R}\right)^{\frac{1}{n}+2} + \frac{2\left(1 - \frac{r_p}{R}\right)^{\frac{1}{n}+3}}{\left(\frac{1}{n}+2\right)\left(\frac{1}{n}+3\right)} - \frac{2\left(1 - \frac{r_p}{R}\right)^{\frac{1}{n}+2}}{\left(\frac{1}{n}+2\right)}\right]$$

R-Components $\begin{vmatrix} \left\{{1 \atop 1\ 1}\right\} & \left\{{1 \atop 1\ 2}\right\} & \left\{{1 \atop 1\ 3}\right\} \\ \left\{{1 \atop 2\ 1}\right\} & \left\{{1 \atop 2\ 2}\right\} & \left\{{1 \atop 2\ 3}\right\} \\ \left\{{1 \atop 3\ 1}\right\} & \left\{{1 \atop 3\ 2}\right\} & \left\{{1 \atop 3\ 3}\right\} \end{vmatrix}$

z- Components $\begin{vmatrix} \left\{{3 \atop 1\ 1}\right\} & \left\{{3 \atop 1\ 2}\right\} & \left\{{3 \atop 1\ 3}\right\} \\ \left\{{3 \atop 2\ 1}\right\} & \left\{{3 \atop 2\ 2}\right\} & \left\{{3 \atop 2\ 3}\right\} \\ \left\{{3 \atop 3\ 1}\right\} & \left\{{3 \atop 3\ 2}\right\} & \left\{{3 \atop 3\ 3}\right\} \end{vmatrix}$

θ-Components $\begin{vmatrix} \left\{{2 \atop 1\ 1}\right\} & \left\{{2 \atop 1\ 2}\right\} & \left\{{2 \atop 1\ 3}\right\} \\ \left\{{2 \atop 2\ 1}\right\} & \left\{{2 \atop 2\ 2}\right\} & \left\{{2 \atop 2\ 3}\right\} \\ \left\{{2 \atop 3\ 1}\right\} & \left\{{2 \atop 3\ 2}\right\} & \left\{{2 \atop 3\ 3}\right\} \end{vmatrix}$ $\left\{{1 \atop 2\ 2}\right\}$ = -r, $\left\{{2 \atop 2\ 1}\right\}$ = $\left\{{2 \atop 1\ 2}\right\} = \left\{{1 \atop r}\right\}$

Physical Components

since $\sqrt{g_{11}}\, v^1 = v_r\ or, v_r = v^1$

$\sqrt{g_{22}}\, v^2 = v_\theta\ or, v_\theta = r v^2$

and $\sqrt{g_{33}}\, v^3 = v_z \; or, v_z = v^3$

Matrix of physical components of shearing stress tensor

$$T'^{\lambda j} = \eta_m \quad (e^{ij})^n = \eta_m \left(g^{jk}\, v^i_{,k} + g^{jk}\, v^j_{,k}\right)^n \; (4.1) \; T'^{ij} = \begin{bmatrix} 0 & 0 & \eta_m \left(\frac{dv}{dr}\right)^n \\ 0 & 0 & 0 \\ \eta_m \left(\frac{dv}{dr}\right)^n & 0 & 0 \end{bmatrix}$$

The covariant derivative of T'^{ij}

$$T'^{ij}_{,j} = \frac{1}{\sqrt{g}} \frac{\delta}{\delta x^j} \left(\sqrt{g T^{ij}} + \begin{Bmatrix} & i & \\ j & & k \end{Bmatrix}\right)$$

According the above facts, the governing tensorial equation can be transformed into cylindrical form which is as follow

The equation of continuity $\frac{\partial V}{\partial z} = 0$ (4.3)

The equation of motion

r- Component $-\frac{\partial p}{\partial r} = 0$ (4.4)

$\theta -$ Component 0=0 (4.5)

z- Component $-\frac{\partial p}{\partial z} + \frac{\eta_m}{r} \frac{\partial}{\partial r}\left[r\left(\frac{\partial vz}{\partial r}\right)^n\right]$ (4.6)

Here this fact has been taken in view that the blood flow is axially symmetric in arteses concerned i.e. $v_\theta = 0$ and v_r, v_z and p do not depend upon θ and also blood flow radially.

$$\frac{\partial p}{\partial t} = \frac{\partial v_r}{\partial t} = \frac{\partial v_\theta}{\partial t} = \frac{\partial v_z}{\partial t} = 0$$

From (4.5) $v_z = v_r$ since v does not depend upon (4.7)
From equation (4.4) p=p(z) (4.8)
because p does not depend upon, θ using equation (4.7) & (4.8) in (4.6) then

$$\frac{\partial p}{\partial z} + \frac{\eta_m}{r} \frac{d}{dr}\left[r\left(\frac{dv}{dr}\right)^n\right] = 0 \quad (4.9)$$

The pressure gradiant $-\frac{dp}{dz}$ = P of blood flow. In arterioles remote from liver can be supposed to be constant, therefore equation (4.9) takes the following form

Observations- Haemoglobin and blood pressure is taken for Nahar Nursing Home Rewa Road, Satna (M.P.) by Dr. RK Nayak and Dr. Chunnu Ram Pandey, Patient name-Priyanka Age-27 years /Female, Diagnosis- major beta thalassemia (Cooley anemia)

SI. No.	Date	HB (Hemoglobin) gm/dl	BP (Blood Pressure) mmhg	Hematocrit ($3 \times HB$)	BP (In Pascal)
1	14.10.21	9.6	110/80	28.8	14665.2/10665.6
2	16.10.21	8.3	105/75	24.4	13998.69/9332.4
3	19.10.21	6.8	100/68	20.4	13332/9065.76
4	21.10.21	8	105/70	24	13998.6/9332.4
5	23.10.21	8.6	110/78	25.8	14665.2/10398.96

Blood Pressure Droop in arterioles= $\left[\frac{\frac{S+D}{2}+D}{3} - \frac{S+D}{2}\right]$

Q=1000 ml/min=0.01666 liter/sec

$R = 1, r_p = \frac{1}{3}$

According to Gustafson Danial R, (1980)

$\eta_p = 0.0015$

According to Glenn Elert (2010) [2]

$\eta_m = 0.035\,(pascal - Sec)$

Length of hepatic arteriole=50 μm=5$\times 10^{-5}$meter

H=20.4, Blood pressure drop=4444 (Pascal Sec.)

Since $\eta_m = \eta_c\, X + \eta_p(1-x)$ where x=$\frac{H}{100}$

Substituting the value of $\eta_m \eta_p$, and H in above relation we get $\eta_c = 0.1657$ again from above relation $\eta_m = 0.001642H + 0.0015$
Substituting the value of r_p *and* R in equation (28) we get

$$Q = \frac{2\pi}{27}\left(\frac{p}{3\eta m}\right)^{\frac{1}{n}} \left[\frac{26n^3+33n^2+9n}{6n^3+11n^2+6n+1}\right]$$

Or, $\frac{27Q}{2\pi} \; \left(\frac{p}{3\eta_m}\right)^{\frac{1}{n}} \left[\frac{26n^3+33n^2+9n}{6n^3+11n^2+6n+1}\right]$

Or, $\left(\frac{P}{3\eta_m}\right) = \left(\frac{27Q}{2\pi A}\right)^n$ where A=$\frac{26n^3+33n^2+9n}{6n^3+11n^2+6n+1}$

Or, P=$\left(\frac{27q}{2\pi a}\right)^n \; 3\,\eta m$

Since P=$-\frac{dp}{dz}$ or, dp=-Pdz

$$p_f - p_i = \left(\frac{27q}{2\pi A}\right)^n \; 3\eta_m(z_f - z_i) \quad (5.2)$$

Where
$p_f - p_i$ pressure drop and $z_f - z_i = length\ hepatic\ arteriole$
Substituting the values of Q,$p_f - p_i, and\ \eta_m$ in above equation and solve by numerical method we get

n=-4.1566, and again $p_f - p_i = 3\eta_m(z - z_i)\left(\frac{27Q}{2\pi A}\right)^n$

Substituting the value of $3\eta_m, Q\ and\ n$ we get

$$p_f - p_i = 11.7521\,H + 10.7357 \quad (5.3)$$

This is relation between haematocrit and blood pressure drop

Table 1: Hematocrit and blood pressure drop

Hematocrit	28.8	24.9	20.4	24	25.8
blood pressure drop	349.1962	303.3630	250.4785	292.7861	313.9399

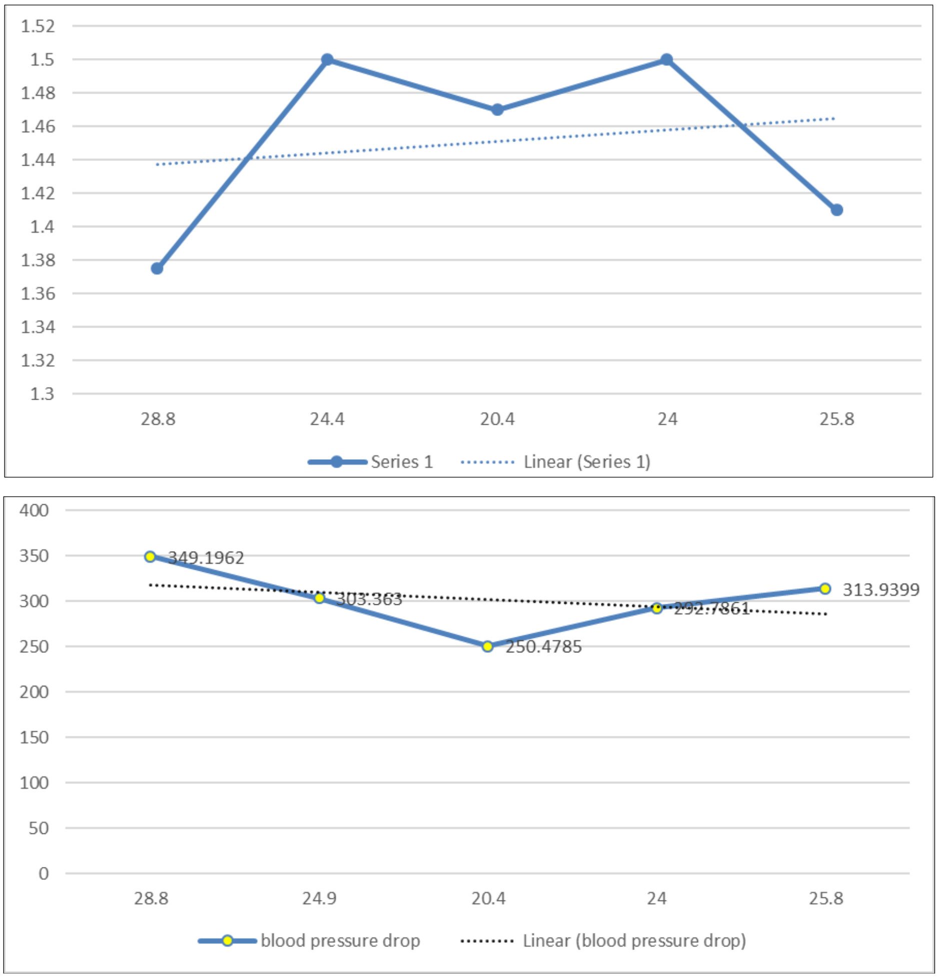

Fig 1: Chart Title

Conclusion

This graph between blood pressure drop and hematocrit in major Beta thalassemia (Cooley amania) patient shows that when hematocrit increased then blood pressure drop also increased.

Acknowledgement

I am Thankful to my guide, Dr. RK Nayak and Chunnu Ram Pandey from Nahar Nursing Home Rewa Road, Satna Madhya Pradesh, India for providing me pathological data of a patient.

References

1. Needs T, Mosquera LFG, Lynch DT. Beta Thalassemia, PMID; c2022. 30285376.
2. Cavazzana-Calco M, Payen E, Negre O, Wang G, Hehir K, Fusil F, *et al.*, Gene Therapy for β-Haemoglobin Opathies; c2010. doi-10.1038/nature09328.
3. Abu Samra W, Auda H, Al-Tonbary KY. Impact of educational programme regarding chelation therapy on the quality of life for β- Thalassemia major children. Hematology; v2015.
4. Maurya P, Upadhyay V, Chaturvedi SK, Kumar D. Mathematical Study and Simulation on Stenosed Carotid Arteries with The Help of a Two-Phase Blood Flow Model, The Canadian Journal of Chemical Engineering; c2022. DOI; 10.1002/cjce.24834.
5. Devi L, Upadhyay V, Chaturvedi SK, Pandey PN. Mathematical Modelling of Two-Phase Blood Flow in Arterioles During- Iron Deficiency Anemia, IJIRT. 2022;9(4):2349-6002.
6. Gupta P, Upadhyay V, Chaturvedi SK, Kumar D, Pandey PN. Mathematical Analysis of the Two-Phase Coronary Blood Flow in Arterioles with Silent Ischemia Disease. International Journal of Creative Research thoughts (IJCRT); ISSN:2320-2882,, 2022, 10(10).
7. Mishra S, Upadhyay V. A Mathematical Study of Two Phase (One Phase is Newtonian and Other is Non-Newtonian) Coronary Blood Flow in Venules Using Herschel Bulkley Model During Angina. 2017 IJIRAS; ISSN:2394-4404;4(1).
8. Joisan K, Bhoraniya R, Chandum AM. Numerical Analysis Two Phase Blood Flow in Idealized Artery with Blockage. Springer Nature Singapore Pte Ltd; c2019. DOI; 10.10071978-981-13-2697-4_29.
9. Kumar D, Kumar P, Rai KN. Numerical solution of Non-linear Dual Phase-Lag-Bio-Heat Transfer Equation within Skin Tissues. Mathematical Biosciences. 2017;293:56-63.
10. Upadhyay V, Chaturvedi SK, Upadhyay A. Discussed A Mathematical Model on The Effect of Stenosis in Two Phase Blood Flow in Arteries Remote from The Heart. Journal of International Academy of Physical Sciences. 2012;16(3):247-257.

11. Anne W, Allison G. Ross and Wilson Anatomy and Physiology in Health and Illness. Churchill Livingstone, 11th edition, London, UK; c2010. ISBN 9780702032288.
12. Uc De, Shaikh AA, Sengupta J. Tensor calculus (II-Edition), Alpha sciencesInternational Ltd, Oxford, UK. ISBN, 2008, 978-1-84265-448-4.
13. Kapur JN. Mathematical Models in Biology and Medicine. EWP, New Delhi; c2008. p. 344-389.
14. Wiat L, Fine J. Applied Bio-fluid Mechanics, Mc Graw-Hill companies; c2007. DOI: 10.1036/0071472177.
15. Guyton AC, JE Hall I. Textbook of Medical Physiological. ElsevierInc, India 11th edition; c2006. ISBN 0-8089-2317-X.
16. Jagan Mazumdar N. Bio-fluid Mechanics, World scientific, 2004, (17). ISBN978-9814713979
17. Upadhyay V. Some Phenomena in two phase blood flow, Ph.D. Thesis, Central University, Allahabad; c2001.
18. Upadhyay V, *et al*. discussed a power law model of two-phase blood flow in arteries remote from the heart; c1999.
19. Fung YC. Biomechanics Mechanical Properties of Living Tissues, Springer-Veray; c1993. DOI:10.1007/978-1-4757-2257-4. York, ISBN978-1-4419-3104-7.
20. Strikwerda JC. Finite Difference Schemes and Partial Differential Equations, Chapman Hall, New York; c1989.
21. Walker HK, Hall WD, Hurst JW. Billett HH. Haemoglobin and Haematocrit, The History of Physical and Laboratory Examination, 3rd Edition, Boston, Chapter; c1990.