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Parameter estimation of Type-II extreme value distribution for censored data

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Abstract
Type-II extreme value distribution has been used fro modeling and analysis of several extreme value events relatled to floods, sea currents, and wind speeds. Probability Weighted Moments (PWMs) and Partial probability weighted moments (PPWMs) are of potential interests for estimating parameters of distributions that may be expressed in inverse form. In the present paper, the method of probability-weighted moments developed by Greenwood *et al.* (1979) and used for estimation of Type-II Extreme value distribution parameters from complete and censored samples. Expressions for Probability-weighted moments and partial probability weighted moments estimators have been derived. Based on the Monte Carlo simulation method, the derived estimators have been compared with the estimators obtained using method of moments and maximum likelihood method in terms of bias and relative efficiency using monte carlo simulation. PWM method is simpler than the other methods like method of moments and maximum likelihood method for the estimation of Fréchet distribution parameters. PWMs estimates result in explicit expressions and therefore, this method may serve as a better alternative for parameter estimation.

Keywords: Fréchet distribution, probability weighted moments, method of maximum likelihood, method of moments, Monte Carlo simulation

Introduction

Traditional methods of parameter estimation include the Method of Moments (MOMs) and the Maximum Likelihood Method (MLE). MLE is, in general, rather complicated both from the computational point of view, because the estimators cannot be obtained in closed form, and from the inferential point of view, because the likelihood function is typically nonregular. The first difficulty has been overcome, in the majority of cases, by the introduction of the expectation–maximization (EM) algorithm (Bee, 2015) [3]. The Probability Weighted Moments (PWMs) were initially introduced by Greenwood *et al.* (1979) [5]. These moments have been successfully applied for estimating parameters of a number of distributions in the statistical literature [Greenwood *et al.* (1979) [5]; Landwehr *et al.* (1979) [5]]. In particular, parameter estimation based on PWMs is of potential interest for distributions which may be expressed in inverse form i.e. if X is a random variable with cumulative distribution function F(x) then X may be written as a function of F as X = X(F).

Fréchet distribution was introduced by a French mathematician named Maurice Fréchet (1878-1973) who had identified one possible limit distribution for the largest order statistic in 1927. The Fréchet or Type-II extreme value distribution has been applied by many researchers to solve important problems related to hydrology, resource management and estimation & forecasting of a number of weather parameters such as temperature, precipitation, wind velocity, flood, drought and rainfall etc. [Reiss and Thomas (1997); Xapsos *et al.* (1998), Palutikof *et al.* (1999), Kotz and Nadarajah (2000) [10], and Embrechts *et al.* (2001)] [17, 22, 14, 4]. Gumbel (1958) [6] and Tiago de Oliveira (1972) [19] respectively, estimated the parameters of this distribution using maximum likelihood method (ML). Hooda *et al.* (1990) [8] extended the results of Tiago de Oliveira to the case of censored samples. The generalization of the standard Fréchet distribution has been introduced by Nadarajah and Kotz (2003) [11] & Abd-Elfattah and Omima (2009) [1].

Hao and Singh (2009) [7] applied the maximum entropy method to the Burr III distribution and compared the results with the MOM, ML and probability weighted moments (PWM); they found no differences on the quantiles for small return period, the differences increased for large period returns. Ahmed *et al*. (2010) [2] have considered ML and Bayesian estimation of the scale parameter of Weibull distribution with known shape and compared their performance under squared error loss. Recently, the extreme value distribution is becoming increasingly important in engineering statistics as a suitable model to represent phenomena with usually large maximum observations. In engineering circles, this distribution is often called the Fréchet model. It is one of the pioneers of extreme value statistics. The Fréchet (extreme value type II) distribution is one of the probability distributions used to model extreme events (Mubarak, 2012) [12]. Estimation of the Fréchet distribution parameters by MOMs and ML method requires an iterative solution of the resulting equations. Also MOMs fails when the shape parameter is less than or equal to 2. The moment estimators for the parameters, quantiles and confidence limits, using the general extreme value distribution for the minima, were presented towards low flow frequency analysis. The procedures to compute the parameters, quantiles for several return periods and their confidence limits (Raynal, 2013) [16]. Recently, Prosdocimi *et. al* (2016) [15] explored different statistical estimation procedures, namely maximum likelihood and partial probability weighted moments, and the strengths and weaknesses of each method and assessed the usefulness of historical data and aims to provide practitioners with useful guidelines to indicate in what circumstances the inclusion of historical data is likely to be beneficial in terms of reducing both the bias and the variability of the estimated flood frequency curves. The guidelines are based on the results of a large Monte Carlo simulation study, in which different estimation procedures and different data availability scenarios were studied. They provided some indication of the situations under which different estimation procedures might give a better performance. Khan *et al.* (2017) [9] focused on regional frequency analysis of extreme precipitation based on monthly precipitation records at 17 stations of Northern areas and Khyber Pakhtunkhwa, Pakistan and developed regional frequency methods based on L-moment and partial L-moments. The L and PL moments are derived for generalized extreme value, generalized logistic, generalized normal, and generalized Pareto distributions.

In the present study, probabilities weighted moments (PWMs) estimators are derived for the parameters of Fréchet distribution. Expressions for Probability-weighted moment estimators of Fre′chet distribution were derived. The resulting estimators have been compared with the estimators obtained from the traditional parameter estimation methods such as method of moments and maximum likelihood method. The inferences were derived from Monte Carlo simulation experiments by generating independent random samples from Fre′chet distribution for varied shapes. Performance of Probability Weighted Moments Estimators has been studied in terms of bias and relative efficiency of the estimators.

Moments and Maximum Likelihood Estimators

If a random variable X follows a Fre′chet distribution with scale parameter b and shape parameter k, the cumulative distribution function F(x) and probability density function f(x) are given by

$$F(x) = \exp\left[-\left(\frac{x}{\beta}\right)^{-k}\right] \quad \text{if } x > 0,\ b,\ k > 0 \tag{1}$$

and

$$f(x) = \left(\frac{k}{\beta}\right)\left(\frac{\beta}{x}\right)^{k} \exp\left[-\left(\frac{x}{\beta}\right)^{-k}\right]. \tag{2}$$

Estimates of Fréchet distribution parameters based on the method of moments and that of maximum likelihood method were studied by Gumbel (1958) [6] and Tiago de Olivera (1972) [19] respectively. For k > 2, moment estimates of the parameters b and k are given by

$$1+V^2 = \frac{\mu_2}{\mu_1^2} = \frac{\Gamma\left(1-\frac{2}{\hat{k}}\right)}{\left[\Gamma\left(1-\frac{1}{\hat{k}}\right)\right]^2} \tag{3}$$

and

$$\hat{\beta} = \frac{\mu_1}{\Gamma(1-\frac{1}{\hat{k}})} \tag{4}$$

Where

$$m_r = \beta\,\Gamma(1-\frac{r}{k})$$

is the r-th moment and V is the coefficient of variation. Using the sample moments in (3), k may be estimated by an iterative solution and then β can be obtained from (4). Given a random sample $X_1, X_2, \ldots X_n$, maximum likelihood estimates of β and k are obtained by simultaneous solution of the equations

$$\frac{n}{\hat{k}} = \sum \ln x_i - n\frac{\sum x_i^{-\hat{k}} \ln x_i}{\sum x_i^{-\hat{k}}} \tag{5}$$

And

$$\hat{\beta} = \left(\frac{n}{\sum x_i^{-\hat{k}}}\right)^{\frac{1}{\hat{k}}} \tag{6}$$

Method of probability weighted moments

The probability weighted moments of a random variable X with distribution function F(x) were defined Greenwood *et al*. (1979) [5] as the quantities

$$M_{prs} = E\left[X^p (F(X))^r (1-F(X))^s\right] \tag{7}$$

Where, p, r and s are real numbers. When the inverse form of the cumulative distribution function F(x) exists i.e. x can be expressed as x = x (F), then

$$M_{prs} = \int_0^1 x^p(F)\ F^r(1-F)^s dF \tag{8}$$

When r and s are non-negative integers, the parameters can be estimated by either of the following two forms of the probability-weighted moments M_{1r0} or M_{10s}. For the Fréchet distribution defined in (1) the inverse form exist and the random variable X can be expressed as

$$X = \beta[-\ln F]^{-\frac{1}{k}}$$

$$M_{1ro} = \int_0^1 x(F)F^r dF = \int_0^1 \beta[-1nF]^{\frac{-1}{k}} F^r\ dF \qquad r = 0, 1, 2$$

$$= \beta(1+r)^{\frac{1}{k}-1}\Gamma\left(1-\frac{1}{k}\right) \tag{9}$$

Where,
G () is the complete gamma function. Setting r = 0 and r =1 we have

$M_{100} = \beta\Gamma\left(1-\frac{1}{k}\right)$ and $M_{110} = 2^{\frac{1}{k}-1}\beta\ \Gamma\left(1-\frac{1}{k}\right)$

The probability weighted moments are then given by

$$\hat{k} = \frac{1n(2)}{\ln(2M_{110})-\ln(M_{100})} \tag{10}$$

and

$$\hat{\beta} = \frac{M_{100}}{\Gamma\left(1-\frac{1n(2M_{110})-\ln(M_{100})}{1n(2)}\right)} \tag{11}$$

Also

$$M_{10s} = E\left[X(F)(1-F(x))^s\right] = \int_0^1 \beta[-\ln(F)]^{-\frac{1}{k}}[1-F]^s dF$$

s = 0, 1,2… (12)

Taking s=0 and s=1 and integrating we have

$$M_{100} = \beta\Gamma\left(1-\frac{1}{k}\right) \text{ and } M_{101} = \beta\left(1-2^{\frac{1}{k}-1}\right)\Gamma\left(1-\frac{1}{k}\right)$$

If one uses the PWMs of type M_{10s}, the estimates of k and b are given in terms of M_{100} and M_{101} as

$$\hat{k} = \frac{1n(2)}{1n\left[2\left(1-\frac{M_{101}}{M_{100}}\right)\right]} \tag{13}$$

And

$$\hat{\beta} = \frac{M_{100}}{\Gamma\left(1-\frac{1n[2(1-M_{101}/M_{100})]}{1n2}\right)} \tag{14}$$

Given an ordered sample $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ of size n from any distribution, Landwehr *et al.* (1979) [5] showed that unbiased estimates of M_{100}, M_{110} and M_{101} respectively given by the following statistics

$$\hat{M}_{100} = \frac{1}{n}\sum_{j=1}^{n} X_{(j)}$$

$$\hat{M}_{110} = \frac{1}{n}\sum_{j=1}^{n}\frac{j-1}{n-1}X_{(j)}$$

And

$$\hat{M}_{101} = \frac{1}{n}\sum_{j=1}^{n}\frac{n-j}{n-1}X_{(j)}$$

Estimation from censored samples
Wang (1990) [20] extended the concept of PWW to partial weighted moments (PPWM), which can be obtained from (10) with slight modifications.

PPWM with single censoring
Under left censoring, PPWM are defined as

$$M^L_{prs} = \int_{F_L}^1 x^p\ (F)\ F^r\ (1-F)^s\ dF \tag{16}$$

For the Fréchet distribution,
when p = 1, s=0 and r = 0, 1, 2….. The PPWM are

$$M^L_{1ro} = \beta(+r)^{-(1-\frac{1}{k})}\Gamma\left[\left(1-\frac{1}{k}\right),-(1+r)\ 1nF_L\right] \tag{17}$$

Where $F_L = F(x_L)$, x_L being the left censoring threshold and

$$\Gamma(a,d) = \int_0^d t^{a-1}e^{-t}dt$$

is the incomplete gamma function. For fixed F_L, PPWM estimates of k and b can be obtained by successive solution of

$$\frac{M^L_{100}}{M^L_{110}} = \frac{\Gamma\left[1-\frac{1}{k},-1nF_L\right]}{2^{-(1-\frac{1}{k})}-\Gamma\left[\left(1-\frac{1}{k}\right),-21nF_L\right]} \tag{18}$$

$$\hat{\beta} = \frac{M_{100}^{L}}{\Gamma\left[\left(1-\frac{1}{k}\right), -1n\ F_L\right]} \quad (19)$$

For an ordered random sample X(1) < X(2) < ……….. < X(n) an unbiased estimator (Wang, 1990) [20] M^L_{1r0} for is

$$\hat{M}_{1ro}^{L} = \frac{1}{n}\sum_{j-1}^{n} \frac{(j-1)(-2).......(j-r)}{(n-1(n-2)......(n-r)} X_{(j)} \quad (20)$$

Where

$X^*_{(J)} = 0$ if $X_{(J)} \le x_L$

$= X_{(J)}$ if $X_{(J)} < x_L$

Similarly the PPWM can be obtained for right censored samples with $F_R = F(x_R)$, x_R now being the right censoring threshold.

PPWM with double censoring

PPWM from a doubly censored sample are defined as

$$M_{prs}^{D} = \int_{F_L}^{F_R} x^p\ (F) F^r (1-F)^s\ dF \quad (21)$$

For Fréchet distribution where p=1, s=0 and r = 0, 1, 2 …….

$$M_{1r0}^{D} = \beta(1+r)^{-(1-\frac{1}{k})} \left[\left\{\Gamma\left(1-\frac{1}{k}\right), -(1+r)1nF_L\right\}, \Gamma\left(1-\frac{1}{k}\right), -(1+r)1nF_R\right] \quad (22)$$

Given F_L and F_R, k and b can be estimated by successive solutions of

$$\frac{M_{100}^{D}}{M_{110}^{D}} = \frac{\Gamma\left[\left(1-\frac{1}{k}\right), 1nF_L\right] - \Gamma\left[\left(1-\frac{1}{k}\right), 1nF_R\right]}{2^{-\left(1-\frac{1}{k}\right)}\left[\Gamma\left\{\left(1-\frac{1}{k}\right), -21nF_L\right\}, -\Gamma\left\{\left(1-\frac{1}{k}\right), -21nF_R\right\}\right]} \quad (23)$$

And

$$\hat{\beta} = M_{100}^{D} / \left[\Gamma\left\{\left(1-\frac{1}{k}\right), -(1+r)1nF_L\right\}, \Gamma\left\{\left(1-\frac{1}{k}\right), -(1+r)1nF_R\right\}\right] \quad (24)$$

Also, an unbiased estimator of M^D_{1r0} is given by statistic defined in with

$X^*_{(J)} = 0$ if $x_L < X_{(J)} \le x_R = 0$ otherwise

Comparative study of estimators through monte carlo simulation

Monte Carlo sampling experiments were performed to assess the relative performance of estimators obtained by various methods of estimation outlined above in method of moments and maximum likelihood as well as in probability weighted moments and partial probability weighted moments. A simulation study was carried out involving generation of 1000 sets of samples by the method of Wichmann and Hill (1982) [21]. Since moment estimates of Fre′chet distribution are defined for k >2, hence, samples of sizes n =10, 20, 50, 100 and 200 with β =1 and k = 3.0 (1.0) 6.0 were generated for the Fréchet distribution in equation (1). For each sample MOM, ML and PWMs estimates were obtained by the procedures using method of moments and maximum likelihood as well as in probability weighted moments and partial probability weighted moments. Since no explicit expressions exist for the estimators, both MOM and ML estimates were obtained iteratively using Newton-Raphson Method. As PWM estimates result in explicit expressions, therefore these estimates were used to give the initial values needed for the iteration process. Estimated values of the parameters were then used to approximate the following performance indices for each sample size

$$E(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N} \hat{\theta}_i \quad (25)$$

And

$$\text{Bias} = E(\hat{\theta}) - \theta \quad (26)$$

$$\text{Relative Bias (\%)} = \frac{1}{N}\left[\sum_{i=1}^{N} \frac{\hat{\theta}}{\theta} - 1\right] \times 100 \quad (27)$$

Also variance and mean-squared error of the estimators are

$$V(\hat{\theta}) = \frac{1}{N}\sum \hat{\theta}_i^2 - \left[E(\hat{\theta})\right]^2 \quad (28)$$

And Mean-Squared Error

$$\text{(MSE)} = [Bias]^2 + V(\hat{\theta}) \quad (29)$$

Where $\hat{\theta}$ is an estimate of the parameter θ = (b or k) and N =1000 being the number of random samples used in the estimation.

Since, method of maximum likelihood is known to provide the minimum variance estimates at least in the asymptotic case, efficiencies of the estimators obtained by MOM and PWMs methods were computed relative to the maximum likelihood estimators. The relative efficiency of two methods in estimating a parameter, for a given sample size of n was worked out by comparing their respective mean-squared errors.

The simulation results regarding relative bias and efficiency of the estimates indicated that for the three methods in consideration, the bias of the estimates of k and β decreases as n increases. For small sample (n = 10, 20) a substantial reduction in bias of k is obtained when k assumes the value between 6 and 8. While for large samples (n = 50, 100, 200) reduction in bias could be achieved for all k values (i.e. 2.5 < k < 8) for n = 200 all the methods gave almost unbiased estimates. For small samples the probability weighted moments estimators are having relatively small bias, where as maximum likelihood the largest. The estimator β has very small but negative bias for all the four methods under consideration. For small samples (n = 10) PWMs and MOM estimators were found relatively more efficient than the ML estimators. While for n > 20 efficiency of ML estimators increased with n. The estimates of k obtained by PWMs have been the most efficient followed by MOM between the two methods *viz.* MOM, and PWMs. For all sample sizes ML estimator of β were most efficient estimators followed by PWMs estimators.

For a quick comparison of the estimates obtained by the three methods the results derived for five random samples with k = 2.5 and β =1 are presented in Table 1. The results in Table 1 are in tune with conclusions derived above.

Table: 1 Estimates based on Five Monte Carlo Samples from Fre′chet Distribution with k =2.5 and β = 1

Sample	M_{100}	M_{110}	M_{101}	PWM		MOM		MLE	
				$\hat{k}$	$\hat{\beta}$	$\hat{k}$	$\hat{\beta}$	$\hat{k}$	$\hat{\beta}$
I	1.528	0.998	0.53	2.599	1.05	3.113	1.146	2.138	1.028
II	1.692	1.153	0.538	2.234	1.051	2.569	1.155	2.277	1.062
III	1.604	1.054	0.55	2.535	1.086	3.133	1.206	2.317	1.078
IV	1.351	0.892	0.459	2.495	0.906	2.813	0.968	2.733	0.922
V	1.437	0.921	0.516	2.794	1.026	3.365	1.111	2.728	1.029
Bias				0.031	0.024	0.499	0.117	-0.061	0.024
MSE				0.034	0.004	0.325	0.020	0.064	0.004

Conclusion

Estimation of Fréchet distribution parameters by PWM method is simpler than the other methods like MOM and ML. PWMs estimates result in explicit expressions and therefore, this method may serve as a better alternative for parameter estimation. It may also be used to start the iterative process needed for the computation of moments, and maximum likelihood estimators. Moreover, the PWM estimators are relatively less biased and have comparable efficiency.

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