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Some geometric properties of a new sequence space of No¨rlund type derived by de la Valle´e-Poussin mean

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Abstract
The aim of this paper is to introduce a generalized modular sequence space of No¨rlund type defined by de la Valle´e-Poussin mean and study some of its topological and geometric properties like *k*–*NUC* property, uniform Opial property of this Ko¨ the sequence space.
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Introduction

Geometric characteristics of Banach spaces, including the Opial property, the Fatou property, and their generalizations, are essential to the theory of metric fixed points. The Opial property was defined by Opial in [16] and it was demonstrated that while the space $L_p[0,2\pi](p \neq 2, 1 < p < \infty)$ does not satisfy this property, $\ell_p(1 < p < \infty)$ does. It has been demonstrated by Franchetti [5] that any infinite dimensional Banach space has an equivalent norm that satisfies the Opial property. Subsequently, Prus [19] introduced uniform Opial property for Banach spaces and studied it. Further research on the uniform Opial property for Cesaro-Orlicz spaces has been done by Cui and Hudzik [2], Petrot and Suantai [18], Mongkolkeha and Kumam [14], Şimşek *et al.* [22] and numerous other researchers.
Uniform convexity was first introduced by Clarkson [1] and it is well known that this means that Banach spaces are reflexive. The notion of nearly uniform convexity of Banach spaces was first presented by Huff [7]. According to Rolewicz [20], if X has the drop property, then the Banach space X is reflexive. This result was further extended by Montesinon [15], who demonstrated that X has drop property if and only if it is reflexive and has property $H. k - NUC$ Banach spaces have been characterized by Kutzarova [9].
In summability theory, Leindler [10] initially defines the (V, λ)-summability using de la Vallée-Poussin's mean. Generalized de la Vallée-Poussin's mean is the basis for several sequence spaces that have been introduced and investigated by Malkowsky and Savas [12]. Several authors have also examined the (V, λ) summable sequence spaces, notably Et [3] and Savaş and Savaş [21]. Subsequently, other scholars, including Et *et al.* [4], Şimşek *et al.* [22, 23] and Şimşek [24] and others, developed different sequence spaces and investigated certain geometric features on those sequence spaces by utilizing the idea of de la Vallée-Poussin mean.
This work presents the definition of de la Vallée-Poussin mean, which defines a new generalized modular sequence space $N(\lambda, p)$ of Nörlund type. Additionally, a few of its geometric and topological aspects are described for this sequence space, including the uniform Opial property and the $k - NUC$ property.

Definitions and Preliminaries

Throughout this paper, $\mathbb{N}, \mathbb{R}, \mathbb{R}^+$ and F denotes the set of natural numbers, of real numbers, of nonnegative real numbers and the scalar field respectively. Let the space of all real sequences

$x=(x(i))_{i=1}^{\infty}$ is denote by ℓ^0 and $(X,\|\cdot\|)$ be a Banach space and being a subspace of ℓ^0. Let $S(X)$ and $B(X)$ of X denotes the unit sphere and closed unit ball respectively.

A Banach space $X=(X,\|\cdot\|)$ is said to be a Köthe sequence space if X is a subspace of ℓ^0 such that ([11, 8]):
1. If $x\in\ell^0, y\in X$ and $|x(i)|\leq|y(i)|$ for all $i\in\mathbb{N}$, then $x\in X$ and $\|x\|\leq\|y\|$.
2. There is an element $x\in X$ such that $x(i)>0$ for all $i\in\mathbb{N}$.

A sequence $(x_n)\subset X$ is said to be ε-separated sequence, if separation of sequence (x_n) denoted by $\text{sep}(x_n)=\inf\{\|x_n-x_m\|: n\neq m\}>\varepsilon$ for some $\varepsilon>0$ (Huff [7]).
A Banach space X is said to be uniformly convex, denoted by (UC), if for each $\varepsilon>0$, there is a $\delta>0$ such that for $x,y\in S(X)$, $\|x-y\|\geq\varepsilon$ implies $\left\|\frac{x+y}{2}\right\|<1-\delta$. For any $x\notin X$, the drop determined by x is the set $D(x,B(X))=\text{conv}(\{x\}\cup B(X))$. A Banach space X has the drop property (D), if for every closed set C disjoint with $B(X)$, there exists an element $x\in C$ such that $D(x,B(X))\cap C=\{x\}$.
A Banach space X is called nearly uniformly convex (NUC) if for every $\varepsilon>0$, there exists $\delta\in(0,1)$ such that for every $(x_n)\subseteq B(X)$ with $\text{sep}(x_n)>\varepsilon$, we have $\text{conv}(x_n)\cup((1-\delta)B(X))\neq\phi$. Huff [7] has proved that every NUC Banach space is reflexive and has property (H).
Kutzarova [9] has given a characterization of k-nearly uniformly convex Banach spaces. Let $k\geq 2$ be an integer. A Banach space X is said to be k-nearly uniformly convex $(k-NUC)$, if for any $\varepsilon>0$, there exists $\delta>0$ such that for any sequence $(x_n)\subset B(X)$ with $\text{sep}(x_n)>\varepsilon$, there are $n_1,n_2,\dots,n_k\in\mathbb{N}$ such that $\left\|\frac{x_{n_1}+x_{n_2}+\cdots+x_{n_k}}{k}\right\|<1-\delta$. It is clear that $k-NUC$ Banach spaces are NUC but the opposite does not hold in general.
An element $x\in X$ is said to be order continuous, if for any sequence $(x_n)\subset X$ such that $x_n(i)\leq|x(i)|$ for each $i\in\mathbb{N}$ and $x_n(i)\to 0(n\to\infty)$, we have $\|x_n\|\to 0$ holds.
A Köthe sequence space X is said to be order continuous if all sequences in X are order continuous. It is easy to see that $x\in X$ is order continuous if and only if

$$\|(0,\dots,0,x(n+1),x(n+2),\dots)\|\to 0 \text{ as } n\to\infty$$

A Banach space X is said to have the Kadec-Klee property (or property (H)) if every weakly convergent sequence on the unit sphere with the weak limit in the sphere is convergent in norm.
A Banach space X is said to have the Opial property ([19]), if for every weakly null sequence $(x_n)\subset X$ and every non-zero $x\in X$, we have

$$\liminf_{n\to\infty}\|x_n\|<\liminf_{n\to\infty}\|x_n+x\|.$$

A Banach space X is said to have the uniform Opial property ([19]), if for each $\varepsilon>0$, there exists $\mu>0$ such that for any weakly null sequence (x_n) in $S(X)$ and $x\in X$ with $\|x\|\geq\varepsilon$, the following inequality holds:

$$1+\mu\leq\liminf_{n\to\infty}\|x_n+x\|.$$

In any Banach space X, an Opial property is important because it ensures that X has a weak fixed point property ([6]). Opial in ([16]) has shown that the space $L_p[0,2\pi](p\neq 2, 1<p<\infty)$ does not have this property but the Lebesgue sequence space $\ell_p(1<p<\infty)$ has.

For a real vector space X, a function $\rho: X\to[0,\infty]$ is called a modular if it satisfies the following conditions:
1. $\rho(x)=0$ if and only if $x=0$.
2. $\rho(\alpha x)=\rho(x)$ for all $\alpha\in F$ with $|\alpha|=1$.
3. $\rho(\alpha x+\beta y)\leq\rho(x)+\rho(y)$ for all $x,y\in X$ and all $\alpha,\beta\geq 0$ with $\alpha+\beta=1$.

Further, the modular ρ is called the convex if,
4. $\rho(\alpha x+\beta y)\leq\alpha\rho(x)+\beta\rho(y)$ holds for all $x,y\in X$ and all $\alpha,\beta\geq 0$ with $\alpha+\beta=1$.

If ρ is a modular in X, we define

$$X_\rho=\{x\in X: \rho(\lambda x)\to 0 \text{ as } \lambda\to 0^+\},$$
$$X_\rho^*=\{x\in X: \rho(\lambda x)<\infty \text{ for some } \lambda>0\}.$$

It is clear that $X_\rho\subseteq X_\rho^*$. If ρ is a convex modular, the functions

$$\|x\|_L=\inf\left\{\lambda>0: \rho\left(\frac{x}{\lambda}\right)\leq 1\right\}$$

And

$$\| x \|_A = \inf_{\lambda>0} \frac{1}{\lambda}(1+\rho(\lambda x))$$

are two norms on X_ρ. If ρ is a convex modular on X, then $X_\rho = X_\rho^*$ and both $\|\cdot\|_L$ and $\|\cdot\|_A$ is a norm on X_ρ for which X_ρ is a Banach space.

The norms $\|\cdot\|_L$ and $\|\cdot\|_A$ are called the Luxemburg norm and the Amemiya norm (Orlicz norm) respectively. In addition,

$$\|\cdot\|_L \le \|\cdot\|_A \le 2\ \|\cdot\|_L$$

for all $x \in X_\rho$ holds ([17]).
A sequence (x_n) in X_ρ is called modular convergent to $x \in X_\rho$ if there exists a $\lambda > 0$ such that $\rho(\lambda(x_n - x)) \to 0$ as $n \to \infty$.

Lemma 2.1. Let $(x_n) \subset X_\rho$. Then $\|x_n\|_L \to 0$ (or equivalently $\|x_n\|_A \to 0$) if and only if $\rho(\lambda x_n) \to 0$ as $n \to \infty$, for every $\lambda > 0$.
Proof. See [[17], p.15, Theorem 1].
Throughout this paper, we assume the sequence $p = (p_n)$ as a bounded sequence of positive real numbers with $p_n > 1, H = \sup_n p_n$ and $M = \max\{1, H\}$.
Besides this, we need the following inequalities in the sequel;

$$|a_n + b_n|^{p_n} \le K(|a_n|^{p_n} + |b_n|^{p_n}) \tag{2.1}$$

$$|a_n + b_n|^{q_n} \le |a_n|^{q_n} + |b_n|^{q_n} \tag{2.2}$$

where $q_n = \frac{p_n}{M} \le 1$ and $K = \max\{1, 2^{H-1}\}$ with $H = \sup_n p_n$.
Notations: For any $x \in \ell^0$ and $i \in \mathbb{N}$, we use the following notations throughout the paper:

$$x|_i = (x(1), x(2), \dots, x(i), 0, 0, \dots), \text{ called the truncation of } x \text{ at } i,$$
$$x|_{\mathbb{N}-i} = (0, 0, \dots, 0, x(i+1), x(i+2), \dots),$$
$$x|_I = \{x = (x(i))_{i=1}^{\infty} : x(i) \ne 0 \text{ for all } i \in I \subseteq \mathbb{N} \text{ and } x(i) = 0 \text{ for all } i \in \mathbb{N} \setminus I\},$$
$$\operatorname{supp} x = \{i \in \mathbb{N} : x(i) \ne 0\}$$

and cl A denotes the closure of a set A.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_{n+1} \le \lambda_n + 1, \lambda_1 = 1$.
The generalized de la Vallée-Poussin means of a sequence $x = (x_n)$ are defined as follows:

$$t_n(x) = \frac{1}{\lambda_n} \sum_{j \in I_n} x_j$$

Where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, \dots$

A sequence $x = (x_n)$ is said to be (V, λ)-summable to a number l if $t_n(x) \to l$ as $n \to \infty$ [10]. If $\lambda_n = n$, then (V, λ)-summability and strongly (V, λ)-summability are reduced to $(C, 1)$-summability and $[C, 1]$-summability, respectively.
Let (t_n) be a sequence of non-negative real numbers with $t_0 > 0$ and write $T_n = \sum_{k=0}^{n} t_k$ for all $n \in \mathbb{N}$. Then the Nörlund mean with respect to the sequence $t = (t_k)$ is defined by the matrix $N^t = (a_{nk}^t)$ which is given by

$$a_{nk}^t = \begin{cases} \frac{t_{n-k}}{T_n} & 0 \le k \le n \\ 0 & k > n \end{cases} \quad \text{for all } k, n \in \mathbb{N}.$$

It is known that the Nörlund matrix N^t is a Toeplitz matrix if and only if $t_n/T_n \to 0$ as $n \to \infty$ and is reduced in the case $t = e = (1, 1, \dots)$ to the matrix C_1 of arithmetic means. Additionally, for $t_n = A_n^{n-1}$ for all $n \in \mathbb{N}$, the method N^t is reduced to the Cesàro method C_r of order $r > -1$ where

$$A_n^t = \begin{cases} \frac{(r+1)(r+2)\cdots(r+n)}{n!} & n = 1, 2, \dots \\ 1 & n = 0 \end{cases}$$

Now we introduce the following sequence space
Let (t_n) be a sequence of non-negative real numbers with $t_0 > 0$ with $T_{\lambda_n} = \sum_{i \in I_n} t_i$ for all $n \in \mathbb{N}$ where $T_{\lambda_n} \to \infty$ as $n \to \infty$. Let $p = (p_n)$ is a bounded sequence of positive real numbers with $p_n \ge 1$ for all $n \in \mathbb{N}$. Then we define the generalized modular sequence space

$N(\lambda,p)=\{x\in \ell^0:\rho(\lambda x)<\infty \text{ for some } \lambda>0\}$

Where

$$\rho(x)=\sum_{n=1}^{\infty}\left|\frac{1}{T_{\lambda_n}}\sum_{i\in I_n} t_{n-i}x(i)\right|^{p_n}$$

with $T_{\lambda_n}=\sum_{i\in I_n}t_i$ where $T_{\lambda_n}\to\infty$ as $n\to\infty$ and $I_n=[n-\lambda_n+1,n]$ for $n\geq 1$.
We use the notations $N^L(\lambda,p)=(N(\lambda,p),\|\cdot\|_L)$ and $N^A(\lambda,p)=(N(\lambda,p),\|\cdot\|_A)$ for brevity.

Main Results

Theorem 3.1. The functional ρ is a convex modular on $N^L(\lambda,p)$.
Proof. Let $x,y\in N^L(\lambda,p)$. It is obvious that $\rho(x)=0$ if and only if $x=0$ and $\rho(\alpha x)=\rho(x)$ for scalar α with $|\alpha|=1$. Let $\alpha\geq 0,\beta\geq 0$ with $\alpha+\beta=1$. By the convexity of the function $t\to|t|^{p_n}$ for all $n\in\mathbb{N}$, we have

$$\begin{aligned}\rho(\alpha x+\beta y)&=\sum_{n=1}^{\infty}\left|\frac{1}{T_{\lambda_n}}\sum_{i\in I_n} t_{n-i}(\alpha x(i)+\beta y(i))\right|^{p_n}\\&=\sum_{n=1}^{\infty}\left|\alpha\frac{1}{T_{\lambda_n}}\sum_{i\in I_n} t_{n-i}(x(i))+\beta\frac{1}{T_{\lambda_n}}\sum_{i\in I_n} t_{n-i}(y(i))\right|^{p_n}\\&\leq \alpha\sum_{n=1}^{\infty}\left|\frac{1}{T_{\lambda_n}}\sum_{i\in I_n} t_{n-i}(x(i))\right|^{p_n}+\beta\sum_{n=1}^{\infty}\left|\frac{1}{T_{\lambda_n}}\sum_{i\in I_n} t_{n-i}(y(i))\right|^{p_n}\\&=\alpha\rho(x)+\beta\rho(y).\end{aligned}$$

Lemma 3.1. For $x\in N^L(\lambda,p)$, the modular ρ on $N^L(\lambda,p)$ satisfies the following properties:
1. If $0<a<1$, then $a^M\rho\left(\frac{x}{a}\right)\leq\rho(x)$ and $\rho(ax)\leq a\rho(x)$,
2. If $a>1$, then $\rho(x)\leq a^M\rho\left(\frac{x}{a}\right)$
3. If $a\geq 1$, then $\rho(x)\leq a\rho(x)\leq\rho(ax)$.

Lemma 3.2. If $\rho\in\Delta_2^s$, then for any $L>0$ and $\varepsilon>0$, there exists $\delta=\delta(L,\varepsilon)>0$ such that $|\,\rho(u+v)-\rho(u)\,|<\varepsilon$ whenever $u,v\in X_\rho$ with $\rho(u)\leq L$ and $\rho(v)\leq\delta$.

Lemma 3.3. Convergence in norm and in modular sense are equivalent in X_ρ if $\rho\in\Delta_2$.

Lemma 3.4. If $\rho\in\Delta_2^s$, then for any $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that $\|\,x\,\|\geq 1+\delta$ whenever $\rho(x)\geq 1+\varepsilon$.

Lemma 3.5. For any $x\in N^L(\lambda,p)$, we have
1. If $\|\,x\,\|_L<1$, then $\rho(x)\leq\|\,x\,\|_L$,
2. If $\|\,x\,\|_L>1$, then $\rho(x)\geq\|\,x\,\|_l$,
3. $\|\,x\,\|_L=1$ if and only if $\rho(x)=1$,
4. $\|\,x\,\|_L<1$ if and only if $\rho(x)<1$,
5. $\|\,x\,\|_L>1$ if and only if $\rho(x)>1$.

Lemma 3.6. For any $x\in N^L(\lambda,p)$, we have
1. If $0<a<1$ and $\|\,x\,\|_L>a$, then $\rho(x)>a^M$,
2. If $a\geq 1$ and $\|\,x\,\|_L<a$, then $\rho(x)<a^M$.

Lemma 3.7. Let (x_n) be a sequence in $N^L(\lambda,p)$.
1. If $\|x_n\|_L\to 1$ as $n\to\infty$, then $\rho(x_n)\to 1$ as $n\to\infty$.
2. If $\rho(x_n)\to 0$ as $n\to\infty$, then $\|\,x\,\|_L\to 0$ as $n\to\infty$.

Lemma 3.8.
For any $x\in N^L(\lambda,p)$ and $\varepsilon\in(0,1)$, there exists $\delta\in(0,1)$ such that $\rho(x)\leq 1-\varepsilon$ implies

$\|\,x\,\|_L\leq 1-\delta.$

Proof. Suppose that the Lemma does not hold, then there exists $\varepsilon>0$ and $x_n\in N^L(\lambda,p)$ such that $\rho(x_n)\leq 1-\varepsilon$ and $\frac{1}{2}\leq\|x_n\|_L\nearrow 1$. Let $a_n=\frac{1}{\|x_n\|_L}-1$. Then, $a_n\to 0$ as $n\to\infty$. Let $L=\sup\{\rho(2x_n):n\in\mathbb{N}\}$. By $\sup_n p_n<\infty$, that is $\rho\in\Delta_2^s$, there exists $K\geq 2$ such that

$$\rho(2u)\leq K\rho(u)+1 \tag{3.1}$$

for every $u \in N^L(\lambda, p)$ with $\rho(u) < 1$.
By equation (3.1), we have $\rho(2x_n) \le K\rho(x_n) + 1 \le K + 1$ for all $n \in \mathbb{N}$. Hence $0 < L < \infty$. By Theorem 3.1 and Lemma (3.1) (iii), we have

$$1 = \rho\left(\frac{x_n}{\|x_n\|_L}\right) = \rho(2a_n x_n + (1 - a_n)x_n)$$
$$\le a_n\rho(2x_n) + (1 - a_n)\rho(x_n)$$
$$\le a_n L + (1 - a_n)(1 - \varepsilon) \to 1 - \varepsilon \text{ as } n \to \infty$$

which leads to a contradiction.

Theorem 3.2. The space $N^L(\lambda, p)$ is a Banach space with respect to the Luxemburg norm.

Proof. Let $(x_l) = (x_l(i))$ be a Cauchy sequence in $N^L(\lambda, p)$ and $\varepsilon \in (0,1)$. Thus there exists $N \in \mathbb{N}$ such that $\|x_l - x_m\|_L < \varepsilon$ for all $l, m \ge N$. By Lemma 3.5(i), we have

$$\rho(x_l - x_m) \le \|x_l - x_m\|_L < \varepsilon \text{ for all } l, m \ge N. \tag{3.2}$$

That is,

$$\sum_{n=1}^{\infty} \left|\frac{1}{T_{\lambda_n}} \sum_{i \in I_n} t_{n-i}(x_l(i) - x_m(i))\right|^{p_n} < \varepsilon \text{ for all } l, m \ge N \tag{3.3}$$

For fixed n, we have $|x_l(i) - x_m(i)| < \varepsilon$ for all $l, m \in N$.

Thus, $(x_l(i))$ is a Cauchy sequence in $\mathbb{R}$ for all $i \in \mathbb{N}$. Since $\mathbb{R}$ is complete, for each $i \ge 1$, there exists $x(i) \in \mathbb{R}$ such that $x_m(i) \to x(i)$ as $m \to \infty$.

Thus, for fixed n and each $i \in I_n$, we have

$(x_l(i) - x(i)) < \varepsilon$ as $m \to \infty$, for all $l \ge N$.

This implies that

$$\rho(x_l - x_m) \to \rho(x_l - x) \text{ as } m \to \infty. \tag{3.4}$$

That is,

$$\sum_{n=1}^{\infty} \left|\frac{1}{T_{\lambda_n}} \sum_{i \in I_n} t_{n-i}(x_l(i) - x_m(i))\right|^{p_n} \to \sum_{n=1}^{\infty} \left|\frac{1}{T_{\lambda_n}} \sum_{i \in I_n} t_{n-i}(x_l(i) - x(i))\right|^{p_n} \text{ as } m \to \infty. \tag{3.5}$$

By equation (3.2), we have $\rho(x_l - x) \le \|x_l - x\|_L < \varepsilon$ for all $l \ge N$ and hence $x_l \to x$ as $l \to \infty$. So we have, $x_l - x \in N^L(\lambda, p)$. Since, $(x_l) \in N^L(\lambda, p)$ and from the linearity of the sequence space $N^L(\lambda, p)$, we get $x = x_l - (x_l - x) \in N^L(\lambda, p)$.
Therefore, the sequence space $N^L(\lambda, p)$ is a Banach space with respect to the Luxemburg norm.

Theorem 3.3. Let $(x_n) \in X_\rho$, then $\|x_n\|_L \to 0$ as $n \to \infty$ if and only if $\rho(\lambda x_n) \to 0$ as $n \to \infty$, for every $\lambda > 0$.
Proof. See [[13], Theorem 1.3(a)].

Lemma 3.9. Let $x \in N^L(\lambda, p)$ and $(x_l) \subseteq N^L(\lambda, p)$. If $\rho(x_l) \to \rho(x)$ as $l \to \infty$ and $x_l(j) \to x(j)$ as $l \to \infty$ for all $j \in \mathbb{N}$, then $x_l \to x$ as $l \to \infty$.

Proof. Let $\varepsilon > 0$. Since $\rho(x) = \sum_{n=1}^{\infty} \left|\frac{1}{T_{\lambda_n}} \sum_{i \in I_n} t_{n-i} x(i)\right|^{p_n} < \infty$, there is $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty} \left|\frac{1}{T_{\lambda_n}} \sum_{i \in I_n} t_{n-i} x(i)\right|^{p_n} < \frac{\varepsilon}{6K} \tag{3.6}$$

where $H = \sup p_n$, $K = \max\{1, 2^{H-1}\}$. Since $\rho(x_l) \to \rho(x)$ and $x_l(j) \to x(j)$ as $l \to \infty$ for all $n \in \mathbb{N}$, we have
$\rho(x_l) - \sum_{n=1}^{n_0} \left|\frac{1}{T_{\lambda_n}} \sum_{i \in I_n} t_{n-i} x_l(i)\right|^{p_n} \to \rho(x) - \sum_{n=1}^{n_0} \left|\frac{1}{T_{\lambda_n}} \sum_{i \in I_n} t_{n-i} x(i)\right|^{p_n}$ as $l \to \infty$.

Thus, there exists $l_0 \in \mathbb{N}$ such that

$$\rho(x_l) - \sum_{n=1}^{n_0} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i} x_l(i)\right|^{p_n} < \rho(x) - \sum_{n=1}^{n_0} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i} x(i)\right|^{p_n} + \frac{\varepsilon}{3K} \quad (3.7)$$

for all $l \geq l_0$. Also, since $x_l(j) \to x(j)$ for all $j \in \mathbb{N}$, we have

$$\sum_{n=1}^{n_0} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i}(x_l(i) - x(i))\right|^{p_n} < \frac{\varepsilon}{3} \text{ for all } l \geq l_0 \quad (3.8)$$

It follows from equation (3.6), equation (3.7) and equation (3.8) that for all $l \geq l_0$,

$$\begin{aligned}
\rho(x_l - x) &= \sum_{n=1}^{\infty} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i}(x_l(i) - x(i))\right|^{p_n} \\
&= \sum_{n=1}^{n_0} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i}(x_l(i) - x(i))\right|^{p_n} + \sum_{n=n_0+1}^{\infty} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i}(x_l(i) - x(i))\right|^{p_n} \\
&< \frac{\varepsilon}{3} + K\left[\sum_{n=n_0+1}^{\infty} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i} x_l(i)\right|^{p_n} + \sum_{n=n_0+1}^{\infty} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i} x(i)\right|^{p_n}\right] \\
&= \frac{\varepsilon}{3} + K\left[\rho(x_l) - \sum_{n=1}^{n_0} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i} x_l(i)\right|^{p_n} + \sum_{n=n_0+1}^{\infty} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i} x(i)\right|^{p_n}\right] \\
&< \frac{\varepsilon}{3} + K\left[\rho(x) - \sum_{n=1}^{n_0} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i} x(i)\right|^{p_n} + \frac{\varepsilon}{3K} + \sum_{n=n_0+1}^{\infty} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i} x(i)\right|^{p_n}\right] \\
&= \frac{\varepsilon}{3} + K\left[\sum_{n=n_0+1}^{\infty} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i} x(i)\right|^{p_n} + \frac{\varepsilon}{3K} + \sum_{n=n_0+1}^{\infty} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i} x(i)\right|^{p_n}\right] \\
&< \frac{\varepsilon}{3} + K\left[2 \cdot \frac{\varepsilon}{6K} + \frac{\varepsilon}{3K}\right] = \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned} \quad (3.9)$$

This shows that $\rho(x_l - x) \to 0$ as $l \to \infty$. Hence, by Lemma 3.7(ii), we have $\|x_l - x\|_L \to 0$ as $l \to \infty$. That is, $x_l \to x$ as $l \to \infty$. This completes the proof.

Theorem 3.4. The space $N^L(\lambda, p)$ has the Kadec-Klee property.
Proof. Let $x \in S(N^L(\lambda, p))$ and $(x_l) \subseteq B(N^L(\lambda, p))$ such that $\|x_l\|_L \to 1$ and $x_l \xrightarrow{w} x$ as $l \to \infty$. From Lemma 3.5(iii), we have $\rho(x) = 1$, so it follows from Lemma 3.7(i) that $\rho(x_l) \to \rho(x)$ as $l \to \infty$. Since $x_l \xrightarrow{w} x$ and the coordinate mapping $\pi_j: N^L(\lambda, p) \to \mathbb{R}$ defined by $\pi_j(x) = x(j)$ is continuous linear function on $N^L(\lambda, p)$, it follows that $x_l(j) \to x(j)$ as $l \to \infty$ for all $j \in \mathbb{N}$. Thus, by Lemma 3.9, we have $x_l \to x$ as $l \to \infty$.

Theorem 3.5. The sequence space $N^L(\lambda, p)$ has $k - NUC$-property for any integer $k \geq 2$.

Remark 3.1. Since the Luxemburg norm $\|\cdot\|_L$ and the Amemiya norm $\|\cdot\|_A$ are equivalent, so all the above results are also true for the space $N^A(\lambda, p)$.

Theorem 3.6. $S(N^A(\lambda, p))$ is a closed subspace of $N^A(\lambda, p)$.
Proof. We know, $S(N^A(\lambda, p)) = \{x \in \ell^0 : \rho(\lambda x) < \infty \text{ for all } \lambda > 0\}$

and $N^A(\lambda, p) = \{x \in \ell^0 : \rho(\lambda x) < \infty \text{ for some } \lambda > 0\}$.

It is easy to prove that $S(N^A(\lambda, p))$ is a subspace of $N^A(\lambda, p)$. Now, we will prove that $S(N^A(\lambda, p))$ is closed in $N^A(\lambda, p)$. That is, we have to show that if $(x_l) \subseteq S(N^A(\lambda, p))$ for each $l \in \mathbb{N}$ and $x_l \to x \in N^A(\lambda, p)$, then $x \in S(N^A(\lambda, p))$.
Since $\|x_l - x\|_A \to 0$, we have by Theorem 3.3, that $\rho(\alpha(x - x_l)) \to 0$ as $l \to \infty$ for all $\alpha > 0$. Hence, there exists $l_1 \in \mathbb{N}$ such that $\rho\left(2\alpha(x - x_{l_1})\right) < 1$ and by $x_{l_1} \in S(N^A(\lambda, p))$, we have $\rho(2\alpha x_{l_1}) < \infty$. Thus

$$\rho(\alpha x)=\sum_{n=1}^{\infty}\left|\frac{1}{T_{\lambda_n}}\sum_{i\in I_n} t_{n-i}\alpha x(i)\right|^{p_n}$$
$$=\sum_{n=1}^{\infty}\left|\frac{1}{2}\frac{1}{T_{\lambda_n}}\sum_{i\in I_n} t_{n-i}\left(2\alpha\left(x(i)-x_{l_1}(i)\right)-2\alpha x_{l_1}(i)\right)\right|^{p_n}$$
$$=\sum_{n=1}^{\infty}\left|\frac{1}{T_{\lambda_n}}\sum_{i\in I_n} t_{n-i}\left(2\alpha\left(x(i)-x_{l_1}(i)\right)-\frac{1}{2}\frac{1}{T_{\lambda_n}}\sum_{i\in I_n} t_{n-i}2\alpha x_{l_1}(i)\right|^{p_n}$$
$$\le\frac{K}{2}\sum_{n=1}^{\infty}\left|\frac{1}{T_{\lambda_n}}\sum_{i\in I_n} t_{n-i}\left(2\alpha\left(x(i)-x_{l_1}(i)\right)\right|^{p_n}+\frac{K}{2}\sum_{n=1}^{\infty}\left|\frac{1}{T_{\lambda_n}}\sum_{i\in I_n} t_{n-i}2\alpha x_{l_1}(i)\right|^{p_n}$$
$$=\frac{K}{2}\rho\left(2\alpha\left(x-x_{l_1}\right)\right)+\frac{K}{2}\rho\left(2\alpha x_{l_1}\right)$$

where $K=\max(1,2^{H-1})$. Since, $\rho\left(2\alpha\left(x-x_{l_1}\right)\right)<\infty$ and $\rho\left(2\alpha x_{l_1}\right)<\infty$ for every $\alpha>0$, we obtain $\rho(\alpha x)<\infty$ for every $\alpha>0$. Therefore, $x\in S(N^A(\lambda,p))$.
Let E be the set of all sequences of $N^A(\lambda,p)$ with finite number of coordinates different from 0.

Lemma 3.10. If $\rho(x)<\infty$, then the distance $d(x,E)$ from x to E is no more than 1.
Proof. See [[18], Lemma 2.2].

Theorem 3.7. If $\lim\inf_{n\to\infty} p_n>1$, then the following assertions are true.
1. $S(N^A(\lambda,p))=cl(E)$, the closure of the set E,
2. $S(N^A(\lambda,p))$ is the subspace of all order continuous elements of $N^A(\lambda,p)$,
3. $S(N^A(\lambda,p))$ is a separable space.

Proof.
1. First we have to show that $S(N^A(\lambda,p))\subset \mathrm{cl}(E)$. Then for any $x\in S(N^A(\lambda,p))$ and $\alpha\ge 1$, we have $\alpha x\in S(N^A(\lambda,p))$. Therefore, by Lemma 3.10, we get $d(\alpha x,E)\le 1$ or $d(x,E)\le\frac{1}{\alpha}$.

Since, α is arbitrary, we find that $x\in \mathrm{cl}(E)$.
Conversely, since Theorem 3.6 asserts that $S(N^A(\lambda,p))$ is a closed linear subspace of $N^A(\lambda,p)$, hence to show $\mathrm{cl}(E)\subseteq S(N^A(\lambda,p))$, it suffices to show that $e_i\in S(N^A(\lambda,p))$ for each $i\in\mathbb{N}$. Let $=\lim\inf_{n\to\infty} p_n>1$. Fix, $i\in\mathbb{N}$ and take any $\alpha>0$. Choose $n_0=\max\{i,\alpha\}$ such that $p_n\ge\gamma$ for all $n\ge n_0$. Thus

$$\rho(\alpha e_i)=\sum_{n=1}^{n_0}\left(\frac{\alpha}{n}\right)^{p_n}+\sum_{n=n_0+1}^{\infty}\left(\frac{\alpha}{n}\right)^{p_n}\le\sum_{n=1}^{n_0}\left(\frac{\alpha}{n}\right)^{p_n}+\sum_{n=n_0+1}^{\infty}\left(\frac{\alpha}{n}\right)^{\gamma}<\infty$$

Hence $e_i\in S(N^A(\lambda,p))$.

2. Obviously, $S(N^A(\lambda,p))$ is a subspace of $N^A(\lambda,p)$. Now, we have to show that each element of $S(N^A(\lambda,p))$ is an order continuous.

Let $x\in S(N^A(\lambda,p))$ be any arbitrary element and $\varepsilon>0$. Since, $x\in S(N^A(\lambda,p))$, there exists $n_0\in\mathbb{N}$ such that

$\rho\left((x-x|_n)/\varepsilon\right)<\varepsilon$ for all $n\ge n_0$.

Therefore, $\|\varepsilon^{-1}(x-x|_n)\|_A\le 1+\rho\left(\left(x-x|_{n_0}\right)/\varepsilon\right)\le 1+\varepsilon$ for all $n\ge n_0$.
This yields $\|x-x|_n\|_A\to 0$ as $n\to\infty$, since ε is arbitrary. So x is an order continuous element. But, x is any arbitrary element of $S(N^A(\lambda,p))$ and hence every element of $S(N^A(\lambda,p))$ is order continuous.

3. From the construction of E, it can be seen that E is a countable dense set. Also, from (i), we have $S(N^A(\lambda,p))$ has atleast one dense set E. Hence $S(N^A(\lambda,p))$ is separable.

Theorem 3.8. If $p_n>1$ for all $n\in\mathbb{N}$ and $\limsup_n p_n<\infty$, then $N^A(\lambda,p)$ has the uniform Opial property.
Proof. Take any $\varepsilon>0$ and $x\in N^A(\lambda,p)$ such that $\|x\|_A\ge\varepsilon$. Let (x_l) be a weakly null sequence in $S(N^A(\lambda,p))$. Since, $\lim\sup_n p_n<\infty$, by Theorem 3.4, there exists a $\delta\in(0,2/3)$ independent of x such that $\rho(x/2)>\delta$. Also, since $\lim\sup_n p_n<\infty$, we have $S(N^A(\lambda,p))=N^A(\lambda,p)$. By Theorem 3.7(ii), x is an order continuous element. Hence, we can find $n_0\in\mathbb{N}$ such that

$\|x|_{\mathbb{N}-n_0}\| < \frac{\delta}{4}$ and

$$\sum_{n=n_0+1}^{\infty} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i} x(i)\right|^{p_n} < \frac{\delta}{8} \tag{3.10}$$

Since $\rho(x/2) \geq \delta$, it follows that

$$\delta \leq \sum_{n=1}^{n_0} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i} \frac{x(i)}{2}\right|^{p_n} + \sum_{n=n_0+1}^{\infty} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i} \frac{x(i)}{2}\right|^{p_n}$$
$$\leq \sum_{n=1}^{n_0} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i} \frac{x(i)}{2}\right|^{p_n} + \frac{\delta}{8}$$

which implies that

$$\sum_{n=1}^{n_0} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in I_n} t_{n-i} \frac{x(i)}{2}\right|^{p_n} \geq \frac{7\delta}{8} \tag{3.10}$$

Since, $x_l \overset{w}{\to} 0$, it follows that $x_l(i) \to 0$ as $l \to \infty$ for each $i \in \mathbb{N}$, so there exists $l_0 \in \mathbb{N}$ such that

$$\|x_l|_{n_0}\|_A < \frac{\delta}{4} \tag{3.11}$$

for all $l \geq l_0$, which implies that $\|x_l|_{\mathbb{N}-n_0}\|_A > 1 - \frac{\delta}{4}$ since $\| x \|_A = 1$.
Now, for all $l \geq l_0$, we have

$$\begin{aligned}\|x + x_l\| &= \|(x + x_l)|_{n_0} + (x + x_l)|_{\mathbb{N}-n_0}\|_A \\ &\geq \|x|_{n_0} + x_l|_{\mathbb{N}-n_0}\|_A - \|x|_{\mathbb{N}-n_0}\|_A - \|x_l|_{n_0}\|_A\end{aligned} \tag{3.12}$$

Now, we consider $\|x|_{n_0} + x_l|_{\mathbb{N}-n_0}\|_A$. Since, $p_n > 1$ for all $n \in \mathbb{N}$, we have there exists $c_l > 0$ such that

$$\|x|_{n_0} + x_l|_{\mathbb{N}-n_0}\|_A = \frac{1}{c_l}\left[1 + \rho\left(c_l\left(x|_{n_0} + x_l|_{\mathbb{N}-n_0}\right)\right)\right]. \tag{3.13}$$

Combining this fact with equation (3.12) and considering the fact that $\rho(x + y) \geq \rho(x) + \rho(y)$ if $\text{supp}(x) \cap \text{supp}(y) = \phi$, we get

$$\|x + x_l\|_A \geq \frac{1}{c_l} + \frac{1}{c_l}\rho\left(c_l x|_{n_0}\right) + \frac{1}{c_l}\rho\left(c_l x_l|_{\mathbb{N}-n_0}\right) - \frac{\delta}{2}$$

Without loss of generality, we may assume that $c_l \geq \frac{1}{2}$ for all l because if $c_l < \frac{1}{2}$, then $\|x + x_l\|_A > 2 - \frac{\delta}{2} > 1 + \delta$.
Since, $2c_l \geq 1$, by the convexity of the function $t \to |t|^{p_n}$, we have $\rho\left(c_l x|_{n_0}\right) \geq 2c_l \rho\left(x|_{n_0}\right)$. Thus, inequality (3.10) and (3.14) implies that

$$\begin{aligned}\|x + x_l\|_A &\geq \|x_l|_{\mathbb{N}-n_0}\|_A + 2\rho\left(\frac{x|_{n_0}}{2}\right)\frac{\delta}{2} \\ &= \|x_l|_{\mathbb{N}-n_0}\|_A + 2\sum_{n=1}^{n_0} \left|\frac{1}{T_{\lambda_n}} \sum_{i\in \mathbb{N}} t_{n-i} \frac{x(i)}{2}\right|^{p_n} - \frac{\delta}{2} \\ &> 1 - \frac{\delta}{4} + \frac{14\delta}{8} - \frac{\delta}{2} = 1 + \delta\end{aligned}$$

which implies that $\liminf_{n\to\infty} \|x + x_l\|_A \geq 1 + \delta$.

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