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Cauchy modified generalized exponential distribution: Estimation and Applications

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Abstract
We present a unique probability model in this study called the Cauchy Modified Generalized Exponential Distribution. This model is formulated by combining the Cauchy family of distributions with the Modified Generalized Exponential Distribution as the baseline distribution. Our objective is to employ this model in the analysis of lifetime data. We've crafted formulas for various statistical functions, like skewness, kurtosis, survival function, quantile function, hazard rate function, distribution function, and probability density function. Additionally, we've integrated visual depictions of the probability density and hazard rate curves. We gathered a dataset that included notable earthquakes (magnitude 7.0 and higher) that the USGS had documented between 1990 and 2018. Our proposed model's effectiveness was evaluated by applying it to a global dataset covering significant earthquakes of the same magnitude range. The model parameters were estimated using maximum likelihood estimation. Several statistical measures were applied in order to confirm the validity of the model, including the Bayesian Information Criterion, Corrected Akaike's Information Criterion, the Hannan-Quinn Information Criterion, and Akaike's Information Criterion. Additionally, Q-Q and P-P plots were employed for validation. We used the Kolmogorov-Smirnov, Anderson-Darling, and Cramer-von Mises tests to evaluate how well our model fit the data. These tests were conducted to determine the suitability of our model for analyzing the provided earthquake data. Our empirical results indicate that, compared to alternative lifetime distributions, our suggested distribution not only exhibits a better fit but also provides increased flexibility for analyzing lifetime data. This study advances our understanding of earthquake patterns and contributes to the ongoing efforts in seismic risk assessment and mitigation strategies. All numerical calculations were performed using the R programming language.

Keywords: Cauchy family of distribution, earthquakes, failure rate function, maximum likelihood estimation, modified generalized exponential distribution

1. Introduction

In recent decades, the exponential distribution has become a common baseline distribution for establishing new probability models. Numerous modifications of exponential distributions can be found in the literature. The Generalized Exponential Distribution (GED) created by (Gupta & Kundu, 2007) [17] is a statistical probability distribution that extends the traditional exponential distribution by incorporating an additional parameter in base line distribution to better capture the characteristics of real-world data. The inclusion of additional parameters results in the creation of new probability models. Typically, these adjusted models offer a more accurate representation of the data compared to conventional models. The GED introduces a shape parameter that changes the hazard function, allowing for a more flexible modeling approach than the usual exponential distribution, which assumes a constant hazard rate. This modification enables the distribution to better accommodate scenarios where the hazard rate varies over time, providing a more accurate representation of diverse phenomena in fields such as reliability engineering, biology, finance, life testing and survival analysis.

While GED distribution is effective for analyzing datasets with a monotone (increasing/ decreasing) hazard function (HF), it cannot be applied to datasets with a unimodal or bathtub-shaped HF and upside-down bathtub shapes, such as those resembling the Weibull or gamma distributions. Several innovative probability models have been created through the modification of exponential distributions found in literature. These distributions encompass a wide range of models, including the extended exponential distribution (Gomez *et al.*, 2014) [15], The modified exponential (ME) distribution (Rasekhi *et al.*, 2017) [26], the New Odd Generalized Exponential - Exponential Distribution (Kumar & Kumar, 2019) [19], the Marshall-Olkin generalized exponential distribution (Ristic & Kundu, 2015), the beta generalized exponential distribution (Barreto-Souza *et al.*, 2010) [6], the Kumaraswamy-Generalized Exponentiated Exponential Distribution (Mohammed, 2014) [22], Modified slashed generalized exponential distribution (Astorga *et al.*, 2020) [5], Weibull generalized exponential distribution (Almongy *et al.*, 2021) [3], Two-parameter modified weighted exponential distribution (Chesneau *et al.*, 2022) [11], Modified upside-down bathtub-shaped hazard function distribution (Chaudhary *et al.*, 2023) [10], and A New Four Parameter Extended Exponential Distribution (Hassan *et al.*, 2022) [18].

These lifetime models might exhibit a hazard rate function (HRF) with a bathtub-shaped pattern. In reality, numerous datasets display this characteristic HRF. Moreover, the literature documents other modifications of the Weibull distribution. These modifications aim to improve the distribution's suitability and versatility in capturing a wide range of patterns observed in survival and reliability analysis. A few examples are the Poisson Modified Weibull distribution (Abd El-Monsef *et al.*, 2022) [2], the Kumaraswamy Modified Weibull distribution (Cordeiro *et al.*, 2014) [12], the Beta Modified Weibull distribution (Silva *et al.*, 2010) [29], and the Modified Weibull distribution (Lai *et al.*, 2003) [20]. These models offer alternative approaches to modeling lifetimes by adjusting the original Weibull distribution. The Weibull distribution with two parameters is provided as.

$$\overline{F}(y,\lambda,\beta)=\exp[-(\lambda,y)]^{\beta} \qquad (1.1)$$

The aforementioned distribution lacks a bathtub hazard rate function (hrf). To address this, the distribution has been adjusted to generate multiple distributions with a bathtub hazard rate function. The usage of the exponentiated Weibull distribution, as suggested by (Mudholkar & Srivastava, 1993) [24], is one such modification. The following new lifespan distributions may be generated by fitting appropriate limits to the beta-integrated distribution, as proposed by (Lai *et al.*, 2016) [21].

$$\overline{F}(y)=\exp[ay^{b}.\exp(\lambda y)] \qquad (1.2)$$

In this context, the generalized exponential distribution suggested by (Gupta & Kundu,1999a) [16] is adapted to create a novel probability model known as the modified generalized exponential distribution which has recommended by (Telee & Kumar, 2023) [9]. The following is the expression for the generalized exponential distribution's cumulative distribution function (PDF).

$$F_{GED}(x,\alpha,\lambda)=(1-e^{-\lambda x})^{\alpha};x>0,\alpha>0,\lambda>0 \qquad (1.3)$$

The Cumulative Distribution Function (CDF) and Probability Density Function (PDF) for the Modified Generalized Exponential (MGE) model by (Telee & Kumar, 2023) [9] can respectively be expressed as.

$$G(x;\alpha,\beta,\lambda)=\left[1-\exp\left(-\lambda x e^{\beta x}\right)\right]^{\alpha} \quad ; \ \alpha>0,\beta>0,\lambda>0,x>0 \qquad (1.4)$$

and

$$g(x;\alpha,\beta,\lambda)=\alpha\lambda\left(1+\beta x\right)\exp\left(\beta x-\lambda x e^{\beta x}\right)\left[1-\exp\left(-\lambda x e^{\beta x}\right)\right]^{\alpha-1} \qquad (1.5)$$

We have constructed an innovative distribution in this study by utilizing the Cauchy family of distributions. A variety of probability models utilizing the Cauchy family of distributions are presented in the literature. These models include the Half-Cauchy exponential extension distribution (Telee & Kumar, 2022) [30], the Arc tan generalized exponential distribution (Chaudhary *et al.*, 2021) [7], the generalized Cauchy family of distributions (Alzaatreh *et al.*, 2016), the Pareto ArcTan (PAT) distribution (Gómez-Déniz and Calderín-Ojeda, 2015) [14], and the Half-Cauchy modified exponential distribution (Chaudhary *et al.*, 2022).

Consider Cauchy family of distribution on a non-negative random variable X such that $x > 0$, $\theta > 0$ and which is defined by.

$$F(x)=1-\frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}\log G(x)\right\} \qquad (1.6)$$

where G(x) is distribution function of the base line distribution.
Corresponding density function of Cauchy family is.

$$f(x)=\frac{2}{\pi}\frac{g(x)}{\theta G(x)}\left[1+\left\{-\frac{1}{\theta}\log G(x)\right\}^{2}\right]^{-1} \qquad (1.7)$$

This study aims to develop a versatile probability distribution and perform data analysis on the occurrence of significant earthquakes (magnitude 7.0 and above) worldwide between 1990 and 2018. The primary goal is to create a probability distribution model that provides enhanced flexibility, facilitating a thorough and insightful examination of the frequency of major earthquakes (7.0+) globally.

The following framework is utilized to present the various segments of this investigation. Section 2 will introduce the Cauchy Modified Generalized Exponential Distribution, providing an elucidation of its mathematical and statistical properties. Advancing to Section 3, we will delve into estimation techniques, including discussions on least-squares (LSE), Cramer-Von-Mises (CVME), and maximum likelihood (MLE). In Section 4, our focus will pivot towards presenting model parameter estimates using real data pertaining to significant earthquakes (magnitude 7.0 and above) worldwide between 1990 and 2018. Additionally, we will exemplify various criteria employed to evaluate the goodness of fit of the proposed model. All numerical calculations were conducted using the R programming language. In the concluding Section 5, this study aims to provide valuable insights into the realm of statistical analysis and modeling. In essence, this study strives to make a meaningful contribution to the broader landscape of statistical analysis and modeling, with particular relevance to the understanding and characterization of earthquake occurrences. The insights gained herein provide a basis for further research and applications in areas such as risk assessment, disaster preparedness, and the development of more accurate predictive models for seismic activity.

2.1 Cauchy modified generalized exponential distribution

We provide a new probability model in this work called the Cauchy Modified Generalized Exponential Distribution. This model is created by compounding the Cauchy family of distributions with the Modified Generalized Exponential Distribution serving as the baseline distribution.

Using (1.4) and (1.5) in (1.6) and (1.7), distribution and density functions of proposed model Cauchy Modified Generalized exponential (CMGE) distribution are respectively defined as.

$$F(x;\alpha,\beta,\lambda,\theta)=1-\frac{2}{\pi}\arctan\left\{-\frac{\alpha}{\theta}\log\left\{1-\exp\left(-\lambda x e^{\beta x}\right)\right\}\right\};\ (\alpha,\beta,\lambda,\theta)>0, x>0 \tag{2.1}$$

$$f(x;\alpha,\beta,\lambda,\theta)=\left(\frac{2\alpha\lambda}{\pi\theta}\right)(1+\beta x)\exp\left(\beta x-\lambda x e^{\beta x}\right)\left\{1-\exp\left(-\lambda x e^{\beta x}\right)\right\}^{-1}$$

$$\left[1+\left\{-\frac{\alpha}{\theta}\log\left\{1-\exp\left(-\lambda x e^{\beta x}\right)\right\}\right\}^{2}\right]^{-1};(\alpha,\beta,\lambda,\theta)>0, x>0 \tag{2.2}$$

2.1 Survival Function: The equation (2.3) represents the survival function of the suggested model.

$$S(x)=\frac{2}{\pi}\arctan\left\{-\frac{\alpha}{\theta}\log\left\{1-\exp\left(-\lambda x e^{\beta x}\right)\right\}\right\};\ (\alpha,\beta,\lambda,\theta)>0, x>0 \tag{2.3}$$

2.2 Hazard Function: The hazard rate function, denoted as the instantaneous failure rate, is a measure of the immediate risk of failure at any given point in time. It is mathematically defined by expression (2.4), encapsulating the dynamic nature of failure probabilities over time.

$$H(x)=\left(\frac{\alpha\lambda}{\theta}\right)(1+\beta x)\exp\left(\beta x-\lambda x e^{\beta x}\right)\left\{1-\exp\left(-\lambda x e^{\beta x}\right)\right\}^{-1}$$

$$\left[1+\left\{-\frac{\alpha}{\theta}\log\left\{1-\exp\left(-\lambda x e^{\beta x}\right)\right\}\right\}^{2}\right]^{-1}\left[\arctan\left\{-\frac{\alpha}{\theta}\log\left\{1-\exp\left(-\lambda x e^{\beta x}\right)\right\}\right\}\right]^{-1} \tag{2.4}$$

2.3 The Quantile function: The quantile function provides valuable insights into the model's descriptive analysis, offering an alternative perspective to the cumulative distribution function (CDF). Equation (2.5) precisely outlines the quantile function for the CMGE.

After x is solved for in equation (2.5), the quantile function with u following a uniform distribution [0, 1] is obtained.

$$\log\left\{1-\exp\left(-\lambda x e^{\beta x}\right)\right\}+\left(\frac{\theta}{2}\tan(1-u)\left(\frac{\pi}{2}\right)\right)=0 \tag{2.5}$$

2.4 Skewness and Kurtosis:

Based on quartiles, the coefficient of skewness can be obtained using the expression

$$S_k=\frac{Q_3+Q_1-2Q_2}{Q_3-Q_1} \tag{2.6}$$

Here, Q_1, Q_2 and Q_3 are lower quartile, 2nd quartile and upper quartile respectively.

Moors (1988) introduced the idea that the kurtosis coefficient is influenced by the octiles, and it can be expressed as:

$$K_M = \frac{Q(0.375)-Q(0.125)-Q(0.625)+Q(0.875)}{Q(0.75)-Q(0.25)} \tag{2.7}$$

The suggested distribution's hazard rate function and probability density function for a range of parameter values are shown in Figure 1. The PDF graph shows a single-peaked distribution with a positive skew, implying a clustering of values around a central point. In contrast, the hazard rate function plot displays both a rising trend and an inverted bathtub shape, indicating varying risk profiles over time.

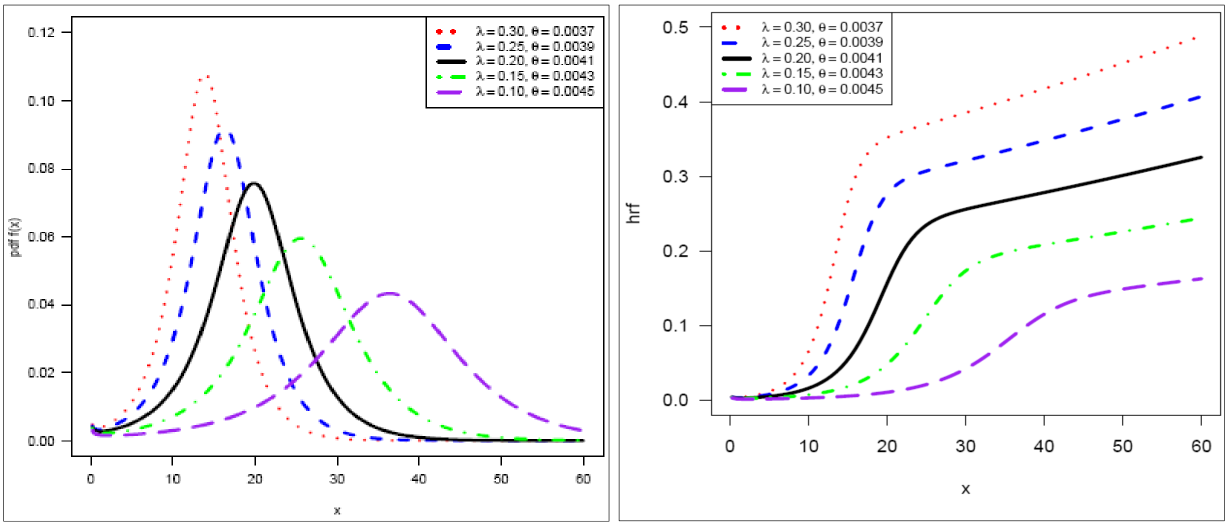

Fig 1: Hazard function (right part) and density function (left part)

3. Parameter estimation

3.1 Method of Maximum Likelihood Estimation (MLE)

Let a random sample with size 'n' drawn from the proposed model be $\underline{x} = (x_1, \ldots, x_n)$, then the log likelihood function can be expressed as,

$$\ell(\alpha,\beta,\lambda,\theta \mid \underline{x}) = n\log\left(\frac{2\alpha\lambda}{\pi\theta}\right) + \sum_{i=1}^{n}\log\left(1+\beta x_i\right) \; \sum_{i=1}^{n}\left(\beta x_i - \lambda x_i e^{\beta x_i}\right) - \sum_{i=1}^{n}\log\left\{1-\exp\left(-\lambda x\, e^{\beta x_i}\right)\right\}$$

$$-\sum_{i=1}^{n}\log\left[1+\left\{-\frac{\alpha}{\theta}\log\left\{1-\exp\left(-\lambda x_i\, e^{\beta x_i}\right)\right\}\right\}^2\right];(\alpha,\beta,\lambda,\theta)>0,\, x>0 \tag{3.1}$$

Partial differentiation (3.1) with regard to α, β, λ, and θ is one method to obtain the partial order derivatives. Setting the nonlinear equations obtained by taking partial derivatives to zero and solving for the unknown parameters (α, β, λ, and θ) can yield the ML estimators for the proposed distribution. As manual calculation of these equations is impractical, one can use appropriate computer software for the solution. Let the parameter vector be represented $\underline{\Delta} = (\alpha, \beta, \lambda, \theta)$, and the corresponding maximum likelihood estimation be denoted by $\hat{\underline{\Delta}} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta})$. The resulting asymptotic normality is expressed as $\left(\hat{\underline{\Delta}} - \underline{\Delta}\right) \rightarrow N_3\left[0, \left(K(\underline{\Delta})\right)^{-1}\right]$. Here, Fisher's information matrix is denoted by $K(\underline{\Delta})$ which is given by,

$$K(\underline{\Delta}) = -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\theta}\right) \\ E\left(\frac{\partial^2 l}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\beta^2}\right) & E\left(\frac{\partial^2 l}{\partial\beta\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\beta\partial\theta}\right) \\ E\left(\frac{\partial^2 l}{\partial\lambda\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\lambda\partial\beta}\right) & E\left(\frac{\partial^2 l}{\partial\lambda^2}\right) & E\left(\frac{\partial^2 l}{\partial\lambda\partial\theta}\right) \\ E\left(\frac{\partial^2 l}{\partial\theta\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\theta\partial\beta}\right) & E\left(\frac{\partial^2 l}{\partial\theta\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\theta^2}\right) \end{pmatrix}$$

Since we don't actually know $\underline{\Delta}$, the asymptotic variance $\left(K(\underline{\Delta})\right)^{-1}$ of the MLE is meaningless. The estimated parameter values are therefore plugged in to approximate the asymptotic variance. The information matrix $K(\underline{\Delta})$ is estimated by the following observed fisher information matrix $O(\hat{\underline{\Delta}})$.

$$O(\hat{\underline{\Delta}}) = -\begin{pmatrix} \left(\frac{\partial^2 l}{\partial\alpha^2}\right) & \left(\frac{\partial^2 l}{\partial\alpha\partial\beta}\right) & \left(\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right) & \left(\frac{\partial^2 l}{\partial\alpha\partial\theta}\right) \\ \left(\frac{\partial^2 l}{\partial\beta\partial\alpha}\right) & \left(\frac{\partial^2 l}{\partial\beta^2}\right) & \left(\frac{\partial^2 l}{\partial\beta\partial\lambda}\right) & \left(\frac{\partial^2 l}{\partial\beta\partial\theta}\right) \\ \left(\frac{\partial^2 l}{\partial\lambda\partial\alpha}\right) & \left(\frac{\partial^2 l}{\partial\lambda\partial\beta}\right) & \left(\frac{\partial^2 l}{\partial\lambda^2}\right) & \left(\frac{\partial^2 l}{\partial\lambda\partial\theta}\right) \\ \left(\frac{\partial^2 l}{\partial\theta\partial\alpha}\right) & \left(\frac{\partial^2 l}{\partial\theta\partial\beta}\right) & \left(\frac{\partial^2 l}{\partial\theta\partial\lambda}\right) & \left(\frac{\partial^2 l}{\partial\theta^2}\right) \end{pmatrix}_{\big|(\hat{\alpha},\hat{\beta},\hat{\lambda},\theta)} = -H(\underline{\Delta})_{\big|(\underline{\Delta}=\hat{\underline{\Delta}})}$$

To enhance the likelihood maximization process, we construct the observed information matrix using the Newton-Raphson method. Subsequently, we derive the variance-covariance matrix from this process as

$$\left[-H(\underline{\Delta})_{\big|(\underline{\Delta}=\hat{\underline{\Delta}})}\right]^{-1} = \begin{pmatrix} \operatorname{var}(\hat{\alpha}) & \operatorname{cov}(\hat{\alpha},\hat{\beta}) & \operatorname{cov}(\hat{\alpha},\hat{\lambda}) & \operatorname{cov}(\hat{\alpha},\hat{\theta}) \\ \operatorname{cov}(\hat{\beta},\hat{\alpha}) & \operatorname{var}(\hat{\beta}) & \operatorname{cov}(\hat{\beta},\hat{\lambda}) & \operatorname{cov}(\hat{\beta},\hat{\theta}) \\ \operatorname{cov}(\hat{\lambda},\hat{\alpha}) & \operatorname{cov}(\hat{\lambda},\hat{\beta}) & \operatorname{var}(\hat{\lambda}) & \operatorname{cov}(\hat{\lambda},\hat{\theta}) \\ \operatorname{cov}(\hat{\theta},\hat{\alpha}) & \operatorname{cov}(\hat{\theta},\hat{\beta}) & \operatorname{cov}(\hat{\theta},\hat{\lambda}) & \operatorname{var}(\hat{\theta}) \end{pmatrix} \quad (3.2)$$

Here, H denotes the Hessian matrix.

Therefore, by utilizing the asymptotic normality of Maximum Likelihood Estimates (MLE) and approximating 100(1-δ)% confidence intervals for α, β, λ, and θ, the following procedure may be used.

$$\hat{\alpha} \pm Z_{\delta/2}\sqrt{\operatorname{var}(\hat{\alpha})}, \; \hat{\beta} \pm Z_{\delta/2}\sqrt{\operatorname{var}(\hat{\beta})}, \; \hat{\lambda} \pm Z_{\delta/2}\sqrt{\operatorname{var}(\hat{\lambda})} \text{ and } \hat{\theta} \pm Z_{\delta/2}\sqrt{\operatorname{var}(\hat{\theta})} \quad (3.3)$$

Here, $Z_{\delta/2}$ represents the upper percentile of standard normal variate.

3.2 Application to Real Dataset

The United States Geological Survey (USGS) created the data set that shows the frequency of notable earthquakes (magnitude 7.0 and above) that occurred between 1990 and 2018 (USGS, 1990-2018). (https://earthquake.usgs.gov/).
18, 16, 13, 12, 13, 20, 15, 16, 12, 18, 15, 16, 13, 15, 16, 11, 11, 18, 12, 17, 24, 20, 16, 19, 12,19, 16, 7, 17
Table 1 shows the summary statistics of the model

Table 1: Summary statistics

Minimum	Q1	Median	Mean	Q3	SD	Skewness	Kurtosis	Maximum
7.000	13.000	16.000	15.410	18.000	3.50	0.009	3.256	24.000

To examine the descriptive characteristics of the data, we generated and presented the TTT plot and boxplot in Figure 2. The boxplot indicates a positive skewness and non-normal distribution of the data. Additionally, the TTT plot reveals that the hazard rate curve consistently increases and exhibits an inverted bathtub shape.

These findings suggest that the data is not symmetrically distributed and exhibits a trend in the hazard rate consistent with a non-decreasing pattern resembling an inverted bathtub. The positively skewed nature observed in the boxplot indicates that the majority of the data points are concentrated towards the lower values, with a tail extending towards higher values. These graphical representations provide valuable insights into the underlying characteristics of the dataset, highlighting both the shape of the distribution and the behavior of the hazard rate over time.

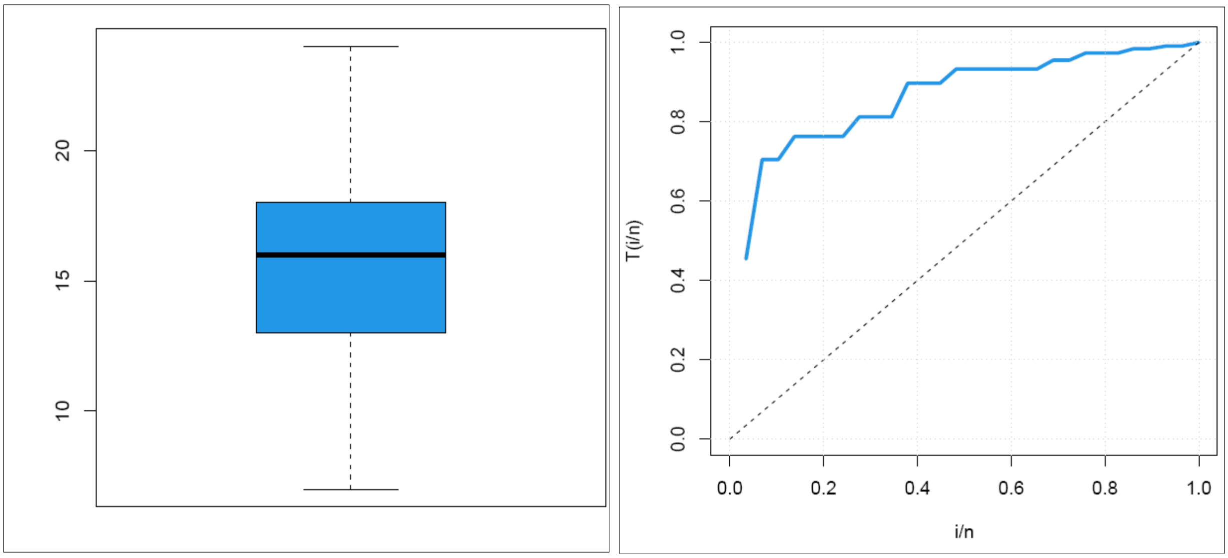

Fig 2: TTT plot (right panel) and boxplot (left panel)

Using maximum likelihood estimation, parameters of the model are estimated and mentioned in table 2. In order to get the MLEs for estimate, the maxLik() function in the R program(R Core Team, 2022) is utilized to maximize the likelihood function. Log-Likelihood value acquired is l = -77.33385.

Table 2: Estimated parameters using MLE

Parameters	MLE	SE
$\hat{\alpha}$	1.8496	5.6349
$\hat{\beta}$	0.0043	0.0315
$\hat{\lambda}$	0.3741	0.3589
$\hat{\theta}$	0.0037	0.0019

We often utilize PDF and CDF plots to evaluate how well a proposed model fits. To gain further insights, it is essential to generate Q-Q and P-P graphs. The P-P plot highlights any lack of fit, while the Q-Q plot can provide information about the fit towards the distribution's tails. Figure 3 illustrates the strong fit of the CMGE model to the data. The combination of PDF and CDF plots along with Q-Q and P-P graphs offers a robust framework for evaluating the goodness of fit of the CMGE distribution.

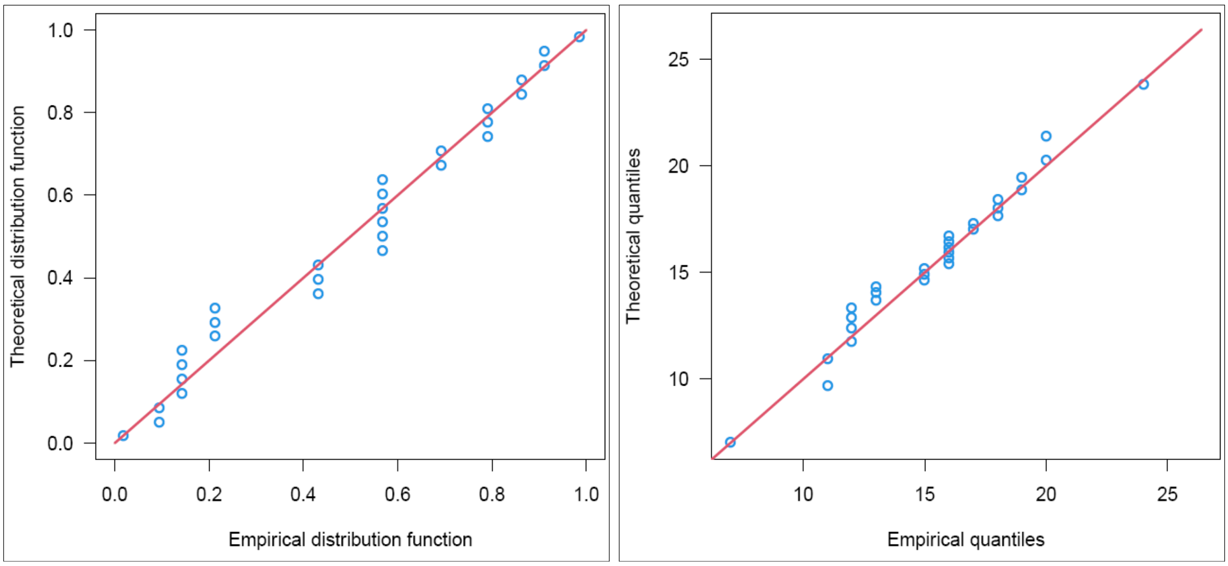

Fig 3: The Q-Q graph (right part) and P-P graph (left part) for CMGE model.

In Figure 4, we have made a comparison between the density plot, the histogram, and the empirical cumulative distribution function (ECDF) vs the fitted cumulative distribution function (CDF).

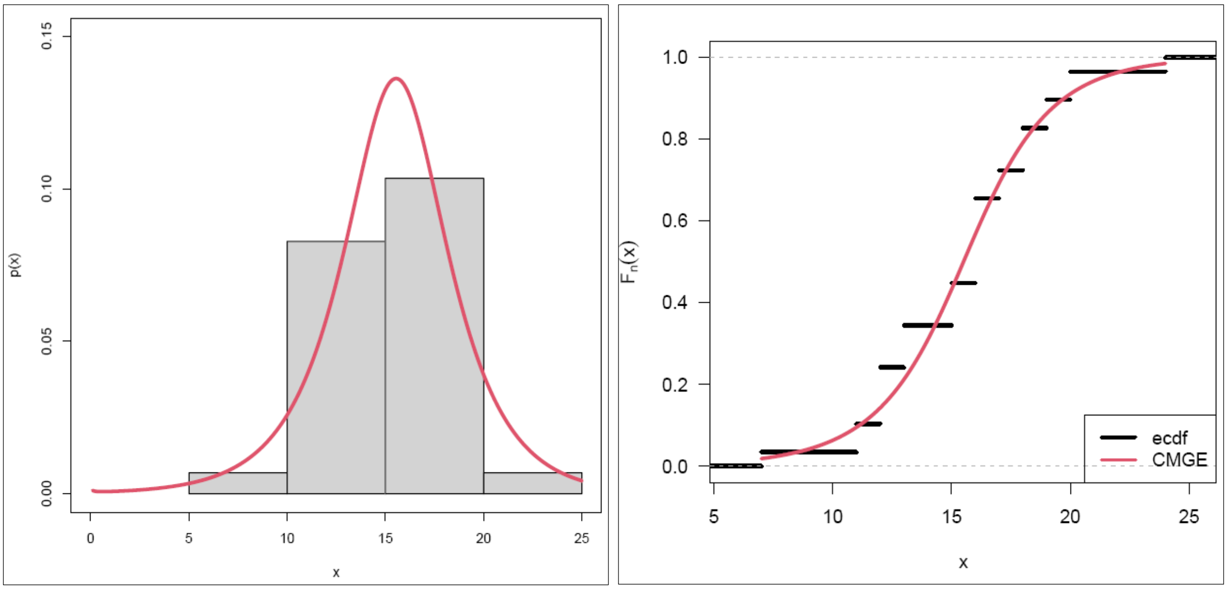

Fig 4: Histogram versus pdf plot (left part) and ECDF versus CDF (left part)

We have chosen a few well-known distributions for comparison in order to show the CMGE distribution's goodness of fit. These are Modified Weibull (MW) (Lai *et al.*, 2003) [20], Odd Lomax Exponential (OLE) distribution (Ogunsanya *et al.*, 2019), Generalized Exponential (GE) distribution (Gupta & Kundu, 1999a) [16], Extended Kumaraswamy Exponential (EKwE) Distribution (Chaudhary *et al.*, 2023) [10].

Table 3 presents the estimated parameter values and standard errors of the proposed models as well as those of competing models. The estimated parameter values and standard errors of the proposed and competing models are essential for understanding and comparing their respective performance. In Table 3, these key statistical measures provide insights into the precision and reliability of the model estimates. Analyzing this information enables a comprehensive evaluation of the proposed models in relation to their competitors, aiding in the assessment of their overall effectiveness and robustness in capturing the underlying relationships within the data.

Table 3: Estimated values of the parameters and their standard error for CMGE and competing models

Model	Alpha	Beta	Lambda	Theta
CMGE	1.8496(5.6349)	0.0043(0.0315)	0.3741(0.3589)	0.0037(0.0019)
MW	0.0057 (0.0040)	0.1510(0.5651)	0.2766(0.0657)	-
GE	48.7789 (24.8900)	-	0.2855(0.0391)	-
EKwE	-	30.9481(18.6062)	0.2618(0.0430)	-
OLE	0.5683(0.1828)	0.0059 (0.0068)	2.6221(0.6692)	-

To evaluate how well the proposed model performs, we compute several information criteria including Hannan-Quinn information criterion (HQIC), the Corrected Akaike information criterion (CAIC), Akaike information criterion (AIC), and Bayesian information criterion (BIC). The findings are displayed in Table 4.

Table 4: CMGE distribution's log-likelihood (LL), BIC, CAIC, AIC, and HQIC

Distributions	LL	AIC	BIC	CAIC	HQIC
CMGE	-77.33385	162.6677	168.1369	164.3344	164.3806
MW	-79.50305	165.0061	169.1080	165.9661	166.2908
GE	-79.8132	163.6263	166.3609	164.0878	164.4827
EKwE	-80.29985	164.5997	167.3343	165.0612	165.4561
OLE	-81.8507	169.7014	173.8033	170.6614	170.9861

The information criterion values in Table 4 are lower than those of the majority of the models that were taken into consideration, indicating that the suggested model fits the real dataset better than most of the other models. Fitted density function, empirical distribution function, the histogram, and estimated distribution function are all displayed for CMGE, MW, GE, EKwE, and OLE distributions in Figure 5.

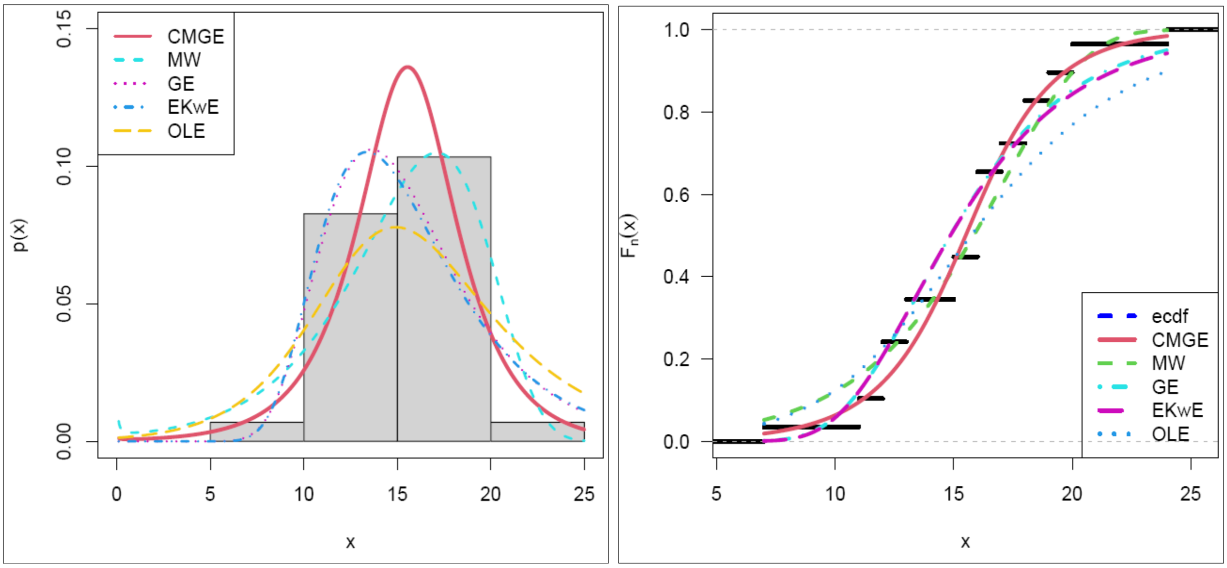

Fig 5: Empirical and estimated distribution functions (right panel) and the density function and histogram of fitted distributions (left panel)

Table 5 presents a goodness-of-fit comparison of the CMGE model with other selected distributions using the Anderson-Darling (W), Kolmogorov-Smirnov (KS), and Cramer-Von Mises (A2) statistics. The CMGE distribution exhibits a higher p-value and a lower test statistic value, suggesting that its findings align more accurately with the distribution and are consequently more reliable compared to those obtained from the other distributions employed for comparison.

Table 5: The goodness-of-fit statistics and associated p-values

Distributions	KS(p-value)	W(p-value)	A^2(p-value)
CMGE	0.1326(0.6882)	0.0704(0.7528)	0.4108(0.8367)
MW	0.1395(0.6254)	0.0973(0.6015)	0.6911(0.5649)
GE	0.1668(0.3945)	0.1284(0.4651)	0.8118(0.4715)
EKwE	0.1691(0.3785)	0.1352(0.4401)	0.8797(0.4260)
OLE	0.1982(0.2048)	0.2353(0.2086)	1.5068(0.1750)

4. Concluding Remarks

In this study, we present a novel probability model termed the Cauchy Modified Generalized Exponential Distribution (CMGE). Constructed by combining the Cauchy family of distributions with the Modified Generalized Exponential Distribution as the baseline, this model manifests a positively skewed and unimodal form. A thorough examination of various statistical properties associated with the proposed model was conducted, revealing its notable flexibility in accommodating escalating hazard functions, including an inverted bathtub-shaped hazard function. These insights came from a thorough graphical examination of the CMGE's Probability Density Function (PDF) and Hazard Rate Function (HRF).

For parameter estimation, we employed Maximum Likelihood Estimation (MLE), providing valuable insights into the accuracy of our model's parameter estimation. Additionally, we evaluated the CMGE distribution's performance by applying it to a real-world dataset comprising significant earthquakes (magnitude 7.0 and above) recorded by the United States Geological Survey (USGS) from 1990 to 2018 worldwide. Results from this application demonstrated the superior fitting performance of the CGME distribution compared to several other commonly used lifetime models, emphasizing its potential utility in analyzing lifetime data, especially in complex and dynamic scenarios such as significant earthquakes.

Our investigation focused on assessing the suitability of the CMGE distribution for modeling and understanding intricate and dynamic scenarios, such as the impact on assets and human lives during catastrophic events. The goodness of fit of our model to the earthquake data was evaluated through the Anderson-Darling, Kolmogorov-Smirnov, and Cramer-von Mises tests. Empirical results indicate that, compared to alternative lifetime distributions, our proposed distribution not only provides a better fit but also offers increased flexibility for analyzing lifetime data. This study contributes to advancing our understanding of earthquake patterns and supports ongoing efforts in seismic risk assessment and mitigation strategies. Our study also underscores the significance of interdisciplinary collaboration between statisticians, geoscientists, and risk management experts. The insights gained from this collaboration can inform the development of more accurate and insightful models that consider both the statistical intricacies and the real-world implications of seismic events. This interdisciplinary approach is vital for addressing the multifaceted challenges associated with natural disasters and other high-impact events.

5. Conclusion

The exponential distribution has evolved as a foundational model, with various modifications enhancing its applicability to real-world data. The Generalized Exponential Distribution (GED) and its derivatives offer improved accuracy in modeling phenomena with varying hazard rates, such as those in reliability engineering, biology, finance, and survival analysis. However, while these models excel with monotonic hazard functions, they may not be suitable for unimodal or bathtub-shaped hazard functions. Several innovative distributions, such as the Cauchy Modified Generalized Exponential Distribution (CMGE), aim to address these limitations, providing flexible and accurate models for complex data patterns.

6. References

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