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Asymptotic behavior of solutions of biharmonic Stefan's problem

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Abstract
In this article we have considered Stefan's problem in one dimensional situation for biharmonic operator. Stefan's problem is very important from the point of view of the study of free boundary value problems and spreading of solutions. Blow up in finite time and spreading of solutions are still less understood phenomena and it becomes necessary to look for and analyze models which offer similar behavior but are significantly different from the traditional Stefan's problem. The elliptic part of Stefan's problem for biharmonic case corresponds to Steklov boundary value problem. In general, Green's function and Neuman's functions change sign on domain of interest. But, for Steklov boundary conditions these fundamental solutions do not change sign and this property is used to prove asymptotic behavior of the free boundary for the biharmonic free boundary. We have proved that free boundary for Stefan's problem corresponding to biharmonic operator grows like $t^{1-\delta}$ for $\delta = \frac{1}{4}$.

Keywords: Stefan's problem, biharmonic operator, Steklov boundary conditions, free boundaries

Introduction
Free boundary value problems arise in the mathematical description of a variety of physical phenomena. They also appear in the theory minimal surfaces and are known as obstacle problems. They are the boundary value problems where the determination of the boundary itself is a part of the solution to be determined. In free boundary value problems, the variable of interest changes its behavior discontinuously across the boundary. Among several examples of free boundary problems, we list a few.

1. Solidification and crystallization of substance when temperature crosses certain threshold value.
2. Dynamics of contact lines and phase transition.
3. Internal gravity waves.

In the present article we want to study asymptotic analysis of solutions of free boundary problems for the bi-harmonic Stefan's problem in one space variable. Stefan's problem is one of the most studied free boundary value problems and interest in it stems from the study of how phases evolve during evolution of, say for example, melting of ice. One of the most studied models for fourth order diffusion is thin film equation. For non-negative finite mass on appropriate domain self-similarity solutions of the thin film equation adequately explain finite time blow up of solutions as well as self-similar spreading. The blow-up dynamics of the it is governed by the interactions between non-linear second order and fourth order terms. One of the approaches to deal with the thin film equation is to study monotone dissipation of its energy functional. One of the major sources of difficulties for the free boundary problems is that the energy functional is not non-negative and changes sign on the domain of interest and hence finite time blow up and infinite time spreading in its dynamics. Therefore, it becomes of some interest to study positive solutions of fourth order diffusion equations to illuminate the dynamics and asymptotic analysis of the solutions as time becomes infinite.

The following problem studied by Bertozzi is worthy of mention here is fourth order diffusion equation involving Laplace-Beltrami operator Δ_S on the surface S.

$$\begin{aligned} u_t &= -\Delta_S \Delta_S u, \\ u(x,0) &= u_0. \end{aligned} \tag{1}$$

Energy functional corresponding to the above diffusion equation is given as follows.

$$\mathrm{E}(u) = \frac{1}{2}\int_S |\Delta_s u|^2 \tag{2}$$

Solutions of parabolic partial differential equations generally depend on the properties of solutions of boundary value problems for corresponding elliptic partial differential equations. It is known that solutions of poly-harmonic partial differential equations do not preserve positivity for positive data. But there are few instances like Steklov boundary value problems for bi-harmonic partial differential equations where non-negativity of solutions for non-negative data is guaranteed owing to non-negative nature of the corresponding Green's function. Bernis *et al*. have considered asymptotic analysis of degenerate parabolic equations. On the lines of these authors, we have taken up the study of asymptotic analysis of solutions of parabolic partial differential equations corresponding to Steklov boundary value problem for which non-negativity of the solution for the non-negative data is guaranteed.

In the present article we have tried to analyze free boundary problems for the fourth order diffusion equation for one space variable which generalizes to Stefan's problem for biharmonic partial differential equation. Boundary value problems involving Laplace operator can be generalized in two ways. One way is to consider p −Laplacian operators and other is by considering polyharmonic operators. We have taken up the study of boundary value problems which involve polyharmonic operators especially because they are in some sense limits of thin film equations without lower order perturbations. The idea is that solutions of thin film equations without lower order perturbations converge to obstacle problem for initial boundary value problem for biharmonic partial differential equations. One of the important topics of research in the analysis of partial differential equations is to study these limit processes in various function spaces. One of the sources of difficulty in this study is understanding the dominance of positivity in the solutions of boundary value problems for poly-harmonic equations.

Main Theorem

We consider the following problem which is followed by main theorem of this article.

$$\begin{gathered} u_{xxxx} = u_t, 0 < x < s(t), t > 0, \\ (u_{xx})_x(0,t) = f(t), f(t) < 0, t > 0, \\ u(x,0) = \phi(x), \phi(x) \geq 0, 0 \leq x \leq b, \phi(b) = 0, \\ u_{xx}(x,0) = \psi(x), \psi'(x) \geq 0, 0 \leq x \leq b, \\ u(s(t),t) = 0, t > 0, \\ (u_{xx})_x(s(t),t) = -\frac{ds(t)}{dt}, t > 0. \end{gathered} \tag{3}$$

Theorem 1. Let $\mathrm{u}(\mathrm{x},\mathrm{t})$ and $\mathrm{s}(\mathrm{t})$ be the solution of the free boundary problem for (3). If $\lim_{t\to\infty} t^{\delta} f(t) = -\gamma$ for some $\gamma > 0$ and $\frac{1}{4} < \delta < 1$, then $\lim_{t\to\infty} \frac{s(t)}{t^{1-\delta}} = \frac{\gamma}{1-\delta}$

Proof. On integrating first equation in (3) with respect to x and t between the limits $0 \leq x \leq s(t)$ and 0 to t respectively we get,

$$\int_0^t \int_0^{s(t)} u_{xxxx}\, dx\, dt - \int_0^{s(t)} \int_0^t u_t\, dt\, dx = 0.$$

$$\int_0^t \big((u_{xx})_x(s(t),t) - (u_{xx})_x(0,t)\big)\, dt - \int_0^{s(t)} \big(u(x,t) - u(x,0)\big)\, dx = 0$$

$$\int_0^t \left(-\frac{ds}{dt} - \psi'(x)\right) dt - \int_0^{s(t)} \big(u(x,t) - u(x,0)\big)\, dx = 0$$

which becomes

$$-s(t) + s(0) - \int_0^t \psi'(x)\, dt - \int_0^{s(t)} u(x,t)\, dx + \int_0^{s(0)} \phi(x)\, dx = 0 \tag{4}$$

Using $s(0) = b$ we get

$$-s(t) + b - \int_0^t \psi'(x)\, dt - \int_0^{s(t)} u(x,t)\, dx + \int_0^b \phi(x)\, dx = 0 \tag{5}$$

Denote by $I(t)$ the integral $\int_0^{s(t)} u(x,t)\, dx$. Since this problem corresponds to Steklov boundary value problem for biharmonic operator, we have $u(x,t) \geq 0$ and therefore we have the following important estimate

$I(t) \geq 0$. (6)

$$\begin{gathered} w_{xxxx} = w_t, 0 < x < s(t), t > 0. \\ w(x,0) = \Phi(x), 0 < x < \infty, \\ (w_{xx})_x(0,t) = f(t) - \varepsilon t > 0, \varepsilon > 0. \end{gathered}$$

Where $\Phi(x)$ is $\phi(x)$ extended by 0 beyond the interval $0 \leq x \leq b$. Solution of the above problem is given by

$$w_\varepsilon(x,t) = -\int_0^t (f(\tau) - \varepsilon)\, N(x,t;0,\tau)\, d\tau + \int_0^b \Phi(\xi) N(x,t;\xi,0)\, d\xi$$

Where N is a Neumann function for the underlying problem which is non-negative (see for example (Sweers 1991)) and therefore we have $(w_\varepsilon)_x \geq 0$ and hence $w_\varepsilon(x,t) \geq 0$ which in particular implies $w_\varepsilon(s(t),t) \geq 0$.

Consider the function $w = w_\varepsilon - u$ in the interval $0 \leq x \leq s(t)$ and $0 \leq t \leq T$ for any $T > 0$. On $x = s(t)$ we have $w \geq 0$ and when $t = 0$ we have $w = 0$. Thus, we have $(w_{xx})_x = -\varepsilon < 0$ and therefore $w \geq 0$. We get $u(x,t) \leq w(x,t), \forall 0 \leq x \leq s(t)$. We have

$$I(t) \leq s(t)\left(\int_0^t |f(\tau)|\, N(x,t;0,\tau)\, d\tau + \int_0^b \Phi(\xi) N(x,t;\xi,0)\, d\xi\right)$$

Note that $N(x,t;\xi,\tau) \leq (t-\tau)^{-1/4}$ (Friedman 1992)which implies

$$I(t) \leq s(t)\left(\int_0^t |f(\tau)|\,(t-\tau)^{1/4}\, d\tau + A(t+1)^{-1/4}\right) \tag{7}$$

for certain constant A which depends on ϕ. Using equation (6) and (7) in (5) we get the equations stated in the theorem. ☐

Conclusion

The study of the asymptotic behavior of solutions of evolution equations is important mainly because we are interested in identifying if the solution of evolution equation goes to steady state solution. This steady state solution is a solution of a corresponding boundary value problem for elliptic partial differential equation. Several instances are known where instead of stabilizing into steady state, these solutions blow up in finite time and exhibit spreading of the solution and expanding contact lines or surfaces which are free boundaries. This peculiar behavior is essentially attributed to non-positivity of solutions even for positive data. Stefan's problem corresponds to Steklov boundary conditions and hence maintain nonnegativity of solutions for nonnegative data. For general boundary conditions we are led to consider obstacle problems which are not well understood for poly-harmonic operators. In this article we have considered Stefan's problem of fourth order diffusion in one space dimension which generalizes to Stefan's problem biharmonic operators in higher space dimensions. Though solutions of biharmonic equations in general change sign for general boundary conditions but they maintain positivity of the solution for positive data for Steklov boundary conditions. The conclusion of the theorem given in this article states that the support of the solution increases as $t^{1-\delta}$ as solution evolves in time and we see spreading of the solution in the biharmonic case.

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