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Certain new subclasses of multivalent meromorphic function

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Abstract
In the present paper, we will introduce and study some new classes of analytic functions using the technique of convolution, certain liner operators will be defined all classes of analytic function. The main object of the present paper is to introduce the certain subclasses of Meromorphic multivalent functions. We obtain some interesting geometric properties of Meromorphic multivalent functions according to coefficient inequality.

Keywords: Univalent, meromorphic, star like, convex, convolution

Introduction

The Riemann mapping theorem in complex analysis gave a new idea to study the complex functions. In which the geometry of complex functions is studied and this new branch called Geometric Function Theory. Latter in 1907 P. Koebe initiate works for univalent function. Other geometric properties like starlike function, convex function, close-to-convex function, spiral like function and multivalent function were defined and studied initially by Robertson, Lowner, Kaplan, Spacek, Libera. Latter many other authors contributed their works in the field of geometric function theory. In the analytic function theory, there have been many differential implications in which characterization of the functions is determined by a differential condition. If $f(z)$ is analytic in a convex domain D and $R\{f'(z)\} > 0$ then f is univalent in D. Most of the special classes of functions for which Bieberbach's conjecture is known to hold depends on the image domain having a specific geometric characterization which can then be expressed in the analytic form. One of the earliest results in this direction concerns the star like functions. A single valued function $f(z)$ is said to be univalent or Schlicht in a domain $D \subset C$ (the complex plane), if it never takes on the same values twice, that is, if $f(z_1) = f(z_2)$ for $z_1, z_2 \in D$ implies that $z_1 = z_2$. In the other word F is one to one on D. In the case the equation $f(z) =$ w has at most one root in D for any complex number w. Let A denote the class of function $f(z)$ normalized by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ Which are analytic in the open unit disk $U = \{z: z \in C, and\ |z| < 1\}$. Also let S denote the subclass of A consisting of function which are univalent in U. Ludwig Bieberbach (1896-1982) in 1916 conjectured one of the most outstanding coefficient estimated in the theory of class S. Bieberbach, actually proved that $f(z) \in s \Rightarrow |a_2| \leq 2$ and further conjectured the general result $f(z) \in s \Rightarrow |a_n| \leq n, n = 2,3, \ldots$ Where the equality holds true only if $f(z)$ is the Koebe function, k $(z) = z(1-z)^{-2} = \sum_{n=1}^{\infty} nz^n$. A set E is said to be Starlike with respect to $\omega_0 \in E$, if the line segment joining ω_0 to every other point $\omega \in E$, lies entirely in E.

The function $f(z)$ is said to be starlike with respect to ω_0, if $f(z)$ maps a domain $D \subseteq C$ onto a domain that is starlike with respect to ω_0. A function $f(z)$ in S is said to be starlike, if $f(z)$ maps D onto a domain with respect to the origin, i.e. the line segment joining $\omega = 0$ to any point of f lies in $f(D)$. The class of such functions is denoted by S^*. A function $f(z)$ in S is said to be starlike of order α $(0 \leq \alpha < 1)$, if and only if $\Re e\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, (z \in U)$.

The class of such functions is denoted by S* (α), (0 ≤ α < 1). We observe, S* (α) ⊆ S* (0) ≡ S*. Thus f (z) ∈ S*, if and only if $\Re e\left\{\frac{zf'(z)}{f(z)}\right\} > 0, (z \in U)$.

The set *E* is said to be convex, if the line segment joining any two points of *E* lies entirely in *E*. This is equivalent to saying that $tz_1 + (1\text{-}t)\, z_2 \in E$ for every $z_1, z_2 \in E$, 0 ≤ t < 1. A function in *S* is said to be convex of order (α) (0 ≤ α < 1), if any only if $\Re e\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, (z \in U)$. *K* (α) denotes such class of convex functions. It is observed that *K* (α) ⊆ *K* (0) ≡ *K*, where *K* being the class of convex functions with respect to origin in *U*. This means that *f* ∈ *K*, if any only if $\Re e\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, (z \in U)$.

Some Related Definitions

Definition 2.1 A function $f(z) \in A$ is said to be close-to-convex in the unit disk *U*, if there exists a convex function *g* (*z*) such that $\Re e\left\{\frac{f'(z)}{g'(z)}\right\} > 0, (z \in U)$, and the class of all such functions is denoted by ***C***. Using Noshiro-Warschawski result, it can be proved that every close-to-convex function is univalent. It is easy to see from the definition that every convex and starlike function is close-to-convex. Thus, we infer that $\kappa \subset S* \subset C \subset S$.

Definition 2.2 A function $f(z) \in A$ is said to be close-to-convex of orderα $(0 \leq \alpha < 1)$, if there exists a convex function *g* (*z*) such that $\Re e\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha\ (0 \leq \alpha < 1; z \in U)$, We denote the class of all such close-to-convex functions of order α $(0 \leq \alpha < 1)$ by $C\ \ (\alpha)$. It has been established by Reade that if $f(z) \in A$ is of the form then$|a_n| \leq n\ (n \in N\{1\})$.

Definitions 2.3 A function $f(z) \in A$ is said to be a λ-spiral like function in *U*, if

$\Re e\left\{e^{\iota\lambda}\frac{zf'(z)}{f(z)}\right\} > 0, \left(|\lambda| < \frac{\pi}{2}\right)$. We denote the class of all such λ-spirallike functions by $Sp(\lambda)$. It was also shown by Spacek that λ-spirallike functions are always univalent in the unit disk *U*.

Definitions 2.4 A function $f(z) \in A$ is said to be a λ-spirallike function of order α $(0 \leq \alpha < 1)$ in *U*, if $\Re e\left\{e^{\iota\lambda}\frac{zf'(z)}{f(z)}\right\} > \alpha, \left(0 \leq \alpha < 1, |\lambda| < \frac{\pi}{2}\right)$. We denote the class of all such type of functions by *Sp* (λ, α) which was introduced by Libera [18].

Definitions 2.5 A function $f(z) \in A$ is said to be a Robertson function of order $\alpha, (0 \leq \alpha < 1)$ in *U*, if $\Re e\left\{e^{\iota\lambda}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \alpha, \left(0 \leq \alpha < 1, |\lambda| < \frac{\pi}{2}\right)$. We denote the class of all such functions by *M* (λ, α) which were studied by Robertson. When $\lambda = 0$, then $M(0, \alpha) = \kappa(\alpha)$.

Main Results

Let Σ^{m+1} be the class of function *f* of the form $f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$. Which are analytic in the punctured unit disk $E* = \{z : z \in C \quad and \quad 0 \leq |z| < 1\} = U \setminus \{0\}$. If *f* and *g* are analytic in E = E ∪ {0}, we say that *f* is subordinate to *g*, written $f\ \pi\ g$ or $f(z)\ \pi\ g(z).$ If there exists a Schwarz function ω in E such that $f(z) = g(\omega(z))$ and $|\omega(z)| < 1.$ For p = 1, the integral operator reduces into the following operator:

$$L^m(\lambda, \lambda) f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left[\frac{\lambda}{\lambda + \lambda(1+k)}\right]^m a_k z^k \tag{1}$$

$$(\lambda > 0, \lambda \geq 0, m \in N_0; z \in E*)$$

$$\lambda(z)\left(L^{m+1}(\lambda, \ell) f(z)\right)' = \ell L^m(\lambda, \ell) f(z) - (\ell + \lambda) L^{m+1}(\lambda, \ell) f(z)\ (\lambda > 0) \tag{2}$$

Now in this article, we introduce some new class of Meromorphic function using the operator $\left(L^{m+1}(\lambda, \lambda)\right)$. To establish our main results we need the following Lemma.

Lemma 3.1 [10, 11]**:** Let the function *h* (*z*) be analytic and convex (univalent) in *E* with *h* (0) = 1, suppose also that the function ϕ (*z*) given by

$\phi(z) = 1 + c_1 z + c_2 z^2 + \dots$ is analytic in E. If $\varphi(z) + \frac{z\varphi'(z)}{\gamma} \prec h(z), \{z \in E, \Re e\{\gamma\} \geq 0, \gamma \neq 0\}$Then,$\varphi(z) \prec \phi(z) = \frac{\gamma}{z^\gamma}\int_0^z t^{\gamma-1} h(t)\, dt \prec h(z), \{z \in E\}$.

Theorem 3.2 Let μ > 0, α ≥ 0, -1 ≤ B ≤ 1, A ∈ ℜe, $f(z) \in MB(\alpha,\lambda,\lambda,\mu,A,B)$ then $\left(zL^{m+1}(\lambda,\lambda) f(z)\right)^{\mu} \prec \frac{\mu\lambda}{\alpha\lambda}\int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\mu\lambda}{\alpha\lambda}-1} du \prec \frac{1+Az}{1+Bz}$.

Proof: Consider the function ϕ (z) defined by $\left(zL^{m+1}(\lambda,\lambda) f(z)\right)^{\mu} = \phi(z), \quad (z \in E)$ Now ϕ (z) is analytic in E with ϕ (0) = 1. Differentiating with respect to *z* after taking logarithm

$$\mu \log z + \mu \log L^{m+1}(\lambda,\lambda) f(z) = \log \phi(z)$$

$$\frac{\mu}{z} + \mu \frac{\left(L^{m+1}(\lambda,\lambda) f(z)\right)'}{L^{m+1}(\lambda,\lambda) f(z)} = \frac{\phi'(z)}{\phi(z)} \tag{3}$$

Multiplying by λz and using recurrence relation, we have

$$\mu\lambda + \mu \frac{\left[\lambda L^{m}(\lambda,\lambda) f(z) - (\lambda+\lambda)(L^{m+1}(\lambda,\lambda) f(z))\right]}{L^{m+1}(\lambda,\lambda) f(z)} = \lambda z \frac{\phi'(z)}{\phi(z)} \quad \lambda(z)\frac{\phi'(z)}{\phi(z)} = -\mu\lambda + \frac{\mu\lambda L^{m}(\lambda,\lambda) f(z)}{L^{m+1}(\lambda,\lambda) f(z)} \tag{4}$$

Equation (1) and (2) gives us

$$\phi(z) + \frac{\alpha\lambda}{\mu\lambda} z\phi'(z) = (1-\alpha)\left(zL^{m+1}(\lambda,\lambda) f(z)\right)^{\mu} + \alpha\left(zL^{m}(\lambda,\lambda) f(z)\right)\left(zL^{m+1}(\lambda,\lambda) f(z)\right)^{\mu-1}$$

Since *f* (*z*) belonging to the class $MB(\alpha,\lambda,\lambda,\mu,A,B)$ so

$$\phi(z) + \frac{\alpha\lambda}{\mu\lambda} z\phi'(z) \prec \frac{1+Az}{1+Bz}$$

On using the Lemma (3.1), with $\gamma = \frac{\mu\lambda}{\alpha\lambda}$

$$\phi(z) \prec \psi(z) = \frac{\mu\lambda}{\alpha\lambda} z^{-\frac{\mu\lambda}{\alpha\lambda}} \int_0^z t^{\frac{\mu\lambda}{\alpha\lambda}-1} dt$$

Or

$$\phi(z) \prec \psi(z) = \frac{\mu\lambda}{\alpha\lambda}\int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\mu\lambda}{\alpha\lambda}-1} du \prec \frac{1+Az}{1+Bz} \tag{5}$$

Corollary 3.3: Let μ > 0, α ≥ 0, ρ ≠ 1. If

$$(1-\alpha)\left(zL^{m+1}(\lambda,\lambda) f(z)\right)^{\mu} + \alpha\left(zL^{m}(\lambda,\lambda) f(z)\right)\left(zL^{m+1}(\lambda,\lambda) f(z)\right)^{\mu-1} \prec \frac{1+(1-2\rho)}{1-z} z \tag{6}$$

z ∈ E then $\left(zL^{m+1}(\lambda,\lambda) f(z)\right)^{\mu} \prec \rho + \frac{\mu\lambda}{\alpha\lambda}(1-\rho)\int_0^1 \left(\frac{1+zu}{1-zu}\right) u^{\frac{\mu\lambda}{\alpha\lambda}-1} du$.

Proof: From theorem (3.2)

$$(1-\alpha)\left(zL^{m+1}(\lambda,\lambda)f(z)\right)^{\mu}+\alpha\left(zL^{m}(\lambda,\lambda)f(z)\right)\left(zL^{m+1}(\lambda,\lambda)f(z)\right)^{\mu-1}\ \pi\ \frac{1+Az}{1+Bz}\ zL^{m+1}(\lambda,\lambda)f(z)\ \pi\ \frac{1+Az}{1+Bz}$$

Now, take A = 1 - 2ρ, B = -1

$$(1-\alpha)\left(zL^{m+1}(\lambda,\ell)f(z)\right)^{\mu}+\alpha\left(zL^{m}(\lambda,\ell)f(z)\right)\left(zL^{m+1}(\lambda,\ell)f(z)\right)^{\mu-1}\prec\frac{1+(1-2\rho)z}{1-z}$$

$$=\frac{\mu\lambda}{\alpha\lambda}\int_0^1\frac{1+(1-2\rho)zu}{1-zu}u^{\frac{\mu\lambda}{\alpha\lambda}-1}du=\frac{\mu\lambda}{\alpha\lambda}\int_0^1\frac{(1-\rho)(1+zu)+\rho(1-zu)}{1-zu}u^{\frac{\mu\lambda}{\alpha\lambda}-1}du=\frac{\mu\ell}{\alpha\lambda}\int_0^1\left[\rho+(1-\rho)\left\{\frac{(1+zu)}{(1-zu)}\right\}\right]u^{\frac{\mu\ell}{\alpha\lambda}-1}du$$

$$=\rho+\frac{\mu\ell}{\alpha\lambda}(1-\rho)\int_0^1\left(\frac{1+zu}{1-zu}\right)u^{\frac{\mu\ell}{\alpha\lambda}-1}du,\ (z\in E)$$

Corollary 3.4 Let μ > 0, α ≥ 0 then

$$MB(\alpha,\lambda,\lambda,\mu,A,B)\subset MB(0,\lambda,\lambda,\mu,A,B) \tag{6}$$

Proof: If $f(z)\in MB(\alpha,\lambda,\lambda,\mu,A,B),$ we have

$$\left(zL^{m+1}(\lambda,\lambda)f(z)\right)^{\mu}\ \pi\ \frac{1+Az}{1+Bz} \tag{7}$$

If α = 0 then $f(z)\in MB(0,\lambda,\lambda,\mu,A,B),$ so

$$\left(zL^{m+1}(\lambda,\lambda)f(z)\right)^{\mu}\ \pi\ \frac{1+Az}{1+Bz} \tag{8}$$

From equation (7) and (8), we have

$$MB(\alpha,\lambda,\lambda,\mu,A,B)\subset MB(0,\lambda,\lambda,\mu,A,B)$$

Theorem 3.5: Let $f\in MB(0,\lambda,\lambda,\mu,\rho)$ for z ∈ E then $f\in MB(\alpha,\lambda,\lambda,\mu,\rho)$ for $|z|<R(\alpha,\lambda,\mu)$. Where $R(\alpha,\lambda,\mu)=\dfrac{\lambda\mu}{\alpha+\sqrt{\alpha^2+\lambda^2\mu^2}}$.

Proof: We have the set $\left(zL^{m+1}(\lambda,\lambda)f(z)\right)^{\mu}=(1-\rho)h(z)+\rho\quad(z\in E)$. Now Differentiating with respect to *z*, we get

$$\mu\log z+\mu\log L^{m+1}(\lambda,\lambda)f(z)=\log\left((1-\rho)h(z)+\rho\right)$$

$$\frac{\mu}{z}+\mu\frac{\left(L^{m+1}(\lambda,\lambda)f(z)\right)'}{L^{m+1}(\lambda,\lambda)f(z)}=\frac{(1-\rho)h'(z)}{(1-\rho)h(z)+\rho}$$

Multiplying by λ*z* and using recurrence relation, we have.

$$-\mu\lambda+\mu\lambda\frac{L^{m}(\lambda,\lambda)f(z)}{L^{m+1}(\lambda,\lambda)f(z)}=\frac{\lambda z(1-\rho)h'(z)}{(1-\rho)h(z)+\rho}$$

Multiplying by α and using the definition of *h* (*z*), we have

$$(1-\alpha)\left(zL^{m+1}(\lambda,\lambda)f(z)\right)^{\mu}+\alpha\left(zL^{m}(\lambda,\lambda)f(z)\right)\left(zL^{m+1}(\lambda,\lambda)f(z)\right)^{\mu-1}-\rho=(1-\rho)\left\{h(z)+\frac{\lambda\alpha}{\mu\lambda}zh'(z)\right\}$$

Or

$$\left[(1-\alpha)\left(zL^{m+1}(\lambda,\lambda)f(z)\right)^{\mu}+\alpha\left(zL^{m}(\lambda,\lambda)f(z)\right)\left(zL^{m+1}(\lambda,\lambda)f(z)\right)^{\mu-1}-\rho\right]\frac{1}{1-\rho}$$

$$=\left\{h(z)+\frac{\lambda\alpha}{\mu\lambda}zh'(z)\right\}$$

Using the following known estimate

$$|zh'(z)|\le\frac{2r}{1-r^{2}}\Re e\{h(z)\},\quad |z|=r<1$$

We have

$$\frac{(1-\alpha)\left(zL^{m+1}(\lambda,\lambda)f(z)\right)^{\mu}+\alpha\left(zL^{m}(\lambda,\lambda)f(z)\right)\left(zL^{m+1}(\lambda,\lambda)f(z)\right)^{\mu-1}-\rho}{1-\rho}=\Re e\left\{h(z)+\frac{\lambda\alpha}{\mu\lambda}zh'(z)\right\}$$

$$\ge\Re e\left\{h(z)-\frac{\lambda\alpha}{\mu\lambda}|zh'(z)|\right\}$$

$$\ge\Re e\{h(z)\}\left\{1-\frac{2\alpha\lambda r}{\mu\lambda(1-r^{2})}\right\}\Theta\ kr^{2}+2rk-k<0\quad r\in(\alpha,\beta)$$

The right-hand side of this inequality is positive if $r<\Re e(\alpha,\lambda,\mu)$ where $\Re e(\alpha,\lambda,\mu)$ is given by (3.2). Consequently, it follows from (3.3) that $f\in MB(0,\lambda,\lambda,\mu,\rho)$ for $|z|<\Re e(\alpha,\lambda,\mu)$.

Conclusion

In the research work, we will discuss some new classes of analytic functions using the technique of convolution, certain liner operators will be defined all classes of analytic function. The main object of the present paper is to introduce the certain subclasses of Meromorphic multivalent functions involving. We discuss some interesting geometric properties according to coefficient inequality.

Future Scope

There are several other operators which can be applied on multivalent meromorphic functions. The theorems which are derived in this research article can be applied practically for solving some functional analysis theoretical problems.

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References

1. Robertson MS. On the theory of univalent functions. Ann Math. 1936;37:374-408.
2. Goodman AW. Univalent function Vols. I and II. Tampa, Florida: Mariner Publ. Co.; c1983.
3. Kaplan W. Close-to-convex schlicht functions. Michigan Math J. 1952;1:169-185.
4. Spacek L. Prispevek k teorli funcki prostych. Casopis Pest Mat. 1932;62:12-19.
5. Libera RJ. Some classes of regular univalent functions. Proc Amer Math Soc. 1965;16:755-758.
6. Duren PL. Univalent Functions. Grundlehender Mathematischen wissenchaften 259. New York, Berlin, Heidelberg, Tokyo: Springer-Verlag; c1983.
7. Owa S, Srivastava HM. Univalent and starlike generalized hypergeometric functions. Canad J Math. 1987;39:1057-77.

8. Reade MO. On close-to-convex univalent functions. Michigan Math J. 1955-56;3:12-59.
9. Bernardi SD. Convex and starlike univalent functions. Trans Amer Math Soc. 1969;135:429-446.
10. Duren PL. Harmonic mappings in the plane. Cambridge Tracks in Mathematics, Vol. 156. Cambridge, Mass, USA: Cambridge University Press; c2004.
11. Dziok J, Srivastava HM. Classes of analytic functions associated with the generalized hypergeometric functions. Appl Math Comput. 1993;103:01-13.
12. Dziok J, Srivastava HM. Some subclasses of analytic functions with fixed arguments of coefficients associated with generalized hypergeometric function. Adv Stud Contemp Math. 2002;5:115-125.
13. Dziok J, Srivastava HM. Certain subclasses of analytic functions associated with the generalized hypergeometric functions. Integral Trans Spec Funct. 2003;14:7-18.
14. El-Ashwah RM. Some properties of certain subclasses of meromorphically multivalent functions. Appl Math Comput. 2008;20:824-832.
15. El-Ashwah RM, Aouf MK. Some properties of certain meromor p-valent functions involving extended multiply transformation. Comput Math Appl. 2010;59:2111-2120.
16. Goluzin GM. Geometric theory of functions of a complex variable. Trans Math Monograph Vol. 26. Providence, R. I.: Amer Math Soc; c1969.
17. Graham I, Kohr G. Geometric function theory in one and higher dimensions. New York: Marcel Dekker Inc.; c2003.
18. Hayman WK. Multivalent functions. Cambridge: Cambridge University Press; c1958.
19. Kwon OS, Kim JA, Owa S. Certain subclasses of meromorphically multivalent functions. J Bihar Math Soc. 1996;17:1-8.
20. Littlewood JE. On inequalities in theory of functions. Proc London Math Soc. 1925;23(2):481-519.
21. Liu JL, Srivastava HM. Subclasses of meromorphically multivalent functions associated with a certain linear operator. Math Comput Modelling. 2004;39:35-44.
22. Liu JL, Srivastava HM. A linear operator and associated families of meromorphically multivalent functions. J Math Anal Appl. 2001;259:566-581.