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Mariamma Antony
Associate Professor, Department of Statistics, Little Flower College, Guruvayoor, University of Calicut Kerala, India

Corresponding Author:
Mariamma Antony
Associate Professor, Department of Statistics, Little Flower College, Guruvayoor, University of Calicut,, Kerala, India

Time series models with tailed generalized geometric Linnik distribution as marginals

Mariamma Antony

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Abstract
Tailed distributions are found to be useful in the study of life testing experiments and clinical trials. Tailed forms of type I and type II generalized geometric Linnik distribution and their asymmetric forms are studied in [1]. The usual technique of transferring data to use a Gaussian model fails in certain situations. Hence a number of non-Gaussian autoregressive models have been introduced by various researchers. A first order autoregressive model with tailed type I generalized geometric Linnik distribution is introduced in this paper. It is shown that the process is not time reversible. The model is extended to higher order cases.

Keywords: Geometric linnik distribution, generalized geometric linnik distribution, tailed distributions, tailed geometric Linnik distributions, autoregressive models

1. Introduction

The study of non-Gaussian autoregressive models began with the pioneering work of [2]. They have considered a first order autoregressive (AR(1)) model with exponential marginal distribution. The model is given by $X_0 = \varepsilon_1$

___and for $n = 1, 2, \ldots$

$$X_n = \rho X_{n-1} + \begin{cases} 0 & w.p. \quad \rho \\ \varepsilon_n & w.p. \quad (1-\rho) \end{cases} \tag{1.1}$$

and '$w.p.$' stands for with probability, $0 \le \rho \le 1$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed exponential random variables.

Type I generalized geometric LInnik and type II generalized geometric Linnik distributions are studied in [3]. Autoregressive models with Geometric Linnik marginal distribution were developed in [4].The models developed can be used for modeling stock price returns, speech waves etc, as an alternative to generalized Linnik laws and Pakes generalized Linnik laws.

Definition 1.1

A random variable X on R is said to have geometric Linnik distribution and write $X \underset{=}{d} GL(\alpha, \lambda)$ if its characteristic function $\phi(t)$ is

$$\phi(t) = \frac{1}{1 + \ln(1 + \lambda |t|^{\alpha})}, t \in R, 0 < \alpha \le 2, \lambda > 0. \tag{1.2}$$

Definition 1.2

A random variable X on R is said to have type I generalized geometric Linnik distribution and write $X \underset{=}{d} GeGL_1(\alpha, \lambda, p)$ if it has the characteristic function.

$$\phi(t) = \frac{1}{1 + p\ln(1+\lambda|t|^{\alpha})}, \quad 0<\alpha\le 2,\ p>0,\ \lambda>0. \tag{1.3}$$

Definition 1.3

A random variable X on R has type II generalized geometric Linnik distribution and write $X \overset{d}{=} GeGL_2(\alpha,\lambda,\tau)$, if it has the characteristic function

$$\phi(t) = \left[\frac{1}{1+ln(1+\lambda|t|^{\alpha})}\right]^{\tau}, \quad t\in R, 0<\alpha\le 2,\ \lambda,\tau>0. \tag{1.4}$$

Note that when $\tau = 1$, Type II generalized geometric Linnik distribution reduces to geometric Linnik distribution.

2. Tailed forms of Type I and Type II generalized geometric Linnik distribution

Tailed distributions have found applications in various fields and were studied by many authors (see, [5], [6] and [7]). Tailed distributions are found to be useful in the study of life testing experiments and clinical trials. These distributions can be used data which exhibit zeros, as in the case of stream flow data of rivers that are dry during part of the year. They are useful for modeling life times of devices, which have some probability for damage immediately when it is put to use. We encounter tailed distributions in life testing experiments where an item fails instantaneously. In clinical trials, sometimes a medicine has no response initially with a certain probability and on a later stage there may be response, the length of the response is described by certain probability distribution Tailed forms of type I and type II generalized geometric Linnik distribution and their asymmetric forms are studied in [1].

Definition 2.1

Let the random variable X has distribution function F(x) and characteristic function $\phi_X(t)$. A tailed random variable U with tail probability θ associated with X is defined by the characteristic function.

$$\phi_U(t) = \theta + (1-\theta)\phi_X(t) \tag{2.1}$$

Definition 2.2

A random variable U is said to have tailed type I generalized geometric Linnik distribution $TGeGL_1(\alpha,\lambda,\tau,\theta)$ if it has the characteristic function.

$$\phi_U(t) = \frac{1+\tau\theta\ln\left(1+\lambda|t|^{\alpha}\right)}{1+\tau\ln\left(1+\lambda|t|^{\alpha}\right)}. \tag{2.2}$$

Definition 2.3

A random variable X is said to have tailed type II generalized geometric Linnik distribution and write $X \overset{d}{=} TGeGL_2(\alpha,\lambda,\tau,\theta)$ distribution if it has the characteristic function.

$$\phi_X(t) = \frac{\theta\left[1+\ln\left(1+\lambda|t|^{\alpha}\right)\right]^{\tau} + (1-\theta)}{\left[1+\ln\left(1+\lambda|t|^{\alpha}\right)\right]^{\tau}}, 0<\alpha\le 2, 0<\theta<1,\ \lambda,\tau>0. \tag{2.3}$$

The tailed type II generalized geometric Linnik distribution being the tailed form of type II generalized geometric Linnik is infinitely divisible.

3. Time series models with tailed generalized geometric Linnik distribution as marginals

Now we develop an AR(1) model with $TGeGL_1$ distribution as marginal.

Consider the model

$$X_n = \begin{cases} \varepsilon_n & w.p. \quad p \\ X_{n-1} + \varepsilon_n & w.p. \quad 1-p \end{cases} \tag{3.1}$$
$$= I_n X_{n-1} + \varepsilon_n$$

Where $\{\varepsilon_n\}$ and $\{I_n\}$ are two independent sequences of independent and identically distributed random variables with I_n, X_{n-1} and ε_n mutually independent such that $P(I_n = 0) = p = 1 - P(I_n = 1)$.

We have the model (3.1) in terms of characteristic functions is

$$\phi_{\varepsilon_n}(t) = \frac{\phi_{X_n}(t)}{p + (1-p)\phi_{X_{n-1}}(t)}.$$

In the stationary case

$$\phi_{\varepsilon_n}(t) = \frac{\phi_X(t)}{p + (1-p)\phi_X(t)}.$$

If X has characteristic function (2.2), then

$$\phi_{\varepsilon_n}(t) = \frac{1 + \tau\theta \ln\left(1 + \lambda |t|^{\alpha}\right)}{1 + \tau c \ln\left(1 + \lambda |t|^{\alpha}\right)}$$, where $c = p + (1-p)\theta.$

$$\phi_{\varepsilon_n}(t) = \frac{\tau\theta}{c} + \left(1 - \frac{\tau\theta}{c}\right)\frac{1}{1 + \tau c \ln\left(1 + \lambda |t|^{\alpha}\right)}.$$

That is,

Hence, if the model (3.1) is stationary with $TGeGL_1(\alpha, \lambda, \tau, \theta)$ marginal distribution, then the distribution of the innovation sequence $\{\varepsilon_n\}$ is $TGeGL_1\left(\alpha, \lambda, \tau c, \frac{\tau\theta}{c}\right)$.

If $X_0 \overset{d}{=} TGeGL_1(\alpha, \lambda, \tau, \theta)$ and $\{\varepsilon_n\}$ are independent and identically distributed as $TGeGL_1\left(\alpha, \lambda, \tau c, \frac{\tau\theta}{c}\right)$, then the characteristic function of X_1 is $\phi_{X_1}(t) = \left[p + (1-p)\phi_{X_0}(t)\right]\phi_{\varepsilon_1}(t)$

$$= \left[p + (1-p)\frac{1 + \tau\theta \ln\left(1 + \lambda |t|^{\alpha}\right)}{1 + \tau \ln\left(1 + \lambda |t|^{\alpha}\right)}\right]\left[\frac{1 + \tau\theta \ln\left(1 + \lambda |t|^{\alpha}\right)}{1 + \tau c \ln\left(1 + \lambda |t|^{\alpha}\right)}\right]$$

$$= \frac{1 + \tau\theta \ln\left(1 + \lambda |t|^{\alpha}\right)}{1 + \tau \ln\left(1 + \lambda |t|^{\alpha}\right)}.$$

That is, $X_1 \overset{d}{=} X_0$.

If $X_{n-1} \overset{d}{=} X_0$, we can prove that $X_n \overset{d}{=} X_0$ and hence the process $\{X_n\}$ is stationary.

Based on this, we now define stationary AR(1) tailed Type I generalized geometric Linnik process as follows.

Let $X_0 \underset{=}{d} TGeGL_1(\alpha,\lambda,\tau,\theta)$ and for $n=1,2,\ldots$

$$X_n = \begin{cases} \varepsilon_n & w.p. \quad p \\ X_{n-1}+\varepsilon_n & w.p. \quad 1-p \end{cases}$$

Where $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $TGeGL_1\left(\alpha,\lambda,\tau c,\frac{\tau\theta}{c}\right)$ random variables where $c=p+(1-p)\theta.$

It can be shown that the process $\{X_n\}$ is not time reversible. For this, consider the characteristic function.

$$\phi_{X_n,X_{n+1}}(t_1,t_2)=E\left(e^{it_1X_n+it_2X_{n+1}}\right)$$

$$=\phi_{\varepsilon_n}(t_2)\left[p\phi_{X_n}(t_1)+(1-p)\phi_{X_n}(t_1+t_2)\right]$$

$$=\frac{1+\tau\theta\ln\left(1+\lambda|t|^{\alpha}\right)}{1+\tau c\ln\left(1+\lambda|t|^{\alpha}\right)}\left[p\frac{1+\tau\theta\ln\left(1+\lambda|t_1|^{\alpha}\right)}{1+\tau\ln\left(1+\lambda|t_2|^{\alpha}\right)}+(1-p)\frac{1+\tau\theta\ln\left(1+\lambda|t_1+t_2|^{\alpha}\right)}{1+\tau\ln\left(1+\lambda|t_1+t_2|^{\alpha}\right)}\right].$$

This expression is not symmetric in t_1 and t_2.

Remark 3. 1

If $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $TGeGL_1\left(\alpha,\lambda,\tau c,\frac{\tau\theta}{c}\right)$, where $c=p+(1-p)\theta$ then (3.1) is asymptotically stationary with $TGeGL_1(\alpha,\lambda,\tau,\theta)$ marginal distribution.

Consider the k^th^ order autoregressive process

$$X_n = \begin{cases} \varepsilon_n & w.p. \quad p \\ X_{n-1}+\varepsilon_n & w.p. \quad p_1 \\ X_{n-2}+\varepsilon_n & w.p. \quad p_2\,. \\ \quad\vdots & \\ X_{n-k}+\varepsilon_n & w.p. \quad p_k \end{cases} \tag{3.2}$$

If the process $\{X_n\}$ is stationary, then in terms of characteristic function, (3.2) is

$$\phi_{\varepsilon_n}(t)=\frac{\phi_X(t)}{p+(1-p)\phi_X(t)}, \quad \text{Where} \quad 1-p=\sum_{i=1}^{k}p_i, \quad 0<p_i<1.$$

Thus a necessary and sufficient condition for the model (3.2) defines a stationary AR(k) process with $TGeGL_1(\alpha,\lambda,\tau,\theta)$ marginal distribution is that $\{\varepsilon_n\}$ is distributed as $TGeGL_1\left(\alpha,\lambda,\tau c,\frac{\tau\theta}{c}\right)$.

4. Conclusion

In conclusion, this study delves into non-Gaussian autoregressive models, initiated by groundbreaking work [2]. The exploration of autoregressive models with geometric Linnik marginal distribution, including type I and type II generalized forms [3], offers valuable insights. Tailed distributions, essential in life testing and clinical trials, find application in the examined models [1]. The developed AR(1) model with tailed type I generalized geometric Linnik distribution as marginal presents a significant contribution. The stationary nature of the model is established, emphasizing the importance of the characteristic function in determining stationarity. The exploration extends to AR(k) processes, establishing necessary and sufficient conditions for stationarity in the context of the generalized geometric Linnik distribution [5-7]. Overall, these models provide alternative tools for modeling diverse phenomena, such as stock price returns and speech waves, enhancing the repertoire of statistical methods.

5. References

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