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Bayesian approach to control charts for process precision

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Abstract
Monitoring process variation is essential to maintain quality of certain product in manufacturing industry. Coefficient of variance (CV) is a relative measure of process variation and inverse coefficient of variance (ICV) measures relative precision. In this paper, control charts for ICV under normal model using Bayesian approach is proposed. The control charts are constructed under noninformative and gamma priors. Their performance is investigated. The proposed control charts are illustrated through examples.

Keywords: Inverse coefficient of variance, normal distribution, prior distribution, posterior distribution, control limits, power.
AMS Classification: 62P30, 62C10.

Introduction

Process variation plays a prime role in process monitoring. Hence monitoring process variation using control charts is inevitable in manufacturing. Sometimes, quality practitioners are interested in calibrating relative change in process variance with respect to process mean known as CV which is unit free. The reciprocal of CV is ICV and it accounts for the measure of precision. The population ICV is given by

$$\tau = \frac{\mu}{\sigma} \qquad (1)$$

where μ is population mean and σ is population standard deviation (sd). The sample ICV is given by

$$\hat{\tau} = \frac{\overline{x}}{s} \qquad (2)$$

where $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is sample mean and $s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2}$ is sample sd. The control chart for ICV is helpful in maintaining quality of certain production processes such as manufacturing of surgical instruments, footwear, gems and jewelry, apparel, etc.

Singh (1993) [9] studied the behavior of population ICV when a random sample of size n is drawn from various distributions including life testing distributions. Sharma and Krishna (1994) [8] constructed asymptotic distribution of sample ICV. Van Zyl and Van der Merwe (2017) [11] developed Bayesian procedure of obtaining control limits for CV using reference and probability matching priors. Kalkur and Rao (2017) [6] developed Bayes estimators for CV and ICV under normal model using various priors. Using Bayesian approach, the control charts were developed under conjugate priors for process average and process variance respectively by Bhat and Gokhale (2014a) and (2014b) [1-2]. These authors developed control chart for process sd using noninformative (NI), conjugate and non-conjugate priors in 2016. Bhat and Malagavi (2021a) [4] and (2021b) [5] respectively developed range based posterior control charts for process average and process variation.

In this paper, we focus on the development of Bayesian control charts for ICV under normal model. We use NI and gamma priors. In section 2, we obtain $E(\tau), V(\tau), E(\tau_r), V(\tau_r)$ and construct control charts for ICV. We evaluate the performance of the proposed control charts using power and average run length (ARL) in section 3 and illustrate them with examples in section 4. Conclusions are given in section 5. Tables supporting our study are given in appendix.

Bayesian control charts for ICV

In this section, we derive posterior joint distribution of $\left(\mu, \frac{1}{\sigma^2}\right)$ and propose posterior control charts (T and T_R control charts) for ICV. We find $E(\tau)$, $V(\tau)$, $E(\tau_r)$ and $V(\tau_r)$ using joint posterior distribution of μ and $\frac{1}{\sigma^2}$ under NI and gamma priors.

Let X_1, X_2,… X_n be a random sample of size n from normal distribution with mean μ and variance σ^2. Then the likelihood function is given by

$$f(\underline{x}|\mu,\sigma^2) = \prod_{i=1}^{n} f(x_i|\mu,\sigma^2) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}. \quad (3)$$

Assuming the prior distribution for μ as $\pi(\mu) = 1$, the posterior distribution of μ is given by

$$\pi(\mu|\overline{x}) = \frac{1}{\left(\frac{\sigma}{\sqrt{n}}\right)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\mu-\overline{x}}{\frac{\sigma}{\sqrt{n}}}\right)^2} \sim N\left(\overline{x}, \frac{\sigma^2}{n}\right). \quad (4)$$

We know that, $z = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)}$. (5)

Hence, the probability density function of $f\left(s^2\middle|\frac{1}{\sigma^2}\right)$ is given by

$$f\left(s^2\middle|\frac{1}{\sigma^2}\right) = \frac{\left(\frac{n-1}{2\sigma^2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} e^{-\left(\frac{n-1}{2\sigma^2}\right)s^2} (s^2)^{\left(\frac{n-1}{2}\right)-1}, s^2 > 0 \sim Gamma\left(\left(\frac{n-1}{2}\right), \left(\frac{n-1}{2\sigma^2}\right)^{-1}\right). \quad (6)$$

We derive the posterior distribution of $\left(\frac{1}{\sigma^2}\middle|s^2\right)$ when $\left(\frac{1}{\sigma^2}\right)$ follows

(i) NI prior, $\pi\left(\frac{1}{\sigma^2}\right) = \frac{1}{\sigma}$ (7)

and (ii) gamma prior, $\pi\left(\frac{1}{\sigma^2}\right) = \frac{\delta^\eta}{\Gamma\eta}\left(\frac{1}{\sigma^2}\right)^{\eta-1} e^{-\frac{\delta}{\sigma^2}}, \sigma^2 > 0.$ (8)

The posterior distribution of $\left(\frac{1}{\sigma^2}\middle|s^2\right)$ under NI prior is given by

$$\pi\left(\frac{1}{\sigma^2}\middle|s^2\right) = \frac{\left(\frac{n-1}{2}s^2\right)^{\frac{n+2}{2}}}{\Gamma\left(\frac{n+2}{2}\right)} e^{-\left(\frac{n-1}{2}s^2\right)\frac{1}{\sigma^2}} \left(\frac{1}{\sigma^2}\right)^{\frac{n+2}{2}-1}, \sigma^2 > 0 \sim Gamma\left(\left(\frac{n+2}{2}\right), \left(\frac{n-1}{2}s^2\right)^{-1}\right). \quad (9)$$

Also, the posterior distribution of $\left(\frac{1}{\sigma^2}\middle|s^2\right)$ under gamma prior is given by

$$\pi\left(\frac{1}{\sigma^2}\middle|s^2\right) = \frac{\left(\frac{n-1}{2}s^2+\delta\right)^{\left(\frac{n-1}{2}+\eta\right)}}{\Gamma\left(\frac{n-1}{2}+\eta\right)} e^{-\left(\frac{n-1}{2}s^2+\delta\right)\frac{1}{\sigma^2}} \left(\frac{1}{\sigma^2}\right)^{\left(\frac{n-1}{2}+\eta\right)-1}, \sigma^2 > 0 \sim Gamma\left(\left(\frac{n-1}{2}+\eta\right), \left(\frac{n-1}{2}s^2+\delta\right)^{-1}\right). \quad (10)$$

Since μ and σ^2 are independent, the posterior joint distribution of $\left(\mu, \frac{1}{\sigma^2}\right)$ is written as

$$\pi\left(\mu, \frac{1}{\sigma^2}\middle|s^2\right) = \pi(\mu|\overline{x}) \times \pi\left(\frac{1}{\sigma^2}\middle|s^2\right). \quad (11)$$

Hence, under NI prior,

$$\pi\left(\mu, \frac{1}{\sigma^2}\middle|s^2\right) = \frac{1}{\left(\frac{\sigma}{\sqrt{n}}\right)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\mu-\overline{x}}{\frac{\sigma}{\sqrt{n}}}\right)^2} \times \frac{\left(\frac{n-1}{2}s^2\right)^{\frac{n+2}{2}}}{\Gamma\left(\frac{n+2}{2}\right)} e^{-\left(\frac{n-1}{2}s^2\right)\frac{1}{\sigma^2}} \left(\frac{1}{\sigma^2}\right)^{\frac{n+2}{2}-1} \quad (12)$$

and under gamma prior,

$$\pi\left(\mu,\frac{1}{\sigma^2}\middle|s^2\right)=\frac{1}{\left(\frac{\sigma}{\sqrt{n}}\right)\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\mu-\bar{x}}{\frac{\sigma}{\sqrt{n}}}\right)^2}\times\frac{\left(\frac{n-1}{2}s^2+\delta\right)^{\left(\frac{n-1}{2}+\eta\right)}}{\Gamma\left(\frac{n-1}{2}+\eta\right)}e^{-\left(\frac{n-1}{2}s^2+\delta\right)\frac{1}{\sigma^2}}\left(\frac{1}{\sigma^2}\right)^{\left(\frac{n-1}{2}+\eta\right)-1}. \quad (13)$$

Using (12), under NI prior,

$$E_{NI}(\tau)=\int_0^\infty\int_{-\infty}^\infty\frac{\mu}{\sigma}\pi\left(\mu,\frac{1}{\sigma^2}\middle|s^2\right)d\mu\, d\left(\frac{1}{\sigma^2}\right)=\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\frac{\hat{\tau}}{\sqrt{\frac{n-1}{2}}} \quad (14)$$

$$and\ V_{NI}(\tau)=\frac{1}{n}+\left(\frac{n+2}{n-1}\right)\hat{\tau}^2-\left(\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\frac{\hat{\tau}}{\sqrt{\frac{n-1}{2}}}\right)^2=\frac{1}{n}+\left\{\left(\frac{n+2}{n-1}\right)-\left(\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\frac{1}{\sqrt{\frac{n-1}{2}}}\right)^2\right\}\hat{\tau}^2. \quad (15)$$

Also, under gamma prior, using (13), we get

$$E_{Ga}(\tau)=\frac{\Gamma\left(\frac{n}{2}+\eta\right)}{\Gamma\left(\frac{n}{2}+\eta-\frac{1}{2}\right)}\frac{\bar{x}}{\sqrt{\left(\frac{n-1}{2}s^2+\delta\right)}} \quad (16)$$

$$\text{and } V_{Ga}(\tau)=\frac{1}{n}+\left\{\left(\frac{n-1}{2}+\eta\right)-\left(\frac{\Gamma\left(\frac{n}{2}+\eta\right)}{\Gamma\left(\frac{n}{2}+\eta-\frac{1}{2}\right)}\right)^2\right\}\frac{\bar{x}^2}{\left(\frac{n-1}{2}s^2+\delta\right)}. \quad (17)$$

The proposed T control chart for ICV is given by

$$LCL_T=E(\tau)-3\sqrt{V(\tau)},\ CL_T=E(\tau)\text{ and }UCL_T=E(\tau)+3\sqrt{V(\tau)}. \quad (18)$$

where $LCL_T$is lower control limit, $UCL_T$is upper control limit and CL_T is central line of T control chart. Under NI prior, the control chart is given by substituting (14), (15) in (18) and under gamma prior by substituting (16), (17) in (18). Further the width, W_T of the T control chart is given by

$$W_T=UCL_T-LCL_T=6\sqrt{V(\tau)}. \quad (19)$$

When sample information is available in terms of extreme observations, the range is $r=x_{(n)}-x_{(1)}$ where $x_{(n)}$ and $x_{(1)}$ are respectively the highest and lowest values of the data. According to Tippett (1925), the sample sd, $s=\frac{r}{d}$ where d is a constant which is a function of sample subgroup size. Hence, $\hat{\tau}_r=\frac{\bar{x}d}{r}$. $E(\tau_r)$ and $V(\tau_r)$ under NI and gamma priors are obtained by substituting $s=\frac{r}{d}$ in appropriate equations. Also, we obtain range based T_R control chart and its width using $E(\tau_r)$ and $V(\tau_r)$.

Performance of the proposed control charts

In this section, we investigate the performance of the proposed T and T_R control charts for ICV in terms of power and ARL. The power and ARL of any control chart is given by

$$Power=1-\beta\text{ and }ARL=\frac{1}{1-\beta} \quad (20)$$

where β is probability of not detecting the shift in $\hat{\tau}'$, where $\hat{\tau}'=\frac{\bar{x}'}{s'}$, $\bar{x}'=\bar{x}+a$ and $s'=s+b$, a and b are non-negative constants.

Under NI and gamma prior, β of T control chart is respectively given by

$$\beta_{NI}=\int_{LCL_\mu}^{UCL_\mu}\frac{1}{\left(\frac{\sigma}{\sqrt{n}}\right)\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\mu-\bar{x}}{\frac{\sigma}{\sqrt{n}}}\right)^2}d\mu\int_{LCL_{\frac{1}{\sigma^2}}}^{UCL_{\frac{1}{\sigma^2}}}\frac{\left(\frac{n-1}{2}s^2\right)^{\frac{n+2}{2}}}{\Gamma\left(\frac{n+2}{2}\right)}e^{-\left(\frac{n-1}{2}s^2\right)\frac{1}{\sigma^2}}\left(\frac{1}{\sigma^2}\right)^{\frac{n+2}{2}-1}d\left(\frac{1}{\sigma^2}\right), \quad (21)$$

$$\beta_{Ga}=\int_{LCL_\mu}^{UCL_\mu}\frac{1}{\left(\frac{\sigma}{\sqrt{n}}\right)\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\mu-\bar{x}}{\frac{\sigma}{\sqrt{n}}}\right)^2}d\mu\int_{LCL_{\frac{1}{\sigma^2}}}^{UCL_{\frac{1}{\sigma^2}}}\frac{\left(\frac{n-1}{2}s^2+\delta\right)^{\left(\frac{n-1}{2}+\eta\right)}}{\Gamma\left(\frac{n-1}{2}+\eta\right)}e^{-\left(\frac{n-1}{2}s^2+\delta\right)\frac{1}{\sigma^2}}\left(\frac{1}{\sigma^2}\right)^{\left(\frac{n-1}{2}+\eta\right)-1}d\left(\frac{1}{\sigma^2}\right) \quad (22)$$

where, $LCL_\mu=\bar{x}-3\frac{\sigma}{\sqrt{n}}$, $UCL_\mu=\bar{x}+3\frac{\sigma}{\sqrt{n}}$, (23)

$$LCL_{\frac{1}{\sigma^2}(NI)}=\frac{\left(\frac{n+2}{2}\right)}{\left(\frac{n-1}{2}s^2\right)}-3\sqrt{\frac{\left(\frac{n+2}{2}\right)}{\left(\frac{n-1}{2}s^2\right)^2}},\ UCL_{\frac{1}{\sigma^2}(NI)}=\frac{\left(\frac{n+2}{2}\right)}{\left(\frac{n-1}{2}s^2\right)}+3\sqrt{\frac{\left(\frac{n+2}{2}\right)}{\left(\frac{n-1}{2}s^2\right)^2}}$$

and $LCL_{\frac{1}{\sigma^2}(Ga)} = \frac{\left(\frac{n-1}{2}+\eta\right)}{\left(\frac{n-1}{2}s^2+\delta\right)} - 3\sqrt{\frac{\left(\frac{n-1}{2}+\eta\right)}{\left(\frac{n-1}{2}s^2+\delta\right)^2}}$, $UCL_{\frac{1}{\sigma^2}(Ga)} = \frac{\left(\frac{n-1}{2}+\eta\right)}{\left(\frac{n-1}{2}s^2+\delta\right)} + 3\sqrt{\frac{\left(\frac{n-1}{2}+\eta\right)}{\left(\frac{n-1}{2}s^2+\delta\right)^2}}$.

Using (20) , we compute power and ARL of T control chart for various values of $\hat{\tau}'$and n. In table 1, we provide values of $\hat{\tau}'$ which are computed for various values of $\overline{x}'$and s'. Here, $\overline{x} = 1, a = 0\ (0.25)\ 2, s = 1$ and $b = 0\ (0.5)\ 4$. The values of $\hat{\tau}'$ for different shifts are plotted in figure 1 using table 1. For n=15, 20, 25, power and ARL of T control chart under NI prior are given in table 2 and plotted in figure 2. Also, under gamma prior, for hyper parameters $\eta = \delta = 0,3,5$ the corresponding power, ARL values are given in tables 3. Using table 3 power and ARL of T control chart are plotted in figure 3 for $\eta, \delta = 3$. Table 4 consists of power and ARL of T_R control chart under NI prior and gamma prior with $\eta, \delta = 3$. In these tables $10E - l$ represents e^{-l}, where l is an integer.
From figure 2 and table 2, it is seen that, for a given value of n, power increases as $\overline{x}'$ increases and decreases as $\hat{\tau}'$ increases. Also, for a specified value of $\hat{\tau}'$, power increases as n increases. For all the values of $\overline{x}'$, for n=15 and increasing $\hat{\tau}'$, ARL increases up to some value of $\hat{\tau}'$ and then decreases. But for n=20, 25, ARL increases as $\hat{\tau}'$ increases for different values of $\overline{x}'$.
From figure 3 and table 3, it is noticed that, for a given value of n, power increases as $\overline{x}'$ increases and decreases as $\hat{\tau}'$ increases. Also, for a specified value of $\hat{\tau}'$, power increases as n increases. For all the values of $\overline{x}'$, for n=15, 20 and increasing $\hat{\tau}'$, ARL increases up to some value of $\hat{\tau}'$ and then decreases. But for n=25, ARL increases as $\hat{\tau}'$ increases.
From tables 2 and 3 it is observed that, power of T control chart increases and ARL decreases with increasing values of n under both priors. Also, power of T control chart increases with increasing values of hyper parameters under gamma prior for a given shift in $\hat{\tau}$. From table 4, it is found that, power of T_R control chart increases with increasing values of n. On comparing T and T_R control charts under NI and gamma priors, power of T_R control chart is higher than power of T control chart. However, for smaller values of n, power and ARL of both T and T_R control charts are almost equal.

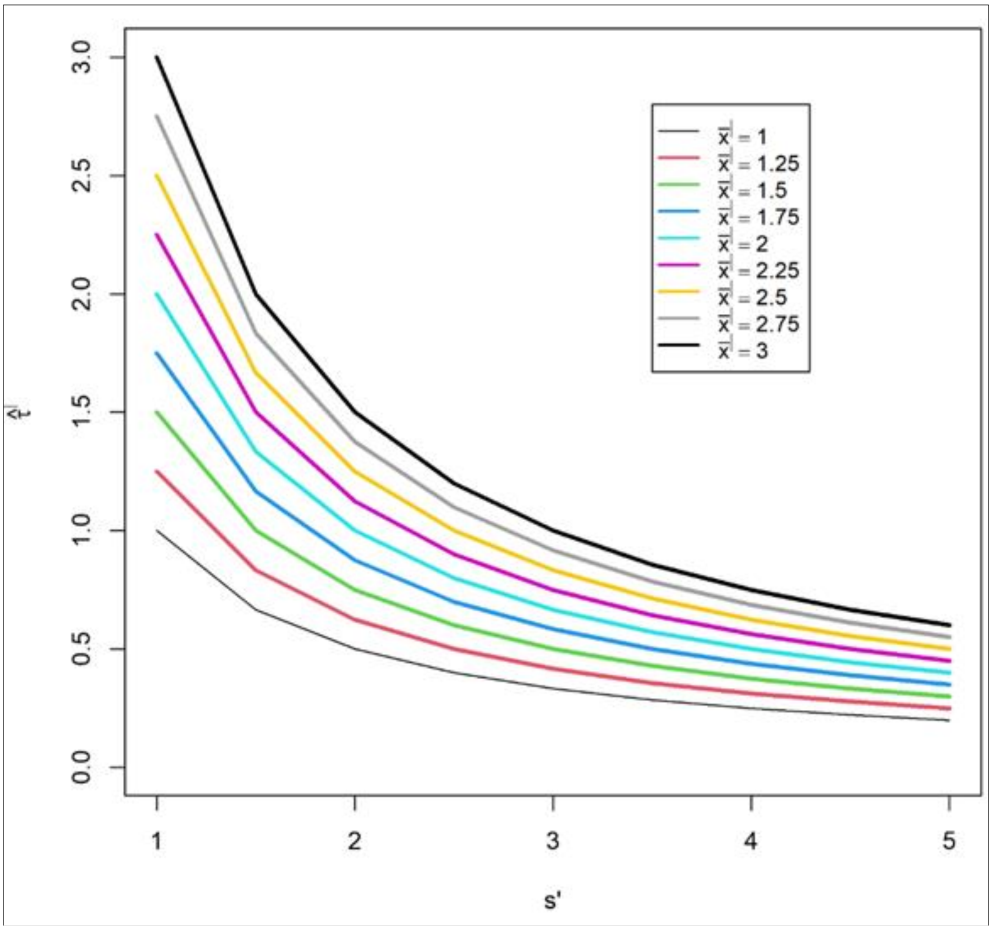

Fig 1: $\hat{\tau}'$ for various values of $\overline{x}'$and s'

From figure 1 and table 1, we observe that, for a given value of s', $\hat{\tau}'$ increases as $\overline{x}'$ increases and decreases as s' increases

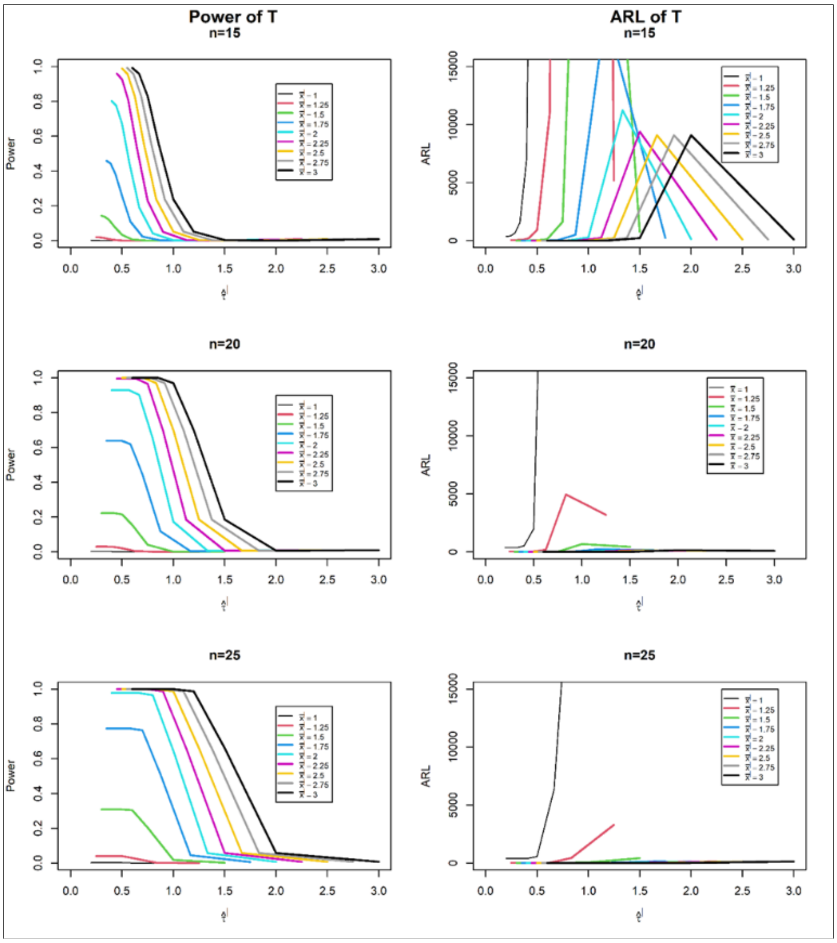

Fig 2: Power and ARL for T control chart under NI prior

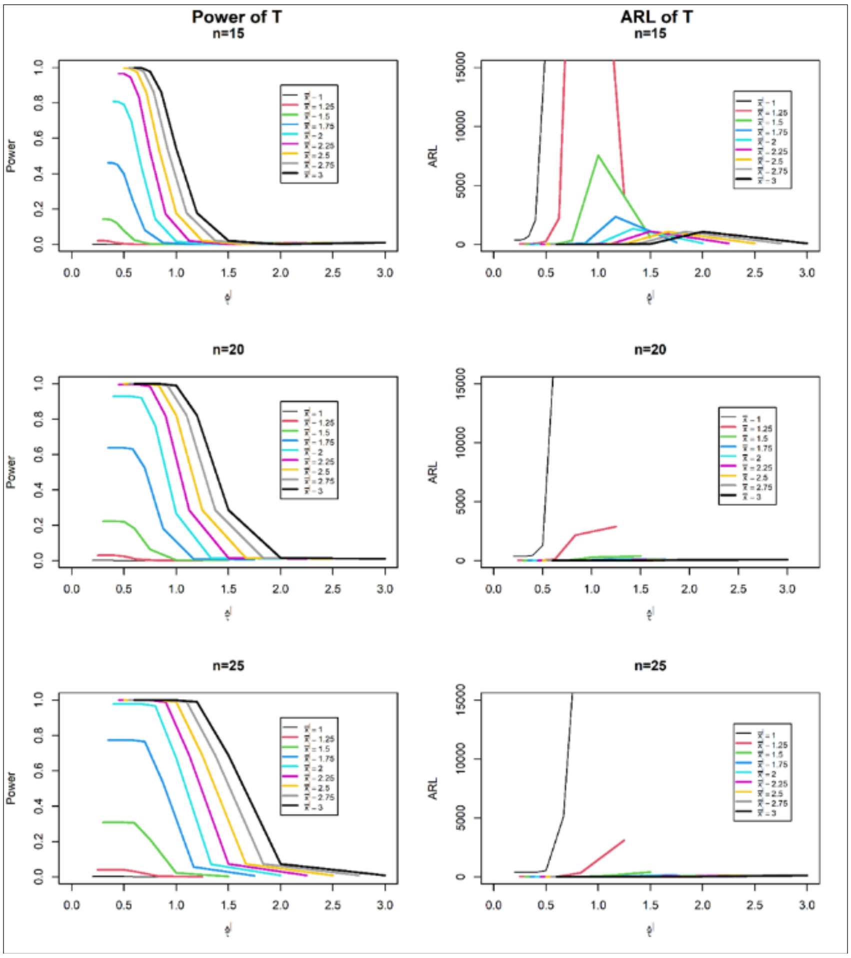

Fig 3: Power and ARL for T control chart under gamma prior for $\eta, \delta = 3$

Illustration

In this section, we illustrate the proposed T and T_R control charts in figures 4 and 5 respectively for examples on hard bake process and automobile engine piston ring due to Montgomery (1996). Here, we make use of first twenty five data points of these data and compute $\hat{\tau}'$ for each subgroup. Then $\hat{\tau}'$ is plotted for various samples. The control limits and width of the control charts are provided in table 5. When subgroup size is 5, d=2.326 and we consider $\eta = 15, \delta = 0.0001$.

Example 1: Hard bake process

The data consists of 25 samples with each sample having subgroup of size 5. It is the measurement of flow width (in micron) in hard-bake process for semi-conductor manufacturing.

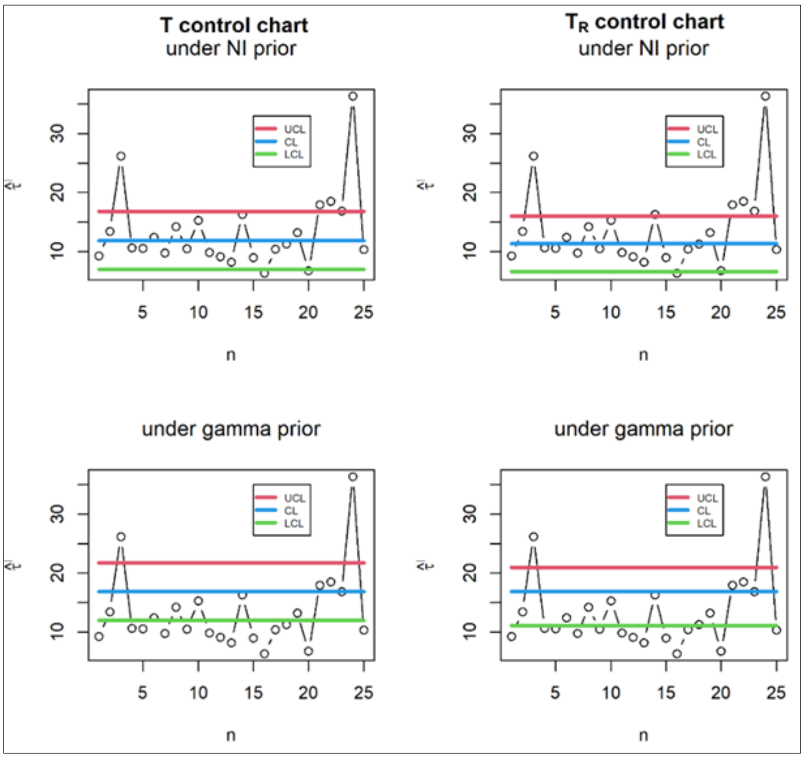

Fig 4: T and TR control charts for hard bake process data

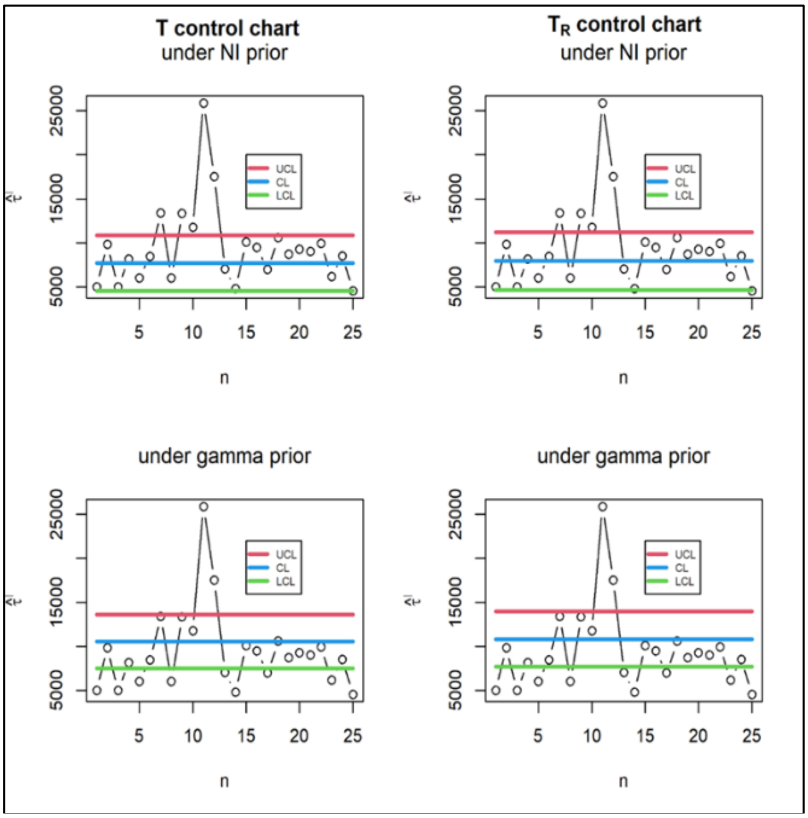

Fig 5: T and TR control charts for automobile engine piston ring data

From figure 4 and table 5, it is observed that, the control limits of T and T_R control charts under NI prior are lower than that of gamma prior. Also, width under gamma prior is wider than width under NI prior. The width of T_R control chart is narrower than width of T control chart under NI prior, whereas widths of both control charts are same under gamma prior.

Example 2: Automobile engine piston ring inside diameter

The data is on measurement of inside diameter (in mm) of piston rings for automobile engine. It contains 25 samples with each of size 5.

Figure 5: T and T_R control charts for automobile engine piston ring data

From figure 5 and table 5, it is seen that, for T and T_R control charts the control limits under gamma prior are higher than control limits under NI prior, but the width under gamma prior is narrower than width under NI prior. Also, width of T_R control chart is wider than width of T control chart under both NI and gamma priors.

Conclusions

In this section, we record our conclusions about the proposed control charts for process ICV, $\tau = \frac{\mu}{\sigma}$, the process precision. Here μ is process mean and σ is process sd.

- A posterior control chart, that is, T control chart for process precision using Bayesian approach is proposed. The sample ICV is $\hat{\tau} = \frac{\bar{x}}{s}$ where $\bar{x}$ is sample mean and s is sample sd.
- Under NI and gamma priors, when process measurements follow normal distribution, the T control chart is developed.
- We obtain posterior joint distribution of $\left(\mu, \frac{1}{\sigma^2} | s^2\right)$ under NI, gamma priors and derive $E(\tau), V(\tau)$ under both priors.
- The control limits and width of T control chart are developed using $E(\tau), V(\tau)$.
- For T control chart, under NI prior, for a given value of n, power increases as shift in $\bar{x}$ increases and decreases as shift in $\hat{\tau}$ increases. For a specified value of shift in $\hat{\tau}$, power increases with increasing value of n and ARL increases with increasing shifts in $\hat{\tau}$ and higher values of n.
- Under gamma prior, for increasing values of n, power increases with increasing shifts in $\bar{x}$ when η and δ take higher values. For a given η, δ and n, the power decreases as shift in $\hat{\tau}$ increases.
- On substituting $s = \frac{r}{d}$, we obtain control limits based on range and develop T_R control chart. The behavior of this control chart is similar to that of T control chart.
- T_R control chart outperforms T control chart for moderate and large values of n. However, performance of both the control charts are the same for small values of n.
- In hard bake process data, under NI prior width of T control chart is wider than width of T_R control chart and under gamma prior width of both control charts are same. Further, in automobile engine piston ring data, the width of T_R control chart is wider than width of T control chart under both priors.
- T control chart is useful when process precision is of prime concern and practitioners have prior information about the process mean and variance, whereas, T_R control chart is helpful when sample information is available in terms of extreme observations.

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Appendix

Table 1: The values of $\hat{\tau}'$ for various values of $\bar{x}'$ and s'

$\bar{x}'$	s'	$\hat{\tau}'$	$\bar{x}'$	s'	$\hat{\tau}'$	$\bar{x}'$	s'	$\hat{\tau}'$
1.00	1.0	1.0000	1.75	1.0	1.7500	2.50	1.0	2.5000
	1.5	0.6667		1.5	1.1667		1.5	1.6667
	2.0	0.5000		2.0	0.8750		2.0	1.2500
	2.5	0.4000		2.5	0.7000		2.5	1.0000
	3.0	0.3333		3.0	0.5833		3.0	0.8333
	3.5	0.2857		3.5	0.5000		3.5	0.7143
	4.0	0.2500		4.0	0.4375		4.0	0.6250
	4.5	0.2222		4.5	0.3889		4.5	0.5556
	5.0	0.2000		5.0	0.3500		5.0	0.5000
1.25	1.0	1.2500	2.00	1.0	2.0000	2.75	1.0	2.7500
	1.5	0.8333		1.5	1.3333		1.5	1.8333
	2.0	0.6250		2.0	1.0000		2.0	1.3750
	2.5	0.5000		2.5	0.8000		2.5	1.1000
	3.0	0.4167		3.0	0.6667		3.0	0.9167
	3.5	0.3571		3.5	0.5714		3.5	0.7857
	4.0	0.3125		4.0	0.5000		4.0	0.6875
	4.5	0.2778		4.5	0.4444		4.5	0.6111
	5.0	0.2500		5.0	0.4000		5.0	0.5500
1.50	1.0	1.5000	2.25	1.0	2.2500	3.00	1.0	3.0000
	1.5	1.0000		1.5	1.5000		1.5	2.0000
	2.0	0.7500		2.0	1.1250		2.0	1.5000
	2.5	0.6000		2.5	0.9000		2.5	1.2000
	3.0	0.5000		3.0	0.7500		3.0	1.0000
	3.5	0.4286		3.5	0.6429		3.5	0.8571
	4.0	0.3750		4.0	0.5625		4.0	0.7500
	4.5	0.3333		4.5	0.5000		4.5	0.6667
	5.0	0.3000		5.0	0.4500		5.0	0.6000

Table 2: Power and ARL of T control chart for various n under NI prior

$\bar{x}'$	s'	$\hat{\tau}'$	Power			ARL		
			n=15	n=20	n=25	n=15	n=20	n=25
1.0	1.0	1.0000	2.46221E-05	2.82939E-05	2.04645E-05	40613.853876	35343.353754	48865.217328
	1.5	0.6667	2.96978E-07	1.81966E-05	0.000156534	3367257.703134	54955.244413	6388.381293
	2.0	0.5000	1.15551E-05	0.000503	0.001782	86541.669940	1988.395680	561.311674
	3.0	0.3333	0.000642	0.002619	0.002700	1557.080660	381.798861	370.405755
	4.0	0.2500	0.002243	0.002700	0.002700	445.865550	370.398347	370.398347
	5.0	0.2000	0.002682	0.002700	0.002700	372.836699	370.398347	370.398347
1.5	1.0	1.5000	0.001311	0.002331	0.002339	762.637809	428.942006	427.585291
	1.5	1.0000	0.000016	0.001499	0.017889	63229.607414	666.960271	55.900250
	2.0	0.7500	0.000615	0.041439	0.203598	1625.060004	24.132018	4.911645
	3.0	0.5000	0.034201	0.215811	0.308531	29.238510	4.633674	3.241161
	4.0	0.3750	0.119441	0.222454	0.308538	8.372363	4.495312	3.241097
	5.0	0.3000	0.142836	0.222454	0.308538	7.001044	4.495312	3.241097
2.0	1.0	2.0000	0.007375	0.009741	0.007408	135.592945	102.656304	134.997328
	1.5	1.3333	0.000089	0.006265	0.056661	11241.887809	159.619891	17.648840
	2.0	1.0000	0.003461	0.173149	0.644868	288.927023	5.775382	1.550706
	3.0	0.6667	0.192365	0.901753	0.977230	5.198452	1.108951	1.023300
	4.0	0.5000	0.671789	0.929508	0.977250	1.488562	1.075838	1.023280
	5.0	0.4000	0.803375	0.929508	0.977250	1.244748	1.075838	1.023280
3.0	1.0	3.0000	0.009120	0.010480	0.007580	109.649237	95.419847	131.926121
	1.5	2.0000	0.000110	0.006740	0.057980	9090.918523	148.367953	17.247327
	2.0	1.5000	0.004280	0.186280	0.659880	233.645102	5.368263	1.515427
	3.0	1.0000	0.237880	0.970140	0.999980	4.203805	1.030779	1.000020
	4.0	0.7500	0.830739	1.000000	1.000000	1.203747	1.000000	1.000000
	5.0	0.6000	0.993459	1.000000	1.000000	1.006584	1.000000	1.000000

Table 3: Power and ARL of T control chart for various hyper parameters under gamma prior

$\eta, \delta = 0$								
$\bar{x}'$	**s'**	**$\hat{\tau}'$**	**Power**			**ARL**		
			n=15	**n=20**	**n=25**	**n=15**	**n=20**	**n=25**
1.00	1.0	1.0000	2.47E-05	2.33E-05	0.000020	40480.693699	42919.854849	49504.950495
	1.5	0.6667	3.132E-06	3.132E-06	0.000076	319308.920125	319308.920125	13157.894737
	2.0	0.5000	6.558E-05	0.000103	0.001210	15249.005654	9688.682902	826.763321
	3.0	0.3333	0.001192	0.001889	0.002697	838.707396	529.480877	370.780289
	4.0	0.2500	0.002512	0.002686	0.002700	398.110843	372.338230	370.398356
	5.0	0.2000	0.002696	0.002700	0.002700	370.936205	370.405755	370.398356
1.50	1.0	1.5000	0.001316	0.001920	0.002305	760.137357	520.893653	433.881644
	1.5	1.0000	0.000167	0.000258	0.008682	5995.911048	3875.269161	115.177557
	2.0	0.7500	0.003492	0.008504	0.138228	286.342397	117.585986	7.234429
	3.0	0.5000	0.063496	0.155618	0.308220	15.749059	6.426006	3.244438
	4.0	0.3750	0.133768	0.221295	0.308538	7.475636	4.518855	3.241097
	5.0	0.3000	0.143568	0.222450	0.308538	6.965357	4.495402	3.241097
2.00	1.0	2.0000	0.007399	0.008022	0.007300	135.148378	124.662580	136.985232
	1.5	1.3333	0.000938	0.001078	0.027500	1066.041085	927.446610	36.363886
	2.0	1.0000	0.019642	0.035535	0.437818	50.910155	28.141200	2.284056
	3.0	0.6667	0.357130	0.650237	0.976243	2.800099	1.537900	1.024335
	4.0	0.5000	0.752373	0.924665	0.977250	1.329128	1.081473	1.023280
	5.0	0.4000	0.807491	0.929489	0.977250	1.238403	1.075860	1.023280
3.00	1.0	3.0000	0.009150	0.008630	0.007470	109.289731	115.874855	133.868809
	1.5	2.0000	0.001160	0.001160	0.028140	862.069860	862.068967	35.536603
	2.0	1.5000	0.024290	0.038230	0.448010	41.169248	26.157468	2.232093
	3.0	1.0000	0.441630	0.699550	0.998970	2.264341	1.429490	1.001031
	4.0	0.7500	0.930389	0.994790	1.000000	1.074819	1.005237	1.000000
	5.0	0.6000	0.998549	0.999980	1.000000	1.001453	1.000020	1.000000
$\eta, \delta = 3$								
$\bar{x}'$	s'	$\hat{\tau}'$	Power			ARL		
			n=15	n=20	n=25	n=15	n=20	n=25
1.0	1.0	1.0000	3.13E-05	3.12E-05	2.18E-05	31958.442394	32069.120982	45955.129944
	1.5	0.6667	2.48E-06	4.16E-05	1.97E-04	402606.899288	24036.232793	5087.190597
	2.0	0.5000	0.000057	0.000774	0.001857	17397.761735	1292.297632	538.534069
	3.0	0.3333	0.001464	0.002674	0.002700	683.026328	373.924455	370.409460
	4.0	0.2500	0.002641	0.002700	0.002700	378.610407	370.398347	370.398347
	5.0	0.2000	0.002700	0.002700	0.002700	370.420573	370.398347	370.398347
1.5	1.0	1.5000	0.001666	0.002569	0.002487	600.108440	389.204522	402.121154
	1.5	1.0000	0.000132	0.003428	0.022465	7560.061756	291.713967	44.514442
	2.0	0.7500	0.003061	0.063760	0.212209	326.691255	15.683875	4.712334
	3.0	0.5000	0.077968	0.220356	0.308528	12.825715	4.538107	3.241194
	4.0	0.3750	0.140658	0.222454	0.308538	7.109461	4.495312	3.241097
	5.0	0.3000	0.143768	0.222454	0.308538	6.955674	4.495312	3.241097
2.0	1.0	2.0000	0.009372	0.010736	0.007877	106.696088	93.146153	126.957785
	1.5	1.3333	0.000744	0.014324	0.071154	1344.138760	69.814281	14.054110
	2.0	1.0000	0.017216	0.266416	0.672143	58.083967	3.753535	1.487779
	3.0	0.6667	0.438530	0.920743	0.977221	2.280344	1.086080	1.023310
	4.0	0.5000	0.791124	0.929508	0.977250	1.264024	1.075838	1.023280
	5.0	0.4000	0.808615	0.929508	0.977250	1.236682	1.075838	1.023280
3.0	1.0	3.0000	0.011590	0.011550	0.008060	86.281366	86.580087	124.069479
	1.5	2.0000	0.000920	0.015410	0.072810	1086.957650	64.892927	13.734377
	2.0	1.5000	0.021290	0.286620	0.687790	46.970457	3.488940	1.453932
	3.0	1.0000	0.542289	0.990570	0.999970	1.844034	1.009520	1.000030
	4.0	0.7500	0.978309	1.000000	1.000000	1.022172	1.000000	1.000000
	5.0	0.6000	0.999939	1.000000	1.000000	1.000061	1.000000	1.000000
$\eta, \delta = 5$								
$\bar{x}'$	s'	$\hat{\tau}'$	Power			ARL		
			n=15	n=20	n=25	n=15	n=20	n=25
1.0	1.0	1.0000	2.02E-05	2.65E-05	2.39E-05	49452.382823	37757.221952	41852.920604
	1.5	0.6667	1.12E-05	8.26E-05	2.59E-04	89038.064266	12112.437781	3867.582201
	2.0	0.5000	0.000252	0.001140	0.002051	3971.674323	877.492472	487.642149
	3.0	0.3333	0.002342	0.002695	0.002700	426.977080	371.058832	370.398347
	4.0	0.2500	0.002699	0.002700	0.002700	370.550273	370.398347	370.398347
	5.0	0.2000	0.002700	0.002700	0.002700	370.398347	370.398347	370.398347
1.5	1.0	1.5000	0.001077	0.002182	0.002731	928.605716	458.237740	366.225594
	1.5	1.0000	0.000598	0.006803	0.029549	1671.936735	147.001708	33.842503
	2.0	0.7500	0.013409	0.093900	0.234356	74.579207	10.649623	4.267015
	3.0	0.5000	0.124724	0.222058	0.308538	8.017680	4.503328	3.241097
	4.0	0.3750	0.143717	0.222454	0.308538	6.958110	4.495312	3.241097

	5.0	0.3000	0.143776	0.222454	0.308538	6.955257	4.495312	3.241097
2.0	1.0	2.0000	0.006057	0.009118	0.008649	165.101156	109.667489	115.624830
	1.5	1.3333	0.003364	0.028424	0.093591	297.261456	35.181101	10.684763
	2.0	1.0000	0.075416	0.392355	0.742290	13.259786	2.548715	1.347183
	3.0	0.6667	0.701508	0.927853	0.977250	1.425501	1.077756	1.023280
	4.0	0.5000	0.808332	0.929508	0.977250	1.237115	1.075838	1.023280
	5.0	0.4000	0.808664	0.929508	0.977250	1.236608	1.075838	1.023280
3.0	1.0	3.0000	0.007490	0.009810	0.008850	133.511487	101.936799	112.994350
	1.5	2.0000	0.004160	0.030580	0.095770	240.384865	32.701112	10.441683
	2.0	1.5000	0.093260	0.422110	0.759570	10.722722	2.369051	1.316534
	3.0	1.0000	0.867489	0.998220	1.000000	1.152752	1.001783	1.000000
	4.0	0.7500	0.999589	1.000000	1.000000	1.000411	1.000000	1.000000
	5.0	0.6000	0.999999	1.000000	1.000000	1.000001	1.000000	1.000000

Table 4: Power and ARL of T_R control chart under NI and gamma prior

NI prior								
$\bar{x}'$	s'	$\hat{t}'$	**Power**			**ARL**		
			n=15	**n=20**	**n=25**	**n=15**	**n=20**	**n=25**
1.0	1.0	1.0000	8.03E-04	2.43E-07	1.34E-04	1245.413225	4115537.192725	7490.360917
	1.5	0.6667	1.19E-06	1.05E-04	6.82E-05	841814.425784	9482.804592	14669.241479
	2.0	0.5000	0.000012	0.001302	0.001281	82863.164954	768.062929	780.706406
	3.0	0.3333	0.000640	0.002696	0.002699	1563.323966	370.917632	370.542859
	4.0	0.2500	0.002245	0.002700	0.002700	445.350905	370.398347	370.398347
	5.0	0.2000	0.002683	0.002700	0.002700	372.761656	370.398347	370.398347
1.5	1.0	1.5000	0.042760	0.000020	0.015257	23.386089	49947.913629	65.542902
	1.5	1.0000	0.000063	0.008689	0.007791	15807.401854	115.087359	128.360258
	2.0	0.7500	0.000643	0.107278	0.146383	1555.985865	9.321539	6.831415
	3.0	0.5000	0.034065	0.222143	0.308417	29.355746	4.501614	3.242361
	4.0	0.3750	0.119579	0.222454	0.308538	8.362699	4.495312	3.241097
	5.0	0.3000	0.142865	0.222454	0.308538	6.999634	4.495312	3.241097
2.0	1.0	2.0000	0.240505	0.000084	0.048325	4.157922	11953.756304	20.693220
	1.5	1.3333	0.000356	0.036307	0.024676	2810.471952	27.543217	40.525931
	2.0	1.0000	0.003615	0.448255	0.463646	276.646009	2.230872	2.156816
	3.0	0.6667	0.191597	0.928207	0.976869	5.219295	1.077346	1.023679
	4.0	0.5000	0.672566	0.929508	0.977250	1.486843	1.075838	1.023280
	5.0	0.4000	0.803537	0.929508	0.977250	1.244498	1.075838	1.023280
3.0	1.0	3.0000	0.297410	0.000090	0.049450	3.362365	11111.111127	20.222447
	1.5	2.0000	0.000440	0.039060	0.025250	2272.729631	25.601639	39.603960
	2.0	1.5000	0.004470	0.482250	0.474440	223.713879	2.073613	2.107748
	3.0	1.0000	0.236930	0.998600	0.999610	4.220660	1.001402	1.000390
	4.0	0.7500	0.831699	1.000000	1.000000	1.202358	1.000000	1.000000
	5.0	0.6000	0.993659	1.000000	1.000000	1.006381	1.000000	1.000000
Gamma Prior with $\eta, \delta = 3$								
$\bar{x}'$	s'	$\hat{t}'$	Power			ARL		
			n=15	n=20	n=25	n=15	n=20	n=25
1.0	1.0	1.0000	1.08E-07	1.13E-06	3.91E-06	9259958.683615	881900.827012	255447.136100
	1.5	0.6667	2.01E-05	1.05E-04	6.54E-04	49651.252995	9568.544235	1528.361243
	2.0	0.5000	0.000329	0.001234	0.002481	3035.803191	810.074244	403.137112
	3.0	0.3333	0.002380	0.002696	0.002700	420.248187	370.928775	370.398347
	4.0	0.2500	0.002699	0.002700	0.002700	370.546566	370.398347	370.398347
	5.0	0.2000	0.002700	0.002700	0.002700	370.398347	370.398347	370.398347
1.5	1.0	1.5000	0.000006	0.000093	0.000447	173881.420389	10703.124349	2235.238969
	1.5	1.0000	0.001073	0.008611	0.074774	932.340056	116.127931	13.373619
	2.0	0.7500	0.017542	0.101715	0.283481	57.005629	9.831406	3.527571
	3.0	0.5000	0.126721	0.222136	0.308538	7.891326	4.501750	3.241097
	4.0	0.3750	0.143719	0.222454	0.308538	6.958040	4.495312	3.241097
	5.0	0.3000	0.143776	0.222454	0.308538	6.955257	4.495312	3.241097
2.0	1.0	2.0000	0.000032	0.000390	0.001417	30915.191474	2561.519208	705.710172
	1.5	1.3333	0.006033	0.035981	0.236837	165.765102	27.792252	4.222322
	2.0	1.0000	0.098665	0.425008	0.897887	10.135298	2.352896	1.113725
	3.0	0.6667	0.712740	0.928179	0.977250	1.403036	1.077379	1.023280
	4.0	0.5000	0.808340	0.929508	0.977250	1.237102	1.075838	1.023280
	5.0	0.4000	0.808664	0.929508	0.977250	1.236608	1.075838	1.023280
3.0	1.0	3.0000	0.000040	0.000420	0.001450	25000.025939	2380.952384	689.655172
	1.5	2.0000	0.007460	0.038710	0.242350	134.048396	25.833118	4.126264
	2.0	1.5000	0.122010	0.457240	0.918790	8.196058	2.187035	1.088388
	3.0	1.0000	0.881379	0.998570	1.000000	1.134586	1.001432	1.000000
	4.0	0.7500	0.999599	1.000000	1.000000	1.000401	1.000000	1.000000
	5.0	0.6000	0.999999	1.000000	1.000000	1.000001	1.000000	1.000000

Table 5: Control limits and width of T and T_R control charts

		Example 1 Hard bake process		Example 2 Automobile engine piston ring	
Control chart	**Prior Control limits**	**NI**	**Gamma ($\eta = 15, \delta = 0.0001$)**	**NI**	**Gamma ($\eta = 15, \delta = 0.0001$)**
T	UCL	16.7826	21.786	10889.7	13599.03
	CL	11.8756	16.8686	7722.652	10547.31
	LCL	6.9686	11.9511	4555.605	7495.589
	W	9.814	9.8349	6334.094	6103.44
T_R	ULC	15.9962	20.9924	11206.8	13963.73
	CL	11.3166	16.0749	7947.527	10830.16
	LCL	6.6371	11.1575	4688.259	7696.605
	W	9.3591	9.8349	6518.536	6267.121