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A new family of lifetime distributions with different directions of hazard rate

Ahlam Alghanmi and Ayman Orabi

Abstract
This paper introduces a new family of lifetime distributions with different directions of hazard rate due to compounding the Lomax with the generalized truncated geometric distributions. This family called the Lomax-Generalized Truncated Geometric Distribution (LGTG) with an integer number equals to the order statistics. The general and some special cases forms of the probability density, probability distribution, reliability and hazard rate functions as well as their properties are derived. The estimation of parameters is presented by simulation study. The application study is illustrated based on real data set.

Keywords: Lomax distribution, lifetime distribution, truncated geometric distribution, order statistics, hazard rate and reliability function

1. Introduction

Hazard rate studies are widely used in many areas such as biological sciences, engineering, physics, economics, business, actuarial science. Lifetime distributions are very related to the study of hazard rate. One of the most common lifetime distributions is Lomax also called the Pareto type II which is proposed by Lomax in 1954.

The Commutative distribution function (CDF) of Lomax is given as

$$F(t) = 1-(\lambda+t)^{-\beta} \qquad t \geq 0 \ \ and \ \ \lambda, \beta > 0 \tag{1}$$

Where λ and β are the scale and shape parameters respectively. The probability density function (pdf) corresponding to equation (1) is given by

$$f(t) = \lambda^{\beta}(\lambda+t)^{-(\beta+1)} \tag{2}$$

Many compound lifetime distributions are introduced by linking the distributions of first or last order statistics with discrete distribution for example, Adamidis and Loukas (1998) [2] introduced the compound exponential-geometric distribution (EG) using first order with decreasing hazard rate, Louzada *et al*. (2011) [14] construct the complementary exponential-geometric (CEG) distribution with increasing hazard rate. In the same way, the several lifetime distributions with decreasing, increasing or bathtub hazard rate are proposed by many authors. For example, Barreto-Souza *et al*. (2011) [8]. Barreto and Bakouch, (2013) [7]. Bakouch *et al*. (2014) [5], Ramos *et al* (2015) [18], and others. Rahmouni and Orabi (2017) [15] introduced the general form of compound Exponential Poisson (EGTP) by linking the distribution of j^{th} order statistics with Poisson distribution also they introduced the general form of compound Exponential Geometric (EGTG) in (2018) and Exponential-Logarithmic (EGTL) in (2023). Several studies interested in the compound of Lomax with different distributions. For instance, The Poisson-Lomax distribution, Al-Zahrania and Sagor (2014) [3], the half-logistic Lomax distribution, Anwar and Zahoor (2018) [4], the beta Marshall-Olkin Lomax distribution, On Maxwell-Lomax distribution, Abiodun and Ishaq (2022) [1], the minimum Lindley Lomax distribution, Khan *et al* (2022) [11].

In this paper, we construct a new family of lifetime distribution with different directions of hazard rate based on mixing Lomax with truncated geometric distribution. We call this family the Lomax-generalized truncated geometric distribution (LGTG). We discuss properties of (LGTG) distribution as probability density function (pdf), cumulative distribution function (cdf), the rth moment, Reliability, hazard rate and its direction and generation of random numbers in general form and some special cases. We introduce the estimation for parameters of LGTG distribution in general form by using likelihood and EM algorithm methods. We apply the LGTG distribution on real data.

2. The New Family of Lifetime Distributions

The new family of lifetime distributions is obtained by compounding the Lomax with generalized truncated geometric distributions as follows: Let $\underline{T} = (t_1, t_2, .. t_s)$ is a random sample with size S that has iid Lomax random as equation (1) and S is a discrete random variable that follows a truncated (j-1) geometric distribution with parameter θ.The probability mass function (PMF) of truncated (j-1) geometric distribution is given as

$$P_j(S=s) = (1-\theta)\theta^{s-j} \; ; \; 0<\theta<1 \, , \; j=1,2,...,S \tag{3}$$

The pdf of the j^{th} order statistic $T_{(j)} = y$ is given by the equation (4) (David, 1981, p. 9; Balakrishnan and Cohen, 1991, p. 12):

$$f_j(y/s,\lambda) = \frac{\beta\lambda^{\beta(s-k+1)}(\lambda+y)^{-(\beta s+1)}[(\lambda+y)^{\beta}-\lambda^{\beta}]^{j-1}\Gamma(s)}{\Gamma(s-j+1)\Gamma(j)(\lambda+y)^{\beta s+1}} \; ; \; y,\lambda,\beta>0 \tag{4}$$

Where $\Gamma(\;)$ is gamma function.

The joint probability density function is obtained from equations (3) and (4) as follows:

$$g(y,s/\theta,\lambda,\beta) = \begin{cases} \dfrac{(1\text{-}\theta)\theta^{s\text{-}j}\beta\lambda^{\beta(s-k+1)}[(\lambda+y)^{\beta}-\lambda^{\beta}]^{j-1}\Gamma(s+1)}{\Gamma(s-j+1)\Gamma(j)(\lambda+y)^{\beta s+1}} & (I)\; j=1,2,...,z \\ \dfrac{(1\text{-}\theta)\theta^{s\text{-}j}\beta\lambda^{\beta(m+1)}[(\lambda+y)^{\beta}-\lambda^{\beta}]^{s-m-1}\Gamma(s+1)}{\Gamma(s-m)\Gamma(m+1)(\lambda+y)^{\beta s+1}} & (II)\; m=0,1,...,z \end{cases} \tag{5}$$

where y in above equation has two definitions: the first is for ascending order statistics i.e. $y=Y_{(j)}$ from $(Y_{(1)}<Y_{(2)}<...<Y_{(s)})$ that is part (I) in the equation and the second is for descending order statistics i.e. $y=Y_{(s-m)}$ from $(Y_{(s)}>Y_{(s-1)}>...>Y_{(1)})$ that is the part (II) in the equation.

The probability density function (pdf) for the new family distributions (LGTG) is the marginal density function of y which given by:

$$g(y/\theta,\lambda,\beta) = \begin{cases} \dfrac{(1\text{-}\theta)\beta j\lambda^{\beta}(\lambda+y)^{\beta-1}[(\lambda+y)^{\beta}\text{-}\lambda^{\beta}]^{j\text{-}1}}{[(\lambda+y)^{\beta}\text{-}\theta\lambda^{\beta}]^{j+1}} & (I)\; j=1,2,...,z \\ \dfrac{(1\text{-}\theta)\beta(m+1)\lambda^{\beta(m+1)}(\lambda+y)^{\beta-1}}{[(\lambda+y)^{\beta}(1-\theta)+\theta\lambda^{\beta}]^{m+2}} & (II)\; m=0,1,...,z \end{cases} \tag{6}$$

Where λ the scale parameter and θ, β are the shape parameters.

The cumulative distribution function (cdf) of y corresponding to the pdf in equation (6) is given by:

$$G(y/\theta,\lambda,\beta) = \begin{cases} \left(\dfrac{(\lambda+y)^{\beta}\text{-}\lambda^{\beta}}{(\lambda+y)^{\beta}\text{-}\theta\lambda^{\beta}}\right)^{j} & (I)\; j=1,2,...,z \\ 1-\left(\dfrac{\lambda^{\beta}}{(\lambda+y)^{\beta}(1-\theta)+\theta\lambda^{\beta}}\right)^{m+1} & (II)\; m=0,1,...,z \end{cases} \tag{7}$$

In Table (1), we present the pdf in equation (6) for some special cases at the first, second and third order statistics and at the last, last-1 and last-2 order statistics.

Table 1: The pdf for some special cases

Order statistic	j, m	Probability density function
first	j=1	$\frac{(1\text{-}\theta)\beta\lambda^{\beta}(\lambda+y)^{\beta-1}}{[(\lambda+y)^{\beta}\text{-}\theta\lambda^{\beta}]^{2}}$
second	j=2	$\frac{2(1\text{-}\theta)\beta\lambda^{\beta}(\lambda+y)^{\beta-1}[(\lambda+y)^{\beta}\text{-}\lambda^{\beta}]}{[(\lambda+y)^{\beta}\text{-}\theta\lambda^{\beta}]^{3}}$
third	j=3	$\frac{3(1\text{-}\theta)\beta\lambda^{\beta}(\lambda+y)^{\beta-1}[(\lambda+y)^{\beta}\text{-}\lambda^{\beta}]^{2}}{[(\lambda+y)^{\beta}\text{-}\theta\lambda^{\beta}]^{4}}$
Last	m=0	$\frac{(1\text{-}\theta)\beta\lambda^{\beta}(\lambda+y)^{\beta-1}}{[(\lambda+y)^{\beta}(1-\theta)+\theta\lambda^{\beta}]^{2}}$
Last-1	m=1	$\frac{2(1\text{-}\theta)\beta\lambda^{2\beta}(\lambda+y)^{\beta-1}}{[(\lambda+y)^{\beta}(1-\theta)+\theta\lambda^{\beta}]^{3}}$
Last-2	m=2	$\frac{3(1\text{-}\theta)\beta\lambda^{3\beta}(\lambda+y)^{\beta-1}}{[(\lambda+y)^{\beta}(1-\theta)+\theta\lambda^{\beta}]^{4}}$

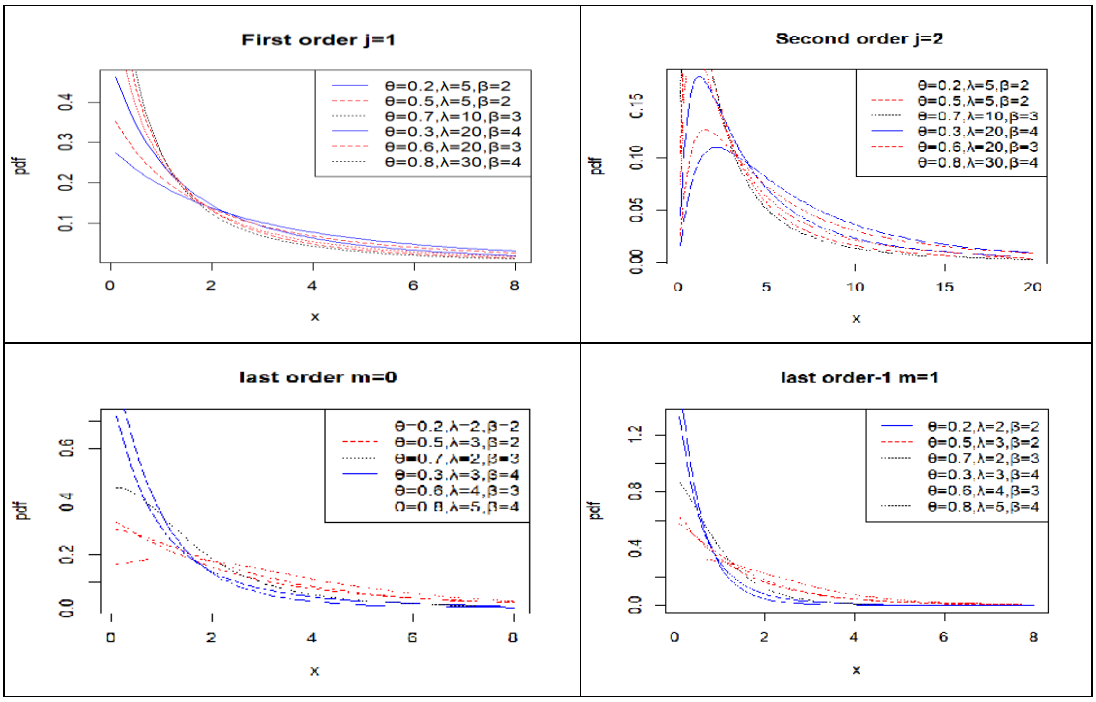


Fig 1: Pdf of the LGTG distribution for j=(1,2) and m=(0,1)

2.1 The r^th^ Moment

If y has the pdf in equation (4), then the r^th^ moment is calculated by:

$$E\left(y^{r}\right)=\begin{cases}\beta(1-\theta)\lambda^{r}\sum_{i=0}^{\infty}\sum_{k=0}^{j-1}\frac{(-1)^{k}\Gamma(i+j+1)\Gamma[\beta(i+k+1)-r]\Gamma(r+1)\theta^{i}}{\Gamma(i+1)\Gamma(k+1)\Gamma(j-k)\Gamma[\beta(i+k+1)+1]} & (I)\ j=1,2,\ldots,z\\ \beta(1-\theta)\lambda^{r}\sum_{i=0}^{\infty}\sum_{k=0}^{i}\frac{(-1)^{k}\Gamma(i+m+2)\Gamma[\beta(i+m+1)-r]\Gamma(r+1)\theta^{i}}{\Gamma(m)\Gamma(k+1)\Gamma(i-k+1)\Gamma[\beta(i+k+1)+1]} & (II)\ m=0,1,\ldots,z\end{cases}$$

2.2 Reliability and Hazard Rate

The reliability function is the probability of being alive just before a duration x, given by R(x) = Pr{Y> y} = 1-G(y) which is the probability that the event under study has not occurred by duration x. So the reliability function corresponding to the pdf in equation (4) is given by:

$$R(y/\theta,\lambda,\beta)=\begin{cases}1-\left(\frac{(\lambda+y)^{\beta}\text{-}\lambda^{\beta}}{(\lambda+y)^{\beta}\text{-}\theta\lambda^{\beta}}\right)^{j} & (I)\ j=1,2,\ldots,z\\ \left(\frac{\lambda^{\beta}}{(\lambda+y)^{\beta}(1-\theta)+\theta\lambda^{\beta}}\right)^{m+1} & (II)\ m=0,1,\ldots,z\end{cases} \qquad (8)$$

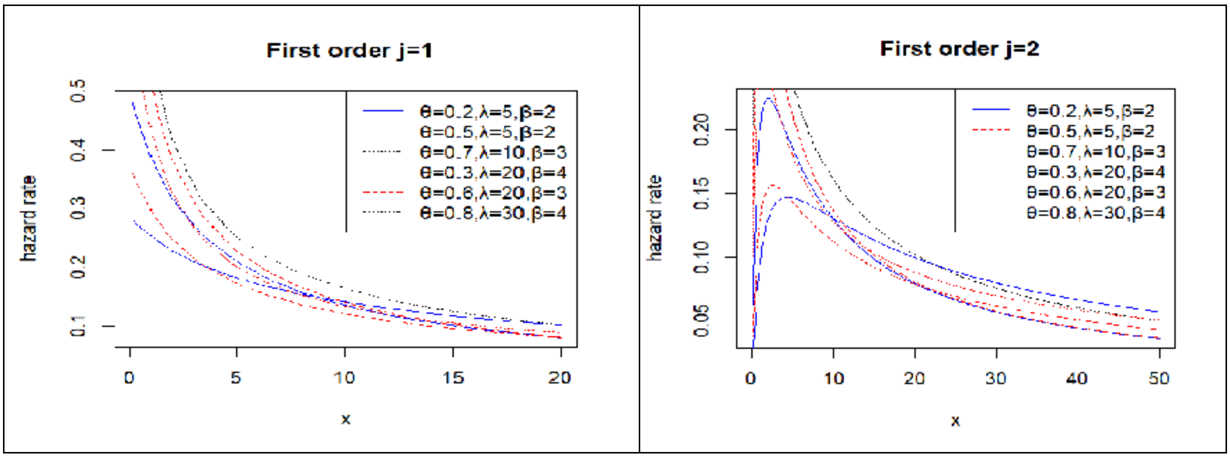
The hazard rate defines as the number of failure times per unit time. Mathematically, it equals the pdf divided by the reliability function. From equations (6) and (8) the hazard rate for (LGTG) distribution is given by:

$$h(y/\theta,\lambda,\beta)=\begin{cases}\dfrac{(1-\theta)j\beta\lambda^{\beta}(\lambda+y)^{\beta-1}[(\lambda+y)^{\beta}-\lambda^{\beta}]^{j-1}}{[(\lambda+y)^{\beta}-\theta\lambda^{\beta}]\{[(\lambda+y)^{\beta}-\theta\lambda^{\beta}]^{j}-[(\lambda+y)^{\beta}-\lambda^{\beta}]^{j}\}} & (I)\ j=1,2,...,z\\ \dfrac{\beta(m+1)(1-\theta)\lambda^{\beta}}{(\lambda+y)[(\lambda+y)^{\beta}(1-\theta)+\theta\lambda^{\beta}]} & (II)\ m=0,1,...,z\end{cases} \qquad (9)$$

From equation (9) in part I and figure 2 we note, if j=1 the hazard rate has decreasing direction for different values of parameters and $\lim_{y\to 0}h(y/\theta,\lambda,\beta)=\dfrac{\beta}{\lambda(1-\theta)}$ but if j>1 the curve of hazard rate has right skewness i.e. it has increasing direction in beginning until reach to the peak after that it reverses its direction and $\lim_{y\to 0}h(y/\theta,\lambda,\beta)=0$. In part II of equation (9), $\lim_{y\to 0}h(y/\theta,\lambda,\beta)=\dfrac{\beta(m+1)(1-\theta)}{\lambda}$ and from figure 2, we note that when the shape parameter $\theta<0.5$ the curve of hazard rate has decreasing direction otherwise it has right skewness.

Table 2: The hazard rate for some special cases

Order statistic	j, m	Hazard rat
First	j=1	$\dfrac{\beta(\lambda+y)^{\beta-1}}{[(\lambda+y)^{\beta}-\theta\lambda^{\beta}]}$
Second	j=2	$\dfrac{2\beta(\lambda+y)^{\beta-1}[(\lambda+y)^{\beta}-\lambda^{\beta}]}{[(\lambda+y)^{\beta}-\theta\lambda^{\beta}][2(\lambda+y)^{\beta}-\lambda^{\beta}(1+\theta)]}$
Third	j=3	$\dfrac{3\beta(\lambda+y)^{\beta-1}[(\lambda+y)^{\beta}-\lambda^{\beta}]^{2}}{[(\lambda+y)^{\beta}-\theta\lambda^{\beta}][3(\lambda+y)^{2\beta}-3\lambda^{\beta}(\lambda+y)^{\beta}(1+\theta)-\lambda^{2\beta}(1+\theta+\theta^{2}]}$
Last	m=0	$\dfrac{\beta(1-\theta)\lambda^{\beta}}{(\lambda+y)[(\lambda+y)^{\beta}(1-\theta)+\theta\lambda^{\beta}]}$
Last-1	m=1	$\dfrac{2\beta(1-\theta)\lambda^{\beta}}{(\lambda+y)[(\lambda+y)^{\beta}(1-\theta)+\theta\lambda^{\beta}]}$
Last-2	m=2	$\dfrac{3\beta(1-\theta)\lambda^{\beta}}{(\lambda+y)[(\lambda+y)^{\beta}(1-\theta)+\theta\lambda^{\beta}]}$



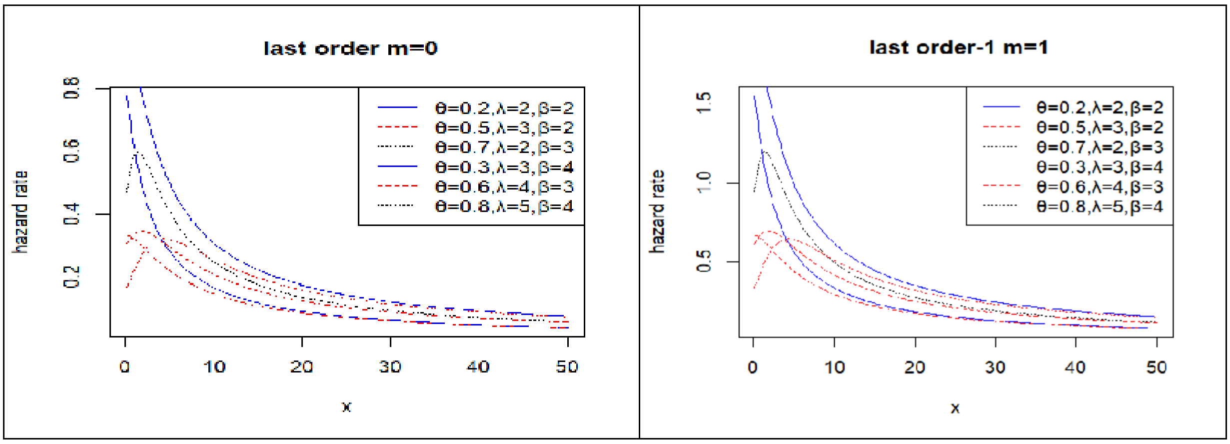


Fig 2: Hazard rate of the LGTG distribution for j=(1,2) and m=(0,1)

2.3. Generation of Random Variable

The generation of random variable Y corresponding the cdf equation in (5) is given by

$$Y=\begin{cases}\lambda\left[\left(\dfrac{1-\theta U^{\frac{1}{j}}}{1-U^{\frac{1}{j}}}\right)^{\frac{1}{\beta}}-1\right] & (I)\ j=1,2,\dots,z\\[2ex] \lambda\left[\left(\dfrac{1-(1-U)^{\frac{1}{m+1}}\theta}{(1-\theta)(1-U)^{\frac{1}{m+1}}}\right)^{\frac{1}{\beta}}-1\right] & (II)\ m=0,1,\dots,z\end{cases} \tag{10}$$

where U is a random variable that follows the standard uniform distribution.

From equation (8) the median $\tilde{Y}$ is

$$\tilde{Y}=\begin{cases}\lambda\left[\left(\dfrac{2^{\frac{1}{j}}-\theta}{2^{\frac{1}{j}}-1}\right)^{\frac{1}{\beta}}-1\right] & (I)\ j=1,2,\dots,z\\[2ex] \lambda\left[\left(\dfrac{(2)^{\frac{1}{m+1}}-\theta}{(1-\theta)}\right)^{\frac{1}{\beta}}-1\right] & (II)\ m=0,1,\dots,z\end{cases}$$

3. Estimation

In this section, we will introduce the estimated parameters θ,λ and β for the LGTG distributions in general form let $(Y_1,Y_2,\dots,Y_s)$ be a random sample that is observed from the LGTG distribution with the pdf in equation (4). The log-likelihood function given is:

$$\ln L(\theta,\lambda,\beta/\underline{Y})=\begin{cases} n\ln(1-\theta)+n\ln(\beta)+n\beta\ln(\lambda)+(\beta-1)\sum_{i=1}^{n}\ln(\lambda+y_i)+(j-1)\sum_{i=1}^{n}\ln[(\lambda+y_i)^{\beta}-\lambda^{\beta}] & \\ -(j+1)\sum_{i=1}^{n}\ln[(\lambda+y_i)^{\beta}-\theta\lambda^{\beta}] & (I)\ j=1,2,\dots,z\\ n\ln(1-\theta)+n\ln(\beta)+n\beta(m+1)\ln(\lambda)+(\beta-1)\sum_{i=1}^{n}\ln(\lambda+y_i) & \\ -(m+2)\sum_{i=1}^{n}\ln[(\lambda+y_i)^{\beta}(1-\theta)+\theta\lambda^{\beta}] & (II)\ m=0,1,\dots,z\end{cases}$$

The first derivative of the log-likelihood function given is:

$$\frac{\partial \ln L(\theta,\lambda,\beta/\underset{-}{Y})}{\partial\theta}=\begin{cases} -n(1-\theta)^{-1}+(j+1)\lambda^{\beta}\sum_{i=1}^{n}[(\lambda+y_i)^{\beta}-\theta\lambda^{\beta}]^{-1} & (I)\ j=1,2,...,z \\ -n(1-\theta)+(m+2)\sum_{i=1}^{n}[(\lambda+y_i)^{\beta}-\lambda^{\beta}][(\lambda+y_i)^{\beta}(1-\theta)+\theta\lambda^{\beta}]^{-1} & (II)\ m=0,1,...,z \end{cases}$$

$$\frac{\partial \ln L(\theta,\lambda,\beta/\underset{-}{Y})}{\partial\lambda}=\begin{cases} n\beta\lambda^{-1}+(\beta-1)\sum_{i=1}^{n}(\lambda+y_i)^{-1}+(j-1)\beta\sum_{i=1}^{n}[(\lambda+y_i)^{\beta-1}-\lambda^{\beta-1}][(\lambda+y_i)^{\beta}-\lambda^{\beta}]^{-1} & \\ -(j+1)\beta\sum_{i=1}^{n}[(\lambda+y_i)^{\beta-1}-\theta\lambda^{\beta-1}][(\lambda+y_i)^{\beta}-\theta\lambda^{\beta}]^{-1} & (I)\ j=1,2,...,z \\ n\beta(m+1)\lambda^{-1}+(\beta-1)\sum_{i=1}^{n}(\lambda+y_i)^{-1} & \\ -(m+2)\beta\sum_{i=1}^{n}[(\lambda+y_i)^{\beta-1}(1-\theta)+\theta\lambda^{\beta-1}][(\lambda+y_i)^{\beta}(1-\theta)+\theta\lambda^{\beta}]^{-1} & (II)\ m=0,1,...,z \end{cases}$$

$$\frac{\partial \ln L(\theta,\lambda,\beta/\underset{-}{Y})}{\partial\beta}=\begin{cases} n\beta^{-1}+n\ln(\lambda)+\sum_{i=1}^{n}\ln(\lambda+y_i)+(j-1)\sum_{i=1}^{n}[(\lambda+y_i)^{\beta}\ln(\lambda+y_i)-\lambda^{\beta}\ln\lambda][(\lambda+y_i)^{\beta}-\lambda^{\beta}]^{-1} & \\ -(j+1)\sum_{i=1}^{n}[(\lambda+y_i)^{\beta}\ln(\lambda+y_i)-\theta\lambda^{\beta}\ln\lambda][(\lambda+y_i)^{\beta}-\theta\lambda^{\beta}]^{-1} & (I)\ j=1,2,...,z \\ n\beta^{-1}+n(m+1)\ln(\lambda)+\sum_{i=1}^{n}\ln(\lambda+y_i)-(m+2) & \\ \sum_{i=1}^{n}[(\lambda+y_i)^{\beta}(1-\theta)\ln(\lambda+y_i)+\theta\lambda^{\beta}\ln\lambda][(\lambda+y_i)^{\beta}(1-\theta)+\theta\lambda^{\beta}]^{-1} & (II)\ m=0,1,...,z \end{cases}$$

From solving the above system of equations we can get the MLE estimator for parameters.

EM algorithm

To start the EM algorithm, we define a hypothetical distribution of complete-data with the joint density function in equation (3). We drive the conditional mass function as.

$$p(s/y,\theta,\lambda,\beta)=\begin{cases} C_j^s[\theta(\lambda+y)^{-\beta}]^{s-j}[1-\theta(\lambda+y)^{-\beta}]^{j+1} & (I)\ j=1,2,...,z \\ C_{m+1}^s\{\theta[1-(\lambda+y)^{-\beta}]\}^{s-j}\{1-\theta[1-(\lambda+y)^{-\beta}]\}^{j+1} & (II)\ m=0,1,...,z \end{cases}$$

E-step

$$E(s/y,\theta,\lambda,\beta)=\begin{cases} \frac{(j+1)\theta\lambda^{\beta}}{(\lambda+y)^{\beta}-\theta\lambda^{\beta}}+j & (I)\ j=1,2,...,z \\ \frac{(m+2)\theta[(\lambda+y)^{\beta}-\lambda^{\beta}]}{(\lambda+y)^{\beta}(1-\theta)+\theta\lambda^{\beta}}+m+1 & (II)\ m=0,1,...,z \end{cases}$$

M-step

$$\theta^{(r+1)}=\begin{cases} \dfrac{(j+1)\sum_{i=1}^{n}\frac{\theta^{(r)}\lambda^{\beta}}{(\lambda+y_i)^{\beta}-\theta^{(r)}\lambda^{\beta}}}{(j+1)\sum_{i=1}^{n}\frac{\theta^{(r)}\lambda^{\beta}}{(\lambda+y_i)^{\beta}-\theta^{(r)}\lambda^{\beta}}+n} & (I)\ j=1,2,...,z \\ \dfrac{(m+2)\sum_{i=1}^{n}\frac{\theta^{(r)}[(\lambda+y_i)^{\beta}-\lambda^{\beta}]}{(\lambda+y)^{\beta}(1-\theta^{(r)})+\theta^{(r)}\lambda^{\beta}}}{(m+2)\sum_{i=1}^{n}\frac{\theta^{(r)}[(\lambda+y_i)^{\beta}-\lambda^{\beta}]}{(\lambda+y)^{\beta}(1-\theta^{(r)})+\theta^{(r)}\lambda^{\beta}}+n} & (II)\ m=0,1,...,z \end{cases}$$

$$\lambda^{(r+1)} = \begin{cases} \dfrac{\beta\sum_{i=1}^{n} y\,[\dfrac{\theta[\lambda^{(r)}]^{\beta}}{(\lambda^{(r)}+y_i)^{\beta}-\theta[\lambda^{(r)}]^{\beta}}+1]}{n} & (Ia)\ j=1 \\ \dfrac{\beta\sum_{i=1}^{n} y\,[\dfrac{\theta[\lambda^{(r)}]^{\beta}}{(\lambda^{(r)}+y_i)^{\beta}-\theta[\lambda^{(r)}]^{\beta}}+1]-n\lambda^{(r+1)}}{(j-1)\beta\sum_{i=1}^{n}\dfrac{y_i[\lambda^{(r+1)}]^{\beta-1}}{(\lambda^{(r+1)}+y_i)^{\beta}-[\lambda^{(r+1)}]^{\beta}}} & (Ib)\ j=2,\ldots,z \\ \dfrac{\beta(m+1)\sum_{i=1}^{n} y_i - n\lambda^{(r+1)}}{\beta\sum_{i=1}^{n}\left[\dfrac{y[\lambda^{(r+1)}]^{\beta-1}}{(\lambda^{(r+1)}+y_i)^{\beta}-[\lambda^{(r+1)}]^{\beta}}\right]\left(\dfrac{\theta[\lambda^{(r)}]^{\beta}}{(\lambda^{(r)}+y_i)^{\beta}-\theta[\lambda^{(r)}]^{\beta}}\right)} & (II)\ m=0,1,\ldots,z \end{cases}$$

$$\beta^{(r+1)} = \begin{cases} \dfrac{n}{\sum_{i=1}^{n}\ln(\dfrac{\lambda}{\lambda+y_i})\left(\dfrac{(j-1)\lambda^{\beta^{(r+1)}}}{(\lambda+y_i)^{\beta^{(r+1)}}-\lambda^{\beta^{(r+1)}}}-(\dfrac{\theta\lambda^{\beta^{(r)}}}{(\lambda+y_i)^{\beta^{(r)}}-\theta\lambda^{\beta^{(r)}}}+1)\right)} & (I)\ j=1,2,\ldots,z \\ \dfrac{n}{\sum_{i=1}^{n}\ln(\dfrac{\lambda}{\lambda+y_i})\left(\dfrac{\lambda^{\beta^{(r+1)}}(\dfrac{\theta\lambda^{\beta^{(r)}}}{(\lambda+y_i)^{\beta^{(r)}}-\theta\lambda^{\beta^{(r)}}})}{(\lambda+y_i)^{\beta^{(r+1)}}-\lambda^{\beta^{(r+1)}}}-(m+1)\right)} & (II)\ m=0,1,\ldots,z \end{cases}$$

4. Simulation Study

The simulation study is done by the following steps:

1. generate 1000 random samples of sizes 20, 50 and 100 from LGTG distribution with parameters (θ,λ,β) = (0.2,3,2), (0.4,5,3), (0.6,8,4), (0.8,10,5) for order statistics j=1,2,3 and m=0,1,2 using equation (10).
2. get the estimation of parameters using MLE estimator.
3. compute the mean square errors (MSE) for each parameter where $MSE = E(\Theta-\hat{\Theta})^2$ compare the MSE for each parameter with sample size

Table 3: The MSE for estimated parameters

MSE for θ							
Parameters (θ,λ,β)	**N**	**Order statistics**			**Reverse order statistics**		
		J=1	**J=2**	**J=3**	**M=0**	**M=1**	**M=2**
(0.2,3,2)	20	0.15439	0.09621	0.13065	0.14478	0.12512	0.17382
(0.4,5,3)		0.15942	0.11689	0.16113	0.18019	0.15191	0.17694
(0.6,8,4)		0.17341	0.13013	0.15501	0.19117	0.13911	0.16958
(0.8,10,5)		0.15965	0.12951	0.17592	0.19544	0.18332	0.19288
(0.2,3,2)	50	0.10621	0.05183	0.06177	0.07754	0.06522	0.02615
(0.4,5,3)		0.11805	0.05596	0.08209	0.07864	0.08498	0.08661
(0.6,8,4)		0.12695	0.06368	0.08731	0.12487	0.11312	0.09017
(0.8,10,5)		0.11161	0.07342	0.12301	0.13766	0.12488	0.13512
(0.2,3,2)	100	0.02297	0.01969	0.04691	0.01112	0.01522	0.01043
(0.4,5,3)		0.04002	0.0259	0.04923	0.02893	0.02937	0.01905
(0.6,8,4)		0.03983	0.03593	0.05399	0.04832	0.05344	0.01941
(0.8,10,5)		0.09329	0.04277	0.05542	0.06529	0.05919	0.02067
MSE for λ							
(0.2,3,2)	20	0.23111	0.25161	0.33591	0.32942	0.33672	0.28173
(0.4,5,3)		0.25977	0.34444	0.34137	0.37566	0.36717	0.30015
(0.6,8,4)		0.27329	0.41403	0.35907	0.39987	0.38183	0.37133
(0.8,10,5)		0.28815	0.44852	0.39624	0.42155	0.41866	0.35904
(0.2,3,2)	50	0.15583	0.20504	0.23345	0.24726	0.23953	0.23466
(0.4,5,3)		0.18622	0.21405	0.26686	0.27452	0.30126	0.25477
(0.6,8,4)		0.19497	0.22721	0.29342	0.27943	0.31107	0.25868
(0.8,10,5)		0.22635	0.22999	0.31942	0.30297	0.32096	0.26493
(0.2,3,2)	100	0.10748	0.10752	0.10664	0.12124	0.10702	0.17576
(0.4,5,3)		0.10898	0.16161	0.12604	0.12446	0.11268	0.13553
(0.6,8,4)		0.11734	0.17395	0.16811	0.17537	0.14639	0.21698

(0.8,10,5)		0.15441	0.20054	0.21556	0.21565	0.16339	0.22774
MSE for β							
(0.2,3,2)	20	0.44314	0.44175	0.50592	0.31668	0.35927	0.35954
(0.4,5,3)		0.47442	0.43253	0.52425	0.42504	0.46707	0.32339
(0.6,8,4)		0.51316	0.51587	0.57943	0.55199	0.48055	0.40047
(0.8,10,5)		0.52923	0.59556	0.56834	0.53936	0.51894	0.40689
(0.2,3,2)	50	0.31106	0.24432	0.26485	0.16777	0.27792	0.29545
(0.4,5,3)		0.31531	0.28219	0.30085	0.24336	0.28879	0.30322
(0.6,8,4)		0.37057	0.30466	0.33058	0.30517	0.31985	0.31962
(0.8,10,5)		0.42454	0.38474	0.39149	0.30787	0.33199	0.30121
(0.2,3,2)	100	0.18915	0.16649	0.16675	0.15583	0.17128	0.17882
(0.4,5,3)		0.20151	0.16952	0.17112	0.15665	0.23558	0.21195
(0.6,8,4)		0.23278	0.17429	0.21911	0.16227	0.23938	0.24776
(0.8,10,5)		0.22062	0.22852	0.22821	0.16393	0.26665	0.23248

Table (3). shows the mean square errors (MSE) for estimated parameters of LGTG distribution for some values of the parameters (θ, λ, β) = (0.2,3,2), (0.4,5,3), (0.6,8,4), (0.8,10,5) and special cases of order statistics j = 1,2,3 and m = 0,1,2 We note from Table 3, the MSE is decreasing by increasing of sample size.

5-Application Example

In this section, we introduce the example of application through the fitting LGTG distribution to real dataset and compare it with Exponential generalized truncated geometric distribution LGTG, Exponential geometric EG, Gamma and Weibull distributions. The real dataset was studied by Kus (2007) [12], Chahkandi and Ganjali (2009) [9], Barreto-Souza and Bakouch (2013) [7] and Rahmouni and Orabi (2017) [15]. The dataset in Table (4) represents 24 observations of "time intervals (in days) between successive earthquakes.

Table 4: The time intervals (in days) between successive earthquakes

1163	3258	323	159	756	409	501	616	389
67	896	8592	2039	217	9	633	461	1821
4863	143	182	2117	3709	979			

Table (5) shows the estimated parameters, calculated values of Kolmogorov-Smirnov (K-S) and their respective p-values due to the goodness of fit for real dataset. From comparison of some special cases of the LGTP distribution with other distributions based on the K-S test at 5% significance level, we note special cases of the LGTP at order statistics j=1,2,3 are the more fitting to the real dataset than others where it has largest p-value.

Table 5: The goodness of fit for real dataset

Distributions	**Estimation of parameters**			**K.S**	**P-value**
	Scale (λ)	**Shape (θ)**	**Shape (β)**		
LGTG(j=1)	2523.17	0.12353	2.5518	0.36103	0.9994100
LGTG(j=2)	443.802	0.13951	1.2914	0.28254	0.9999980
LGTG(j=3)	165.433	0.13654	0.9914	0.29629	0.9999930
LGTG(m=0)	1058.28	0.37065	1.8823	0.34427	0.9997805
LGTG(m=1)	1101.46	0.41679	1.0927	0.34715	0.9997414
LGTG(m=2)	888.172	0.58528	0.8351	0.34723	0.9997402
EGTP(j=1)	2770.08	2.61700	-	0.44091	0.9820000
EGTP(j=2)	1798.56	4.56000	-	0.72505	0.6680000
EGTP(j=3)	1367.99	6.15200	-	0.89651	0.3980000
EGTP(j=4)	1131.22	7.64200	-	0.98469	0.2880000
EG	30303.03	0.73690	-	0.47226	0.9690000
Weibull	2000.00	0.71170	-	0.60502	0.8328000
Gamma	1231.53	0.78540	-	0.49186	0.9690000

6. Conclusion

By compounding the Lomax with the generalized truncated geometric distributions, we establish a new family of lifetime distributions with different directions of hazard rates called the Lomax-generalized truncated geometric distribution (LGTG). The probability density function, cumulative distribution, r^{th} moment, reliability function, hazard rate and generation of random variable are studied as the defining attributes of the new family. The LGTG distribution parameters are estimated using the maximum likelihood method and these estimators are assessed using numerical simulations. The LGTG distribution is applied on real data.

7. References

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