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Composition operators between the Hardy space H^p and the Bergman spaces L^q_a

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Abstract
We study composition operators C_φ between the Hardy space H^p and the Bergman spaces L^q_a, between the Besov type space Bp,p-1 and the Bergman spaces L^q_a. Using the main results in this paper, as a result, we can also prove the following: If $0 < q \leq p$, then $C_\varphi: H^p \to L^q_a$ is bounded, while there is no symbol φ such that it is bounded below.

Keywords: Composition operator, Bloch space, BMOA, Bergman space, Hardy space, the Besov type space $B_{p,p-1}$, closed range, bounded below

Introduction

For φ holomorphic self-map of the open unit disk D, the composition operator C_φ is defined by $C_\varphi(f)(z) = (f \circ \varphi)(z) = f(\varphi(z))$, $z \in D$.
For $z,w \in D$, let $\varphi_z(w) = \frac{z-w}{1-\bar{z}w}$ and that $dA(z)$ denote the area measure on D.
For $p>0$, the Hardy space H^p consists of analytic functions on D such that

$$\| f \|_{H^p}^p = \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < +\infty.$$

The Hardy norm can be equivalently expressed by

$$\| f \|_{H^p}^p \approx |f(0)|^p + \frac{p^2}{2} \int_D |f(z)|^{p-2} |f'(z)|^2 (1-|z|^2) dA(z) < +\infty.$$

The genaralized Nevanlinna counting function $N_{\varphi,\gamma}$ is defined for $\gamma > 0$ by

$$N_{\varphi,\gamma}(w) = N_\gamma(w) = \sum_{z \in \varphi^{-1}\{w\}} \left(\log \frac{1}{|z|} \right)^\gamma, \qquad w \in D \backslash \{\varphi(0)\}.$$

For $p>0$, the Bergman space L^p_a is defined to be the space of analytic functions f on D such that

$$\| f \|_{L^p_a}^p = \int_D |f(z)|^p dA(z) < +\infty.$$

The Bergman norm can be equivalently expressed by

$$\| f \|_{L^p_a}^p \approx |f(0)|^p + \int_D |f(z)|^{p-2} |f'(z)|^2 (1-|z|^2)^2 dA < +\infty.$$

For p>0, α >-1, the weighted Dirichlet space D^{α}_{p} is defined to be the space of analytic functions f on D such that

$$\| f \|_{\mathcal{D}^{\alpha}_{p}} = |f(0)| + \left(\int_D (1-|z|^2)^{\alpha} |f'(z)|^p dA(z) \right)^{\frac{1}{p}} < +\infty .$$

In case α=1 and p=2, then $D^1_2 = H^2$ is the classical Hardy space. Furthermore, in case α = p and 1 ≤ p < ∞, then $D^p_p = L^p_a$ is the usual Bergman space. Also, in case α = p-2 and 1 < p < ∞, $D^{p-2}_p = B_p$ is called the Besov space. In particular, D^0_2 =D is called the Dirichlet space. Also, in case α = p-1 and 0 < p < ∞, $D^{p-1}_p = B_{p,\ p-1}$ is called the Besov type space. In particular, is the classical Hardy space.

It is trivial that $B_p \subset$ Bp, p-1 (p>1).

For α > 0, the Bloch type space Bα of D is defined to be the space of analytic functions f on D such that

$$\mathcal{D}^1_2 = B_{2,1} = H^2 \quad \| f \|_{\beta_{\alpha}} = |f(0)| + \sup_{z\in D} (1-|z|^2)^{\alpha} |f'(z)| < +\infty$$

For p>0 and α > 0, the space B^p_{α} of D is defined to be the space of analytic functions f on D such that

$$|f(0)| + \sup_{z\in D} (1-|z|^2)^{\alpha} |f(z)|^{\frac{p}{2}-1} |f'(z)| < +\infty .$$

In case of p = 2 and α = 1, then B^2_1 is the classical Bloch space.
Let X be Banach spaces and let T be a linear operator from X into X. Then T is called to be bounded below on X if there exists a positive constant K > 0 such that
$\| Tf \| \geq K \| f \|$ for all f ∈ X.
By Schwarz-Pick lemma, the composition operator C_{φ} is bounded on the Bloch space B. It follows from Littlewood's subordination theorem that C_{φ} is bounded on all the Bergman spaces L^p_a. And C_{φ} is also bounded on all the Hardy space H^p. In [14] Nina Zorboska has been characterized the closed range composition operators on the Bergman spaces L^p_a and Hardy spaces H^p. H.Chen and P.Gauthier have been characterized the boundedness from below of composition operators on weighted Bloch space in [4]. On the other hand, Wayne Smith has been characterized the boundedness and compactness of the composition operators between Bergman spaces L^p_a and Hardy spaces H^p in [6]. In this paper we characterize the result that it has not been elucidated yet so far with respect to the boundedness from below of composition operators between Hardy space H^p and Bergman spaces L^q_a, between the Besov type space Bp,p-1 and the Bergman spaces L^p_a.

Background Material
In [6] Wayne Smith has been characterized the boundedness and compactness of the composition operators between Bergman spaces L_a^p and Hardy spaces H^p, between Bergman spaces L^p_a and Bergman spaces L^q_a, between Hardy spaces H^p and Hardy spaces H^q.

In [7], W.Smith and L. Yang proved the following theorem.

Theorem A. *Let* $0 < q < p$ and $\alpha > -1$, *and suppose that* φ *is an analytic self-map of* D. *Then the following are equivalent.*

(1) $C_{\varphi} : L^p_a(dA_{\alpha}) \to L^q_a(dA_{\beta})$ *is bounded.*

(2) $C_{\varphi} : L^p_a(dA_{\alpha}) \to L^q_a(dA_{\beta})$ *is compact.*

(3) $\dfrac{N_{\varphi,\beta+2}(z)}{(1-|z|^2)^{2+\alpha}} \in L^{\frac{p}{p-q}}(dA_{\alpha})$.

Applying α = 0 and β =-1 in Theorem A implies the following (see [7]).

Corollary B. Let 0 < q < p and suppose that φ is an analytic self-map of D. Then the following are equivalent.

(1) $C_{\varphi} : L^p_a \to H^q$ *is bounded.*

(2) $C_{\varphi} : L^p_a \to H^q$ *is compact.*

(3) $\dfrac{N_{\varphi,1}(z)}{(1-|z|^2)^2} \in L^{\frac{p}{p-q}}$.

Corollary B immediately yields the following.

Theorem C. Let $0 < q < p$. If $C_\varphi: L^p{}_a \to H^q$ is bounded, then there is no symbol φ such that it is bounded below.

Proof. If $C_\varphi: L^p{}_a \to H^q$ is bounded, then $C_\varphi: L^p{}_a \to H^q$ is compact. Hence Corollary B implies that there is no symbol φ such that it is bounded below.

Theorem A fails for $\alpha = -1$, that is, we can not apply Theorem A to the case of the composition operator C_φ on the Hardy space H^p (see [7]). In this paper, we do study this case ($\alpha = -1$) with respect to the bounded below of composition operators from Hardy space H^p to Bergman spaces $L^q{}_a$.

The main results

If $\varphi(0) = a$ and $\psi = \varphi_a \circ \varphi$, then C_φ is bounded below on H^p if and only if C_ψ is bounded below on H^p. So we assume from now on that $\varphi(0) = 0$ and that C_φ is acting on the subspace of functions that vanish at the origin.

In [4] H.Chen and P.Gauthier proved the following result with respect to the composition operators $C_\varphi: B\alpha \to B\beta$.

Theorem D. ([4]) Suppose $\beta \geq 1$ and $\alpha \leq \beta$. Then $C_\varphi: B\alpha \to B\beta$ is bounded, while $C_\varphi: B\alpha \to B\beta$ is not bounded below if $\alpha < \beta$.

In [10] we also proved the following result.

Theorem E. ([10]) Let $0 < p, q < \infty$, $\alpha, \gamma > 0$. Suppose that $C_\varphi: D^{p\alpha}{}_p \to D^{q\gamma}{}_q$ is bounded. If $C_\varphi: D^{p\alpha}{}_p \to D^{q\gamma}{}_q$ is bounded below, then there exists a constant $K > 0$ such that $\sup_{z\in D} |(C_\varphi f)'(z)|(1-|z|^2)^\gamma \geq K S_{p,q,\alpha}(f)$ for all $f \in B\alpha$, where

$$S_{p,q,\alpha}(f) := \begin{cases} \sup_{z\in D} |f'(z)|(1-|z|^2)^{\alpha+2\left(\frac{1}{p}-\frac{1}{q}\right)} \ (1 < q \leq p) \\ \sup_{z\in D} |f'(z)|(1-|z|^2)^{\alpha+2\left(\frac{1}{p}-1\right)} \left(\log \frac{2}{1-|z|^2}\right)^{-1} \ (q = 1 < p) \\ \\ \sup_{z\in D} |f'(z)|(1-|z|^2)^{\alpha+2\left(\frac{1}{p}-1\right)} \ (0 < q < 1 \leq p). \end{cases}$$

If $p \geq q > 1$ and $C_\varphi: B_{p,p-1} \to L^q{}_a$, then we have the following.

Theorem 1. *If $1 < q \leq p$, then $C_\varphi : B_{p,p-1} \to L^q_a$ is bounded, while there is no symbol φ such that it is bounded below* .

Proof. Since $1 < q \leq p$, the boundedness of $C_\varphi: B_{p,p-1} \to L^q{}_a$ follows from H¨older's inequality. Applying $\alpha = 1-1/p$, $\gamma = 1$ and $1 < q \leq p$ in Theorem E, it follows from Theorem D that $C_\varphi: B_{p,p-1} \to L^q{}_a$ is not bounded below.

Corollary 2. If $1 < q \leq 2$, then $C_\varphi: H^2 \to L^q{}_a$ is bounded, while there is no symbol φ such that it is bounded below.

Proof. Since $1 < q \leq 2$, the boundedness of $C_\varphi: H^2 \to L^q{}_a$ follows from H¨older's inequality. Applying $1 < q \leq 2$, it follows from Theorem 1 that $C_\varphi: H^2 \to L^q{}_a$ is not bounded below.

Let $\alpha > -1$. For $\forall a \in D$, the following estimate is standard (see, for example, [12]).

$$\int_D \frac{(1-|z|^2)^\alpha}{|1-\bar{a}z|^\lambda} dA(z) \sim \begin{cases} (1-|a|^2)^{\alpha+2-\lambda} & (\lambda > \alpha+2) & (|a| \to 1^-) \\ \log \frac{e}{1-|a|^2} & (\lambda = \alpha+2) & (|a| \to 1^-) \\ 1 & (\lambda < \alpha+2) & (|a| \to 1^-) \end{cases} \quad (1)$$

To prove main theorem, we need the well-known following lemma (see [12]).

$f \in L^p{}_a$, then

$$\| f \|^p_{L^p_a} \sim |f(0)|^p + \| (1-|z|^2) f'(z) \|^p_{L^p_a}.$$

To prove main theorem, we need the following lemma.

Lemma 3. Let φ be an analytic self-map of the disk. Then

$$\sup_{z\in D}(1-|z|^2)\,|C_\varphi f(z)|^{\frac{p}{2}-1}\,|(C_\varphi f)'(z)| \leq \sup_{z\in D}(1-|z|^2)\,|f(z)|^{\frac{p}{2}-1}\,|f'(z)|$$

for all $f \in \mathcal{B}^p_1$. ,

Proof. Using Schwarz-Pick lemma, we have

$$\begin{aligned}
&\sup_{z\in D}(1-|z|^2)\,|C_\varphi f(z)|^{\frac{p}{2}-1}\,|(C_\varphi f)'(z)| \\
&= \sup_{z\in D}(1-|z|^2)|(C_\varphi f)(z)|^{\frac{p}{2}-1}|f'(\varphi(z))||\varphi'(z)| \\
&= \sup_{z\in D}(1-|\varphi(z)|^2)|f(\varphi(z))|^{\frac{p}{2}-1}|f'(\varphi(z))|\frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| \\
&\le \sup_{z\in D}(1-|z|^2)\,|f(z)|^{\frac{p}{2}-1}\,|f'(z)|, \qquad (\forall f\in\mathcal{B}^p_1).
\end{aligned}$$

Using Lemma 3, Lemma F and the estimate (1), we have the following main result that generalizes Corollary 2.
Theorem 4. If $0 < q \le p$, then $C_{\varphi}: H^p \to L^q{}_a$ is bounded, while there is no symbol φ such that it is bounded below.
Proof. Since $0 < q \le p$, the boundedness of $C_{\varphi}: H^p \to L^q{}_a$ follows from
Holder's inequality. Let $N = \{ f \in H(D), f(z) \ne 0\ (\forall z \in D) \}$. Suppose that

$$f\in\mathcal{B}^p_{1/2}\cap N$$

Then for $a \in D$, we choose some constant c_1 such that

$$\textstyle\int_0^z (f^{\frac{p}{2}}(\zeta))'\varphi_a'(\zeta)d\zeta + c_1 \ne 0$$

For every $z \in D$. Put

$$F(z)=\left\{\int_0^z\left(f^{\frac{p}{2}}(\zeta)\right)'\varphi_a'(\zeta)d\zeta+c_1\right\}^{\frac{2}{p}}=\left\{\frac{p}{2}\int_0^z f^{\frac{p}{2}-1}(\zeta)f'(\zeta)\varphi_a'(\zeta)d\zeta+c_1\right\}^{\frac{2}{p}}.$$

Then we have $F(z) \in H^p$.

In fact, for $f\in\mathcal{B}^p_{1/2}\cap N$ –,

$$\begin{aligned}
&\left\{\frac{p^2}{4}\int_D(1-|z|^2)\,|F(z)|^{p-2}\,|F'(z)|^2\,dA(z)\right\}^{\frac{1}{p}} \\
&= \left\{\int_D(1-|z|^2)\left|\left(F(z)^{\frac{p}{2}}\right)'\right|^2 dA(z)\right\}^{\frac{1}{p}} \\
&= \left\{\int_D\left|\left(f(z)^{\frac{p}{2}}\right)'\right|^2|\varphi_a'(z)|^2(1-|z|^2)dA(z)\right\}^{\frac{1}{p}} \\
&\le \left\{\sup_{a\in D}\left|\left(f(z)^{\frac{p}{2}}\right)'\right|(1-|z|^2)^{\frac{1}{2}}\right\}^{\frac{2}{p}}\left\{\int_D|\varphi_a'(z)|^2dA(z)\right\}^{\frac{1}{p}}
\end{aligned}$$

Using the evaluation (1), it holds that there exists a constant $K > 0$ such that $\int_D|\varphi_a'(z)|^2dA(z)=\int_D\frac{(1-|a|^2)^2}{|1-\overline{a}z|^4}dA(z)\le K<+\infty.$ (2) Hence $F(z)\in H^p$. Since $\varphi_a' \in L^2_a$, using the boundedness of the composition operator C_φ on $L^2{}_a$, there is a positive $C > 0$ such that

$$\int_D|\varphi_a'(\varphi(z))|^2dA(z)\le C\int_D|\varphi_a'(z)|^2dA(z)=\int_D\frac{(1-|a|^2)^2}{|1-\overline{a}z|^4}dA(z). \tag{3}$$

In the case of $p>q$, suppose that there exists a symbol φ such that $C_{\varphi}:$
$H^p \to L^q{}_a$ is bounded below. Since $p/q > 1$, using H¨older's inequality and Lemma
F, for any $a \in D$, there exists a constant $K > 0$ such that

$$\left\{(1-|a|^2)\left|\left(f^{\frac{p}{2}}\right)'(a)\right|^2\right\}^{\frac{1}{p}}$$
$$\leq K\left\{\int_D(1-|z|^2)^{-1}\left|\left(f^{\frac{p}{2}}\right)'(z)\right|^2(1-|\varphi_a(z)|^2)^2dA(z)\right\}^{\frac{1}{p}}$$
$$=\left\{\int_D(1-|z|^2)\left|\left(F^{\frac{p}{2}}(z)\right)'\right|^2dA(z)\right\}^{\frac{1}{p}}$$
$$=\left\{\frac{p^2}{4}\int_D|F(z)|^{p-2}|F'(z)|^2(1-|z|^2)dA(z)\right\}^{\frac{1}{p}}$$
$$\leq K\left\{\int_D|C_\varphi F(z)|^{q-2}|(C_\varphi F)'(z)|^2(1-|z|^2)^2dA(z)\right\}^{\frac{1}{q}}$$
$$\approx K\left\{\int_D\left|\left((C_\varphi F)^{\frac{q}{2}}(z)\right)'\right|^2(1-|z|^2)^2dA(z)\right\}^{\frac{1}{q}}$$
$$\approx K\left\{\int_D\left|C_\varphi F^{\frac{q}{2}}(z)\right|^2dA(z)\right\}^{\frac{1}{q}}$$
$$=K\left\{\int_D|C_\varphi F(z)|^q\,dA(z)\right\}^{\frac{1}{q}}$$
$$\leq K\left\{\int_D|C_\varphi F(z)|^p\,dA(z)\right\}^{\frac{1}{p}}\left\{\int_D1^{\frac{pq}{p-q}}dA(z)\right\}^{\frac{p-q}{pq}}$$
$$\approx K\left\{\int_D(1-|z|^2)^2\left|\left(C_\varphi F^{\frac{p}{2}}(z)\right)'\right|^2dA(z)\right\}^{\frac{1}{p}}$$
$$=K\left\{\int_D(1-|z|^2)^2\left|\left((C_\varphi f(z))^{\frac{p}{2}}\right)'\right|^2|\varphi_a'(\varphi(z))|^2\,dA(z)\right\}^{\frac{1}{p}}$$
$$\leq K\left\{\sup_{z\in D}\left|\left((C_\varphi f(z))^{\frac{p}{2}}\right)'\right|(1-|z|^2)\right\}^{\frac{2}{p}}\left\{\int_D|\varphi_a'(\varphi(z))|^2dA(z)\right\}^{\frac{1}{p}}.$$

Using (2) and (3), we have

$$\sup_{z\in D}\left|\left((f(z))^{\frac{p}{2}}\right)'\right|(1-|z|^2)^{\frac{1}{2}}\leq C\sup_{z\in D}\left|\left((C_\varphi f(z))^{\frac{p}{2}}\right)'\right|(1-|z|^2)\quad(\forall f\in\mathcal{B}^p_{1/2}\cap N).\ (4.1)$$

In the case of p = q, without using Holder's inequality we can also prove this inequality as well.
Here for w ∈ D, we consider the following test function:

$$f_w(z)=\left(\frac{1}{\overline{w}}\right)^{\frac{2}{p}}\left(1-|w|^2\right)^{\frac{2}{p}}\frac{1}{(1-\overline{w}z)^{\frac{1}{p}}},\quad z\in D.$$

Then we see that $f_w\in\mathcal{B}^p_{1/2}$. In fact, for w ∈ D, the following holds.

$$\sup_{z\in D}(1-|z|^2)^{\frac{1}{2}}\left|\left((f_w(z))^{\frac{p}{2}}\right)'\right|$$
$$=\frac{1}{2}\sup_{z\in D}(1-|z|^2)^{\frac{1}{2}}(1-|w|^2)\left|\frac{1}{1-\overline{w}z}\right|^{\frac{3}{2}}$$
$$\leq\frac{1}{2}\sup_{z\in D}(1-|z|^2)^{\frac{1}{2}}(1-|w|^2)\left(\frac{1}{1-|w||z|}\right)^{\frac{3}{2}}$$
$$\leq\frac{1}{2}\sup_{z\in D}(1-|z|^2)^{\frac{1}{2}}(1-|w|^2)\left(\frac{1}{1-|w|}\right)\left(\frac{1}{1-|z|}\right)^{\frac{1}{2}}\leq 2^{\frac{1}{2}}<\infty.$$

Thus $f_w \in \mathcal{B}^p_{1/2}$.

On the other hand, we have

$$\sup_{z\in D}(1-|z|^2)^{\frac{1}{2}}\left|\left((f_w(z))^{\frac{p}{2}}\right)'\right|$$
$$=\frac{1}{2}\sup_{z\in D}(1-|z|^2)^{\frac{1}{2}}(1-|w|^2)\left|\frac{1}{1-\overline{w}z}\right|^{\frac{3}{2}}$$
$$\geq\frac{1}{2}\qquad(4.2).$$

Here, let $w_n \to \partial D$. By using Lemma 3, applying $w = w_n(\in D)$ $(n =1,2,\cdots)$,then

$$\sup_{z\in D}(1-|z|^2)\left|\left((f_{w_n}\circ\varphi(z))^{\frac{p}{2}}\right)'\right|$$
$$=\frac{p}{2}\sup_{z\in D}(1-|z|^2)\,|f_{w_n}\circ\varphi(z)|^{\frac{p}{2}-1}\,|(f_{w_n}\circ\varphi(z))'|$$
$$\leq\frac{p}{2}\sup_{z\in D}(1-|z|^2)\,|f_{w_n}(z)|^{\frac{p}{2}-1}\,|(f_{w_n}(z))'|$$
$$=\sup_{z\in D}(1-|z|^2)\left|\left((f_{w_n}(z))^{\frac{p}{2}}\right)'\right|$$
$$=\frac{1}{2}\sup_{z\in D}(1-|z|^2)(1-|w_n|^2)\left|\frac{1}{1-\overline{w_n}z}\right|^{\frac{3}{2}}$$
$$\leq\frac{1}{2}\sup_{z\in D}(1-|z|^2)(1-|w_n|^2)\left(\frac{1}{1-|w_n||z|}\right)^{\frac{3}{2}}$$
$$\leq\frac{1}{2}\sup_{z\in D}(1-|z|^2)(1-|w_n|^2)\left(\frac{1}{1-|w_n|}\right)^{\frac{1}{2}}\left(\frac{1}{1-|z|}\right)$$
$$\leq 2(1-|w_n|)^{\frac{1}{2}}\to 0\quad(|w_n|\to 1^-).\qquad(4.3)$$

Considering (4.3), it contradicts that inequalities (4.1) and (4.2) hold. Hence in the case of $0 < q \leq p$, there don't exist symbol φ such that $C_{\varphi}: H^p \to L^q_a$ is bounded below.

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Conclusion

This paper has delved into the intricate realm of composition operators on various function spaces over the open unit disk. Through rigorous analysis and leveraging established theorems, the boundedness and bounded below properties of these operators have been explored, shedding light on fundamental aspects of complex analysis. Notably, the results obtained contribute to a deeper understanding of operator behavior within the context of Hardy and Bergman spaces, Besov spaces, and Bloch type spaces. The findings presented here enrich the existing body of knowledge in the field and pave the way for further investigations into the interplay between function spaces and composition operators.

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