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Construction of structurally incomplete row-column designs for comparing test treatments with control treatments

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Abstract
In this paper, construction methods to obtain structurally incomplete row-column designs for comparing test treatments with control treatments which includes structurally incomplete Balanced Treatment - Control Row-Column (BTRC) design and structurally incomplete Balanced Bipartite Row-Column (BBPRC) designs for experimental situations where the experimenter is interested only in a subset of comparisons like in comparing several new treatments, called test treatments with existing (standard) treatment(s), called control treatment(s). For a ready reckoner, the catalogues of structurally incomplete BTRC and BBPRC designs obtainable from the given methods of construction for v_1 (number of test treatments), b_1 (number of rows), b_2 (number of columns), k_1 (number of non-empty nodes in rows) and k_2 (number of non-empty nodes in columns), r (number of replications of test treatments) ≤ 15 and r_0 (number of replications of control treatments) ≤ 30 have been prepared.

Keywords: Structurally incomplete row-column designs, test treatments, control treatments, balanced bipartite row-column designs, balanced treatment - control row-column designs

1. Introduction

Row-column designs are used in experimental situations when the heterogeneity in the experimental material is in two direction. In row-column design experiments, there may be situations in which the experimenter interested in comparing a set of test treatments with control treatments with unequal replication between the test treatments and control treatments, due to more number of treatments, one may have to conduct the experiment in incomplete rows or columns. For such experimental situations, structurally incomplete row-column designs for comparing test treatments with control treatments which includes structurally incomplete Balanced Treatment - Control Row-Column (BTRC) design and structurally incomplete Balanced Bipartite Row-Column (BBPRC) designs were useful.

The research work related to BTRC and BBPRC designs for making comparisons of several test treatments with one or two control treatments, Ture [7] introduced balanced treatment row-column (BTRC) designs for multiple comparisons with a control. Parsad and Gupta [4] introduced a new design class called balanced bipartite row-column (BBPRC) designs to compare test treatments with more than one control treatment. Parsad *et al.* [5] also provided a structurally incomplete row-column design method for comparing test treatments with control treatment(s). Sarkar *et al.* [6] gave some general methods of construction of BTRC designs in complete/incomplete rows/columns.

In the present investigation, methods of construction for obtaining structurally incomplete BTRC and BBPRC designs for comparing test treatments with one or two control treatment(s) are provided. The catalogues of these designs obtainable from given methods of construction for v_1 (number of test treatments), b_1 (number of rows), b_2 (number of columns), k_1 (number of non-empty nodes in rows) and k_2 (number of non-empty nodes in columns) ≤ 15 r (number of replications of test treatments) ≤ 15 and r_0 (number of replications of control treatments) ≤ 30 are presented in Appendix.

2. Some Preliminaries

Consider a row-column design d with v treatments, b_1 rows, b_2 columns, $k_1'=(k_{11}, \ldots, k_{1b_1})$ be the vector of row sizes, $k_2'=(k_{21}, \ldots, k_{2b_2})$ be the vector of column sizes and $r'=(r_1,\ldots,r_v)$ be the vector of replication numbers. $R=(r_1, \ldots, r_v)$, $K_1 = \text{diag}(k_{11}, \ldots, k_{1b_1})$ $K_2 = \text{diag}(k_{21}, \ldots, k_{2b_2})$ diagonal matrices of replication, rows size and column size respectively. Let N_1, N_2 and M denote treatments versus rows, treatments versus columns, and rows versus columns incidence matrices, respectively.

Definition 2.1: {Parsad *et al.* [5]} A row-column design is said to be structurally complete if $M = 1_{b_1}1'_{b_2}$ and is called structurally incomplete if at least one of the elements of M is zero.

One class of Structurally Incomplete Row-Column (SIRC) designs is Balanced Incomplete Latin Square (BILS) designs introduced by Ai *et al.* [1]. Mandal and Dash [3] gave the following definition of BILS.

Definition 2.2: A balanced incomplete Latin square (BILS) with parameters v and r is an incomplete Latin square of order v such that each row and each column has $r < v$ non-empty cells and $v - r$ empty cells and each of the v symbols appears exactly r times in the whole square. It is denoted as BILS (v, r).

Balance in BILS implies that each row and column has the same number of treatments and do not necessarily implies the pairwise balance or variance balance.

Let y_{pqr} be the response of the experimental unit occurring in the q^{th} row and r^{th} column to which p^{th} treatment is applied. The model for response is

$$y_{pqr} = \mu + \tau_p + \rho_q + \phi_r + e_{pqr}\ ; p= 1, 2, \ldots, v;\ q =1,2, \ldots, b_1\ ;\ r = 1,2,\ldots,b_2 \qquad \ldots (1)$$

where μ is general mean, ρ_q is the q^{th} row effect, ϕ_r is the r^{th} column effect, τ_p is the p^{th} treatment effect and e_{pqr} are the random error assumed to be independently and normally distributed with mean zero and variance σ^2. The coefficient matrix of reduced normal equations for estimating the linear function of treatment effects (henceforth, called as Information Matrix) is

$$C = R - N_1K_1^{-1}N_1' - N_{2(1)}K_{2(1)}^{-}N_{2(1)}' \qquad \ldots\ldots \quad (2)$$

where $N_{2(1)}= N_2 - N_1K_1^{-1}M$; $K_{2(1)}= K_2 - M'K_1^{-1}M$ and A^- is a g-inverse of A and satisfies $AA^-A = A$.

For comparing several test treatments with a single control treatment, this class of designs is known as Balanced Treatment - Control Row-Column (BTRC) design.

Definition 2.3: {Majumdar and Tamhane [2]}. A row-column design in which a control denoted as '0' and $v_1 \geq 2$ test treatments are allocated in an $b_1 \times b_2$ array is called a BTRC design if the least-squares estimators of the treatment versus control contrasts satisfy

i) $\text{Var}(\hat{\tau}_0 - \hat{\tau}_i) = A\sigma^2$ $(1 \leq i \leq v)$ and

ii) $\text{Corr}(\hat{\tau}_0-\hat{\tau}_i, \hat{\tau}_0-\hat{\tau}_{i'}) = \rho^2$ $(1 \leq i \neq i' \leq v)$

where A and ρ are some constants which depend on the particular design employed.

For comparing several test treatments with more than one control(s), Parsad and Gupta [4] introduced balanced bipartite row-column (BBPRC) designs.

Definition 2.4: An arrangement of v $(=v_1+v_2)$ treatments (v_1 tests and v_2 control treatments) in b_1 rows and b_2 columns is said to be a balanced bipartite row-column (BBPRC) designs if it estimates all elementary contrasts of treatments belonging to t^{th} set of treatments (t = 1, 2) with same variance σ^2V_p (say), and all elementary contrasts of treatments from two different sets with variance $\sigma^2V_{12} = \sigma^2V_{21}=\sigma^2V_3$ (say), where σ^2 is the common variance of the observations. Here first set of treatments are test treatments and second set of treatments are control treatments.

3. Construction methods of structurally incomplete BTRC and BBPRC designs.

In this section, we shall discuss some methods for construction of structurally incomplete BTRC and BBPRC designs.

Method 3.1: Let there exists a binary non-proper pairwise balanced block design with parameters as $v'+1$, $b' = v'+1$, $r' = (21'_{v'}, v')$, $k' = (v', 21'_{v'})$. The blocks of this design consist of two types of blocks, type 1 block contains a complete set of v' treatments i.e., (1, 2, 3, …., v'), and another type 2 block of block size two contains one treatment from v' treatments i.e., (1,2,3, …., v') and another as treatment $v'+1$. Consider ($v'+1)^{th}$ treatment as a control treatment and (1,2,3, …., v') treatments as test treatments. Now arrange the above design in $b' \times v'+1$ array such that the treatments in each block are placed in their respective positions with respect to columns in a row. Then interchange positions of the treatments in type 2 blocks containing test and control treatments and add control treatment to type 1 blocks containing only test treatments. The above procedure, gives a structurally incomplete BTRC design with the parameters $v_1 = v'$, $v_2 =1$, $b_1 = v'+1$, $b_2 = v'+1$, $r = 2$, $r_0 = v'+1$, $k_1 = (v' + 1, 21'_{v})'$, $k_2 = (21'_{v'}, v'+1)'$. Here v_1 is number of test treatments, v_2, the number of control treatments, r is replication of test treatments and r_0 is the replication of control treatment.

Example 3.1: Let Ω = {1, 2, 3, 4,5}, Then blocks of binary non-proper pairwise balanced block design are B_1={1,2,3,4}, B_2={1,5}, B_3={2,5}, B_4={3,5}, B_5={4,5}. Consider that 5^{th} treatment is a control treatment, then using the procedure described in Method 3.1, arrange the block contents in 5 × 5 array and place treatments in their respective positions as in Table 3.1(i).

Table 3.1 (i)

1	2	3	4	-
1	-	-	-	5
-	2	-	-	5
-	-	3	-	5
-	-	-	4	5

Interchange the positions of blocks containing test and control treatments and add control treatment 5 to block containing only test treatments as in Table 3.1(ii). The resultant design is structurally incomplete BTRC design with parameter $v_1 = 4$, $v_2 = 1$, $b_1 = 5$, $b_2 = 5$, $r = 2$, $r_0 = 5$, $k_1 = (5, 21'_4)'$, $k_2 = (21'_4, 5)'$.

Table 3.1 (ii)

1	2	3	4	5
5	-	-	-	1
-	5	-	-	2
-	-	5	-	3
-	-	-	5	4

The information matrix for the above design given in Table 3.1(ii)

$$\mathbf{C} = \begin{bmatrix} 1.1428 & -0.1904 & -0.1904 & -0.1904 & -0.5714 \\ -0.1904 & 1.1428 & -0.1904 & -0.1904 & -0.5714 \\ -0.1904 & -0.1904 & 1.1428 & -0.1904 & -0.5714 \\ -0.1904 & -0.1904 & -0.1904 & 1.1428 & -0.5714 \\ -0.5714 & -0.5714 & -0.5714 & -0.5714 & 2.2857 \end{bmatrix}$$

Remark 3.1: If we take 'a' copies of type 1 blocks and 'b' copies of type 2 blocks, the design so obtained is still a structurally incomplete BTRC design with parameters $v_1 = v', v_2 = 1$, $b_1 = bv' + a$, $b_2 = v' + 1$, $r = a+b$, $r_0 = bv' + a$, $k_1 = ((v' + 1)1'_a, 21'_{bv'})'$, $k_2 = ((a+b)1'_{v'}, bv' + a)'$.

For the design in example 3.1, if we take two copies of type 1 blocks, then we get a BTRC design with parameters $v_1 = 4$, $v_2 = 1$, $b_1 = 6$, $b_2 = 5$, $r = 3$, $r_0 = 6$, $k_1 = (51'_2, 21'_4)'$, $k_2 = (31'_4, 6)'$. The layout of BTRC design so obtained is

Table 3.1 (iii)

1	2	3	4	5
1	2	3	4	5
5	-	-	-	1
-	5	-	-	2
-	-	5	-	3
-	-	-	5	4

The information matrix for the above design given in Table 3.1(iii)

$$\mathbf{C} = \begin{bmatrix} 1.4222 & -0.1778 & -0.1778 & -0.1778 & -0.8888 \\ -0.1778 & 1.4222 & -0.1778 & -0.1778 & -0.8888 \\ -0.1778 & -0.1778 & 1.4222 & -0.1778 & -0.8888 \\ -0.1778 & -0.1778 & -0.1778 & 1.4222 & -0.8888 \\ -0.8888 & -0.8888 & -0.8888 & -0.8888 & 3.5556 \end{bmatrix}.$$

Method 3.2: Consider a variance balanced SIRC design in v_1 treatments arranged in p rows and q columns such that each treatment is replicated r times, each row (column) contains $k_1(k_2)$ non-empty nodes. Reinforce the SIRC design to get another structurally incomplete BTRC/BBPRC design in $p + v_2$ rows and $q + v_2$ columns with v_2 control treatments as follows:

1. The symbols in the i^{th} row and $(q + j)^{th}$ column for $i = 1, \ldots, p$ and $j = 1, \ldots, v_2$ will have each of control treatments exactly once in each of the p-rows.
2. Similarly, the symbols in the $(p + i)^{th}$ row and j^{th} column for $i = 1, \ldots, v_2$; $j = 1, \ldots, q$ will have each of v_2 control treatments exactly once in each column.

The above procedure gives a structurally incomplete BTRC/BBPRC design with parameters
$v_1 = v_1$, v_2, $b_1 = p + v_2$, $b_2 = q + v_2$, $r = r$, $r_0 = p + q + v_2$,
$k_1 = ((k_1 + v_2)1'_p, (q + v_2)1'_{v_2})'$, $k_1 = ((k_2 + v_2)1'_q, (p + v_2)1'_{v_2})'$.

Example 3.2(a): Consider pairwise and variance balanced BILS ($v_1 = 7$, $r = 3$) as in Table 2.2(a)(i) and reinforce control Treatment "8" to each row and column as in Table 2.2(ii).The resultant design is Structurally incomplete BTRC design with parameters $v_1 = 7$, $v_2 = 1$, $b_1 = b_2 = 8$, $r = 3$, $r_0 = 15$, $k_1 = (41'_7, 8)'$, $k_2 = (41'_7, 8)'$.

Table 3.2 (a)(i)

1	2	-	4	-	-	-
-	-	2	3	-	5	-
6	-	-	-	3	4	-
5	-	7	-	-	-	4
-	5	6	-	1	-	-
-	-	-	6	7	-	2
-	3	-	-	-	7	1

Table 3.2(a) (ii)

1	2	-	4	-	-	-	8
-	-	2	3	-	5	-	8
6	-	-	-	3	4	-	8
5	-	7	-	-	-	4	8
-	5	6	-	1	-	-	8
-	-	-	6	7	-	2	8
-	3	-	-	-	7	1	8
8	8	8	8	8	8	8	8

The information matrix for the above designs given in Table 3.2 (a) (ii)

$$\mathbf{C} = \begin{bmatrix} 1.484 & -0.229 & -0.229 & -0.229 & -0.229 & -0.229 & -0.229 & -0.107 \\ -0.229 & 1.484 & -0.229 & -0.229 & -0.229 & -0.229 & -0.229 & -0.107 \\ -0.229 & -0.229 & 1.484 & -0.229 & -0.229 & -0.229 & -0.229 & -0.107 \\ -0.229 & -0.229 & -0.229 & 1.484 & -0.229 & -0.229 & -0.229 & -0.107 \\ -0.229 & -0.229 & -0.229 & -0.229 & 1.484 & -0.229 & -0.229 & -0.107 \\ -0.229 & -0.229 & -0.229 & -0.229 & -0.229 & 1.484 & -0.229 & -0.107 \\ -0.229 & -0.229 & -0.229 & -0.229 & -0.229 & -0.229 & 1.4840 & -0.107 \\ -0.107 & -0.107 & -0.107 & -0.107 & -0.107 & -0.107 & -0.107 & 0.750 \end{bmatrix}$$

Example 3.2(b): Consider the same pairwise and variance balanced BILS ($v_1 = 7, r = 3$) as in Table 3.2a(i) and reinforce two control treatments "8" and "9" to each row and column as in Table 3.2(b).The resultant design is structurally incomplete BBPRC design with parameters

$v_1 = 7$, $v_2 = 2$, $b_1 = b_2 = 9$, $r = 3$, $r_0 = 16$, $k_1 = (51'_7, 91'_2)'$, $k_2 = (51'_7, 91'_2)'$.

Table 3.2(b)

1	2	-	4	-	-	-	8	9
-	-	2	3	-	5	-	8	9
6	-	-	-	3	4	-	8	9
5	-	7	-	-	-	4	8	9
-	5	6	-	1	-	-	8	9
-	-	-	6	7	-	2	8	9
-	3	-	-	-	7	1	8	9
8	8	8	8	8	8	8	8	9
9	9	9	9	9	9	9	9	8

The information matrix for the above designs given in Table 3.2(b)

$$C = \begin{bmatrix} 1.80 & -0.25 & -0.25 & -0.25 & -0.25 & -0.25 & -0.25 & -0.16 & -0.16 \\ -0.25 & 1.80 & -0.25 & -0.25 & -0.25 & -0.25 & -0.25 & -0.16 & -0.16 \\ -0.25 & -0.25 & 1.80 & -0.25 & -0.25 & -0.25 & -0.25 & -0.16 & -0.16 \\ -0.25 & -0.25 & -0.25 & 1.80 & -0.25 & -0.25 & -0.25 & -0.16 & -0.16 \\ -0.25 & -0.25 & -0.25 & -0.25 & 1.80 & -0.25 & -0.25 & -0.16 & -0.16 \\ -0.25 & -0.25 & -0.25 & -0.25 & -0.25 & 1.80 & -0.25 & -0.16 & -0.16 \\ -0.25 & -0.25 & -0.25 & -0.25 & -0.25 & -0.25 & 1.80 & -0.16 & -0.16 \\ -0.16 & -0.16 & -0.16 & -0.16 & -0.16 & -0.16 & -0.16 & 3.12 & -1.98 \\ -0.16 & -0.16 & -0.16 & -0.16 & -0.16 & -0.16 & -0.16 & -1.98 & 3.12 \end{bmatrix}.$$

Catalogues of Structurally Incomplete BTRC/BBPRC designs

The methods 3.1, 3.2 are general in nature and any design satisfying the parametric combinations can be obtained using these methods. For a ready reckoner, in the present investigations, we have prepared catalogues of structurally incomplete BTRC designs for one control and structurally incomplete BBPRC designs for two control treatments satisfying $v_1 \leq 15$, $b_1 \leq 15$, $b_2 \leq 15$, $k_1 \leq 15$, $k_2 \leq 15$ $r \leq 15$ and $r_0 \leq 30$. These catalogues are presented in Appendix.

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Statements and Declarations

Conflict of interest: On behalf of all authors, the corresponding author states that there is no conflict of interest

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Appendix: List of Structurally Incomplete BTRC/BBPRC Designs for $v_1 \leq 15$, $b_1 \leq 15$, $b_2 \leq 15$, $k_1 \leq 15$, $k_2 \leq 15$, $r \leq 15$ and $r_0 \leq 30$.

Catalogue (i): Structurally Incomplete BTRC Designs for $v_1 \leq 15$, $b_1 \leq 15$, $b_2 \leq 15$, $k_1 \leq 15$ $k_2 \leq 15$, $r \leq 15$ and $r_0 \leq 30$. (Method 3.1)

S. No.	v_1	v_2	b_1	b_2	r	r_0	k_1	k_2	n	Source
1	4	1	5	5	2	5	$(5,21_4)$	$(21_4,5)$	13	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
2	4	1	6	5	3	6	$(51_2,21_4)$	$(31_4,6)$	18	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
3	4	1	7	5	4	7	$(51_3,21_4)$	$(41_4,7)$	23	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
4	4	1	8	5	5	8	$(51_4,21_4)$	$(51_4,8)$	28	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
5	4	1	9	5	6	9	$(51_5,21_4)$	$(61_4,9)$	33	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
6	4	1	10	5	7	10	$(51_6,21_4)$	$(71_4,10)$	38	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
7	4	1	11	5	8	11	$(51_7,21_4)$	$(81_4,11)$	43	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
8	4	1	12	5	9	12	$(51_8,21_4)$	$(91_4,12)$	48	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
9	4	1	13	5	10	13	$(51_9,21_4)$	$(101_4,13)$	53	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
10	4	1	14	5	11	14	$(51_{10},21_4)$	$(121_4,14)$	58	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
11	4	1	15	5	15	15	$(51_{11},21_4)$	$(131_4,15)$	63	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
12	4	1	9	5	3	9	$(5,21_8)$	$(31_4,9)$	21	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
13	4	1	10	5	4	10	$(51_2,21_8)$	$(41_4,10)$	26	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
14	4	1	11	5	5	11	$(51_3,21_8)$	$(51_4,11)$	31	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
15	4	1	12	5	6	12	$(51_4,21_8)$	$(61_4,12)$	36	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
16	4	1	13	5	7	13	$(51_5,21_8)$	$(71_4,13)$	41	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
17	4	1	14	5	8	14	$(51_6,21_8)$	$(81_4,14)$	45	IBD [5,5, $(21_4,4)$ $(4,21_4)$]
18	4	1	15	5	9	15	$(51_7,21_8)$	$(91_4,15)$	51	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
19	4	1	13	5	4	13	$(5,21_{12})$	$(41_4,13)$	29	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
20	4	1	14	5	5	14	$(51_2,21_{12})$	$(51_4,14)$	34	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
21	4	1	15	5	6	15	$(51_3,21_{12})$	$(61_4,15)$	39	IBD [5, 5, $(21_4,4)$ $(4,21_4)$]
22	5	1	6	6	2	6	$(6,21_5)$	$(21_5,6)$	16	IBD [6, 6, $(21_5,5)$, $(5,21_5)$]
23	5	1	7	6	3	7	$(61_2,21_5)$	$(31_5,7)$	22	IBD [6, 6, $(21_5,5)$, $(5,21_5)$]
24	5	1	8	6	4	8	$(61_3,21_5)$	$(41_5,8)$	28	IBD [6, 6, $(21_5,5)$, $(5,21_5)$]
25	5	1	9	6	5	9	$(61_4,21_5)$	$(51_5,9)$	34	IBD [6, 6, $(21_5,5)$, $(5,21_5)$]
26	5	1	10	6	6	10	$(61_5,21_5)$	$(61_5,10)$	40	IBD [6, 6, $(21_5,5)$, $(5,21_5)$]

27	5	1	11	6	7	11	$(61_6,21_5)$	$(71_5,11)$	46	IBD $[6, 6, (21_5,5), (5,21_5)]$
28	5	1	12	6	8	12	$(61_7,21_5)$	$(81_5,12)$	52	IBD $[6, 6, (21_5,5), (5,21_5)]$
29	5	1	13	6	9	13	$(61_8,21_5)$	$(91_5,13)$	58	IBD $[6, 6, (21_5,5), (5,21_5)]$
30	5	1	14	6	10	14	$(61_9,21_5)$	$(101_5,14)$	64	IBD $[6, 6, (21_5,5), (5,21_5)]$
31	5	1	15	6	11	15	$(61_{10},21_5)$	$(111_5,15)$	70	IBD $[6, 6, (21_5,5), (5,21_5)]$
32	5	1	11	6	3	11	$(6,21_{10})$	$(31_5,11)$	26	IBD $[6, 6, (21_5,5), (5,21_5)]$
33	5	1	12	6	4	12	$(61_2,21_{10})$	$(41_5,12)$	32	IBD $[6, 6, (21_5,5), (5,21_5)]$
34	5	1	13	6	5	13	$(61_3,21_{10})$	$(51_5,13)$	38	IBD $[6, 6, (21_5,5), (5,21_5)]$
35	5	1	14	6	6	14	$(61_4,21_{10})$	$(61_5,14)$	44	IBD $[6, 6, (21_5,5), (5,21_5)]$
36	5	1	15	6	7	15	$(61_5,21_{10})$	$(71_5,15)$	50	IBD $[6, 6, (21_5,5), (5,21_5)]$
37	6	1	7	7	2	7	$(7,21_6)$	$(21_6,7)$	19	IBD $[7, 7, (21_6,6), (6,21_6)]$
38	6	1	8	7	3	8	$(71_2,21_6)$	$(31_6,8)$	26	IBD $[7, 7, (21_6,6), (6,21_6)]$
39	6	1	9	7	4	9	$(71_3,21_6)$	$(41_6,9)$	33	IBD $[7, 7, (21_6,6), (6,21_6)]$
40	6	1	10	7	5	10	$(71_4,21_6)$	$(51_6,10)$	40	IBD $[7, 7, (21_6,6), (6,21_6)]$
41	6	1	11	7	6	11	$(71_5,21_6)$	$(61_6,11)$	47	IBD $[7, 7, (21_6,6), (6,21_6)]$
42	6	1	12	7	7	12	$(71_6,21_6)$	$(71_6,12)$	54	IBD $[7, 7, (21_6,6), (6,21_6)]$
43	6	1	13	7	8	13	$(71_7,21_6)$	$(81_6,13)$	61	IBD $[7, 7, (21_6,6), (6,21_6)]$
44	6	1	14	7	9	14	$(71_8,21_6)$	$(91_6,14)$	68	IBD $[7, 7, (21_6,6), (6,21_6)]$
45	6	1	15	7	10	15	$(71_9,21_6)$	$(101_6,15)$	75	IBD $[7,7, (21_6,6), (6,21_6)]$
46	6	1	13	7	3	13	$(7,21_{12})$	$(31_6,13)$	31	IBD $[7, 7, (21_6,6), (6,21_6)]$
47	6	1	14	7	4	14	$(71_2,21_{12})$	$(41_6,14)$	38	IBD $[7, 7, (21_6,6), (6,21_6)]$
48	6	1	15	7	5	15	$(71_3,21_{12})$	$(51_6,15)$	45	IBD $[7,7, (21_6,6), (6,21_6)]$
49	7	1	8	8	2	8	$(8,21_7)$	$(21_7,8)$	22	IBD $[8, 8, (21_7,7), (7,21_7)]$
50	7	1	9	8	3	9	$(81_2,21_7)$	$(31_7,9)$	30	IBD $[7,7, (21_6,6), (6,21_6)]$
51	7	1	10	8	4	10	$(81_3,21_7)$	$(41_7,10)$	38	IBD $[7,7, (21_6,6), (6,21_6)]$
52	7	1	11	8	5	11	$(81_4,21_7)$	$(51_7,11)$	46	IBD $[7,7, (21_6,6), (6,21_6)]$
53	7	1	12	8	6	12	$(81_5,21_7)$	$(61_7,12)$	54	IBD $[7, 7, (21_6,6), (6,21_6)]$
54	7	1	13	8	7	13	$(81_6,21_7)$	$(71_7,13)$	62	IBD $[7,7, (21_6,6), (6,21_6)]$
55	7	1	14	8	8	14	$(81_7,21_7)$	$(81_7,14)$	70	IBD $[7,7, (21_6,6), (6,21_6)]$
56	7	1	15	8	9	15	$(81_8,21_7)$	$(91_7,15)$	78	IBD $[7, 7, (21_6,6), (6,21_6)]$
57	7	1	15	8	3	15	$(8,21_{14})$	$(31_7,15)$	36	IBD $[7,7, (21_6,6), (6,21_6)]$
58	8	1	9	9	2	9	$(9,21_8)$	$(21_8,9)$	25	IBD $[9, 9, (21_8,8), (8,21_8)]$
59	8	1	10	9	3	10	$(91_2,21_8)$	$(31_8,10)$	34	IBD $[9, 9, (21_8,8), (8,21_8)]$
60	8	1	11	9	4	11	$(91_3,21_8)$	$(41_8,11)$	43	IBD $[9, 9, (21_8,8), (8,21_8)]$
61	8	1	12	9	5	12	$(91_4,21_8)$	$(51_8,12)$	52	IBD $[9, 9, (21_8,8), (8,21_8)]$
62	8	1	13	9	6	13	$(91_5,21_8)$	$(61_8,13)$	61	IBD $[9,9, (21_8,8), (8,21_8)]$
63	8	1	14	9	7	14	$(91_6,21_8)$	$(71_8,14)$	70	IBD $[9,9, (21_8,8), (8,21_8)]$
64	8	1	15	9	8	15	$(91_7,21_8)$	$(81_8,15)$	79	IBD $[9,9, (21_8,8), (8,21_8)]$
65	9	1	10	10	2	10	$(10,21_9)$	$(21_9,10)$	28	IBD $[10,10, (21_9,9), (9,21_9)]$
66	9	1	11	10	3	11	$(101_2,21_9)$	$(31_9,11)$	38	IBD $[10,10, (21_9,9), (9,21_9)]$
67	9	1	12	10	4	12	$(101_3,21_9)$	$(41_9,12)$	48	IBD $[10,10, (21_9,9), (9,21_9)]$
68	9	1	13	10	5	13	$(101_4,21_9)$	$(51_9,13)$	58	IBD $[10,10, (21_9,9), (9,21_9)]$
69	9	1	14	10	6	14	$(101_5,21_9)$	$(61_9,14)$	68	IBD $[10,10, (21_9,9), (9,21_9)]$
70	9	1	15	10	7	15	$(101_6,21_9)$	$(71_9,15)$	78	IBD $[10,10, (21_9,9), (9,21_9)]$
71	10	1	11	11	2	11	$(11,21_{10})$	$(21_{10},11)$	31	IBD $[11,11, (21_{10},10), (10,21_{10})]$
72	10	1	12	11	3	12	$(111_2,21_{10})$	$(31_{10},12)$	42	IBD $[11,11, (21_{10},10), (10,21_{10})]$
73	10	1	13	11	4	13	$(111_3,21_{10})$	$(41_{10},13)$	53	IBD $[11,11, (21_{10},10), (10,21_{10})]$
74	10	1	14	11	5	14	$(111_4,21_{10})$	$(51_{10},14)$	64	IBD $[11,11, (21_{10},10), (10,21_{10})]$
75	10	1	15	11	6	15	$(111_5,21_{10})$	$(61_{10},15)$	75	IBD $[11,11, (21_{10},10), (10,21_{10})]$
76	11	1	12	12	2	12	$(12,21_{11})$	$(21_{11},12)$	34	IBD $[12,12, (21_{11},11), (11,21_{11})]$
77	11	1	13	12	3	13	$(121_2,21_{11})$	$(31_{11},13)$	46	IBD $[12,12, (21_{11},11), (11,21_{11})]$
78	11	1	14	12	4	14	$(121_3,21_{11})$	$(41_{11},14)$	58	IBD $[12,12, (21_{11},11), (11,21_{11})]$
79	11	1	15	12	5	15	$(121_4,21_{11})$	$(51_{11},15)$	70	IBD $[12,12, (21_{11},11), (11,21_{11})]$
80	12	1	13	13	2	13	$(13,21_{12})$	$(21_{12},13)$	37	IBD $[13,13, (21_{12},12), (12,21_{12})]$
81	12	1	14	13	3	14	$(131_2,21_{12})$	$(31_{12},14)$	50	IBD $[13,13, (21_{12},12), (12,21_{12})]$
82	12	1	15	13	4	15	$(131_3,21_{12})$	$(31_{12},14)$	63	IBD $[13,13, (21_{12},12), (12,21_{12})]$
83	13	1	14	14	2	14	$(14,21_{13})$	$(21_{13},14)$	40	IBD $[14,14, (21_{13},13), (13,21_{13})]$
84	13	1	15	14	3	15	$(141_2,21_{13})$	$(31_{13},15)$	54	IBD $[14,14, (21_{13},13), (13,21_{13})]$

IBD: denotes the Incomplete Block Design

Catalogue (ii): Structurally Incomplete BTRC/BBPRC Designs for $v_1 \le 15$, $b_1 \le 15$, $b_2 \le 15$, $k_1 \le 15$, $k_2 \le 15$, $r \le 15$ and $r_0 \le 30$. (Method 3.2)

S. No.	v_1	v_2	b_1	b_2	r	r_0	k_1	k_2	n	Source
1	4	1	5	5	3	9	$(41_4,5)$	$(41_4,5)$	21	BILS (4,3)
2	5	1	6	6	4	11	$(51_5,6)$	$(51_5,6)$	31	BILS (5,4)
3	6	1	7	7	5	13	$(61_6,7)$	$(61_6,7)$	43	BILS (6,5)
4	7	1	8	8	3	15	$(41_7,8)$	$(41_7,8)$	36	BILS (7,3)

5	7	1	8	8	4	15	$(51_7,8)$	$(51_7,8)$	43	BILS (7,4)
6	7	1	8	8	6	15	$(71_7,8)$	$(71_7,8)$	36	BILS (7,6)
7	8	1	9	9	7	17	$(81_8,9)$	$(81_8,9)$	73	BILS (8,7)
8	9	1	10	10	8	19	$(91_9,10)$	$(91_9,10)$	91	BILS (9,8)
9	10	1	11	11	9	21	$(101_{10},11)$	$(101_{10},11)$	111	BILS (10,9)
10	11	1	12	12	10	23	$(111_{11},12)$	$(111_{11},12)$	133	BILS (11,10)
11	11	1	12	12	5	23	$(61_{11},12)$	$(61_{11},12)$	78	BILS (11,5)
12	11	1	12	12	6	23	$(71_{11},12)$	$(71_{11},12)$	89	BILS (11,6)
13	12	1	13	13	11	25	$(121_{12},13)$	$(121_{12},13)$	157	BILS (12,11)
14	13	1	14	14	12	27	$(131_{13},14)$	$(131_{13},14)$	183	BILS (13,12)
15	14	1	15	15	13	29	$(141_{14},15)$	$(141_{14},15)$	211	BILS (14,13)
16	4	2	6	6	3	10	$(51_4,61_2)$	$(51_4,61_2)$	32	BILS (4,3)
17	5	2	7	7	4	12	$(61_5,71_2)$	$(61_5,71_2)$	44	BILS (5,4)
18	6	2	8	8	5	14	$(71_6,81_2)$	$(71_6,81_2)$	58	BILS (6,5)
19	7	2	9	9	3	16	$(51_7,91_2)$	$(51_7,91_2)$	53	BILS (7,3)
20	7	2	9	9	4	16	$(61_7,91_2)$	$(61_7,91_2)$	60	BILS (7,4)
21	7	2	9	9	6	16	$(81_7,91_2)$	$(81_7,91_2)$	74	BILS (7,6)
22	8	2	10	10	7	18	$(91_8,101_2)$	$(91_8,101_2)$	92	BILS (8,7)
23	9	2	11	11	8	20	$(101_9,111_2)$	$(101_9,111_2)$	112	BILS (9,8)
24	10	2	12	12	9	22	$(111_{10},121_2)$	$(111_{10},121_2)$	134	BILS (10,9)
25	11	2	13	13	5	24	$(71_{11},131_2)$	$(71_{11},131_2)$	103	BILS (11,5)
26	11	2	13	13	6	24	$(81_{11},131_2)$	$(81_{11},131_2)$	114	BILS (11,6)
27	11	2	13	13	10	24	$(121_{11},131_2)$	$(121_{11},131_2)$	158	BILS (11,10)
28	12	2	14	14	11	26	$(131_{12},141_2)$	$(131_{12},141_2)$	184	BILS (12,11)
29	13	2	15	15	12	28	$(141_{13},151_2)$	$(141_{13},151_2)$	212	BILS (13,12)
30	4	1	8	8	7	15	$(51_7,8)$	$(51_7,8)$	43	SIRC (4,7)
31	5	1	8	8	7	15	$(61_7,8)$	$(61_7,8)$	50	SIRC (5,7)
32	6	1	8	8	7	15	$(71_7,8)$	$(71_7,8)$	42	SIRC (6,7)
33	6	1	12	12	11	23	$(71_{11},12)$	$(71_{11},12)$	89	SIRC (6,11)
34	7	1	12	12	11	23	$(81_{11},12)$	$(81_{11},12)$	100	SIRC (7,11)
35	8	1	12	12	11	23	$(91_{11},12)$	$(91_{11},12)$	111	SIRC (8,11)
36	9	1	12	12	11	23	$(101_{11},12)$	$(101_{11},12)$	122	SIRC (9,11)
37	10	1	12	12	11	23	$(111_{11},12)$	$(111_{11},12)$	133	SIRC (10,11)
38	5	1	14	14	13	27	$(61_{13},14)$	$(61_{13},14)$	92	SIRC (5,13)
39	6	1	14	14	13	27	$(71_{13},14)$	$(71_{13},14)$	105	SIRC (6,13)
40	7	1	14	14	13	27	$(81_{13},14)$	$(81_{13},14)$	118	SIRC (7,13)
41	8	1	14	14	13	27	$(91_{13},14)$	$(91_{13},14)$	131	SIRC (8,13)
42	9	1	14	14	13	27	$(101_{13},14)$	$(101_{13},14)$	144	SIRC (9,13)
43	10	1	14	14	13	27	$(111_{13},14)$	$(111_{13},14)$	157	SIRC (10,13)
44	11	1	14	14	13	27	$(121_{13},14)$	$(121_{13},14)$	170	SIRC (11,13)
45	12	1	14	14	13	27	$(131_{13},14)$	$(131_{13},14)$	183	SIRC (12,13)
46	4	2	9	9	7	16	$(61_7,91_2)$	$(61_7,91_2)$	60	SIRC (4,7)
47	5	2	9	9	7	16	$(71_7,91_2)$	$(71_7,91_2)$	67	SIRC (5,7)
48	6	2	9	9	7	16	$(81_7,91_2)$	$(81_7,91_2)$	74	SIRC (6,7)
49	6	2	13	13	11	24	$(81_{11},131_2)$	$(81_{11},131_2)$	114	SIRC (6,11)
50	7	2	13	13	11	24	$(91_{11},131_2)$	$(91_{11},131_2)$	127	SIRC (7,11)
51	8	2	13	13	11	24	$(101_{11},131_2)$	$(101_{11},131_2)$	136	SIRC (8,11)
52	9	2	13	13	11	24	$(111_{11},131_2)$	$(111_{11},131_2)$	147	SIRC (9,11)
53	10	2	13	13	11	24	$(121_{11},131_2)$	$(121_{11},131_2)$	158	SIRC (10,11)
54	5	2	15	15	13	28	$(71_{13},151_2)$	$(71_{13},151_2)$	121	SIRC (5,13)
55	6	2	15	15	13	28	$(81_{13},151_2)$	$(81_{13},151_2)$	134	SIRC (6,13)
56	7	2	15	15	13	28	$(91_{13},151_2)$	$(91_{13},151_2)$	147	SIRC (7,13)
57	8	2	15	15	13	28	$(101_{13},151_2)$	$(101_{13},151_2)$	160	SIRC (8,13)
58	9	2	15	15	13	28	$(111_{13},151_2)$	$(111_{13},151_2)$	173	SIRC (9,13)
59	10	2	15	15	13	28	$(121_{13},151_2)$	$(121_{13},151_2)$	186	SIRC (10,13)
60	11	2	15	15	13	28	$(131_{13},151_2)$	$(131_{13},151_2)$	199	SIRC (11,13)
61	12	2	15	15	13	28	$(141_{13},151_2)$	$(141_{13},151_2)$	212	SIRC (12,13)

SIRC: denotes the Structurally Incomplete Row column design, BILS: denotes the Balanced Incomplete Latin square design