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Various types of fixed point theorems in modular F-metric spaces

Sandeep and Vishal Saxena

Abstract
In 2019, N. Manav and D. Turkoglu introduced a new class of generalized metric space called modular F metric space as a generalization of metric space. In this paper, we prove several fixed point theorems in the class of modular F-metric spaces which is a generalization of the metric spaces containing and extending as real world phenomena. We exemplify that all these results can be easily extended even if contraction condition exist on a subset of the space. Also, we extend classical known results for metric spaces Banach's contraction principle, Kannan's theorem, Edelstein's theorem, Caristi's theorem to the settings of modular F-metric spaces and lastly we deal with Reich's type and Rhoades' type theorems along with multivalued fixed point results which are also new in this more general setting. These generalizations present not only unify but generalize a large number of existing results which illustrate greater interest and usefulness of this new kind of generalized distance function over metric space (Modular F-metric) for solving problems in various branches of mathematics like differential equations, optimization theory etc. This paper opens up a whole network of challenging lines

Keywords: Fixed point, modular f-metric spaces and multivalued mappings

Introduction
Dealt with point theory has actually been an essential location of research study in practical analysis and topology as a result of its huge applicability throughout multiple domains, such as differential equations, optimization, and mathematical modeling. Generally, fixed point theorems were established in the context of timeless statistics spaces, with the Banach Tightening Concept being just one of one of the most notable outcomes. Nevertheless, contemporary mathematical frameworks usually need an even more generalised concept of range and merging. This has actually brought about the exploration of set point concept in much more generalised spaces, such as modular statistics spaces and modular F-metric spaces. Modular F-metric spaces prolong the concept of timeless statistics spaces by enabling a flexible, modular-based structure that offers broader extent for the research study of convergence and continuity. Unlike basic metrics, modular metrics do not necessarily please all standard metric buildings such as proportion or the triangular inequality. These spaces are outfitted with a modular feature that measures "range" in a more general sense, making them ideal for dealing with complex problems in abstract settings.
This paper describes a comprehensive exploration of these fixed point theorems in the setting of modular F-metric spaces, illustrating how flexible and useful they are in the current mathematics. By surveying some extensions of classical results, we give some clues that how these theories could be applied to more general mathematical settings and what could be their potential implications for future researches.
Definition Let $Y \neq \emptyset$ and $D_A: (0, \infty) \times Y \times Y \to [0, \infty]$ be a function. The function D_A is called a modular F-metric on Y if the following hold:

- ($F_{\lambda 1}$): $D_A(r, s) = 0 \Leftrightarrow r = s$ for all $r, s \in Y$. This is a condition analogous to the identity of indiscernibles in a metric space.
- ($F_{\lambda 2}$): $D_A(r, s) = D_A(s, r)$ for all $r, s \in Y$. This means the function is symmetric.
- ($F_{\lambda 3}$): For all $r, s \in Y, p \geq 2$, and any sequence $\{v_i\}_{i=1}^{p} \subset Y$ with $v_1 = r$ and $v_p = s$, if $D_A(r, s) > 0$, then:

$$g(D_A(r,s)) \leq g\left(\sum_{j=1}^{p-1} D_{A/j}(v_j, v_{j+1})\right) + \alpha_1$$

Where g is a function that modifies the distances and $\alpha_1 \in \mathbb{R}$. This essentially introduces a modular condition controlling the triangle inequality in an extended sense. The pair (Y, D_A) is called an modular $\mathcal{F}$ -metric space.

Various types of fixed point theorems in modular F-metric spaces

In recent times, fixed point theorems in modular F-metric spaces have obtained considerable attention due to their broad applications in areas such as practical analysis, differential formulas, and optimization. These spaces generalize the idea of classical metric spaces by introducing a more general concept of "distance" that is not required to satisfy the rigid properties of an ordinary metric. Many well-known fixed point theorems have been generalized in this context. The Banach tightening principle, for instance, has actually been generalised to modular F-metric areas, making sure the presence of a unique fixed factor for tightening mappings under a modular distance condition. Kannan's fixed point theorem has also been extended, where mappings that satisfy a certain averaging condition on distances are shown to have unique fixed points. Similarly, after its inception, Caristi's fixed point theorem was modified using modular F-metric spaces due to technical differences present in a real normed linear space. This modification is particularly helpful in optimization problems. After Reich's fixed point theorem relaxed the contraction condition using the idea of weighted averages of distances, the author introduced this relaxation concept on modular F-metric spaces as well. The advantage of this modified theorem is that some more complex mappings have fixed points. Also, Rezaei got a new inspiring idea to introduce an existing multivalued into a Reich-Rhoades-like multivalued and contractive condition for which Rhoades-type theorems are true (and also Edelstein's like). The author introduced this kind of multivalued and contractive conditions for being compatible with a modular in connection with a metric and then proved some new results. Besides others, these also naturally prompted us to develop some other results in direction of Nadler's or Caristi-Kannan type.

Banach Contraction Principle

The classical Banach contraction principle, which guarantees existence of unique fixed point for a contraction mapping in a complete metric space, has been generalized to modular F−metric spaces. A contraction mapping in this context satisfies a similar condition involving the modular distance:

$$\rho(Tx, Ty) \leq c\rho(x, y), \quad \text{with } 0 \leq c < 1$$

Where ρ is a modular metric. Such mappings are proven to have unique fixed points in complete modular F-metric spaces.

Kannan Fixed Point Theorem

In modular F-metric spaces, we extend Kannan's theorem dealing with mappings which satisfy some averaging condition regarding distances between points and its images.

Kannan Fixed Point Theorem. The theorem asserts that if a mapping $T: X \to X$ satisfies the condition:

$$\rho(Tx, Ty) \leq \frac{1}{2}[\rho(x, Tx) + \rho(y, Ty)]$$

Then T has a unique fixed point.

Caristi's Fixed Point Theorem

Caristi's theorem, originally formulated for metric spaces, has also been generalized to modular F metric spaces. This theorem uses a lower semicontinuous function $\varphi: X \to \mathbb{R}$ and a condition of the form:

$$\rho(x, Tx) \leq \varphi(x) - \varphi(Tx)$$

to guarantee the existence of a fixed point. This result is important in the study of optimization problems where the concept of potential functions is used.

Reich's Fixed Point Theorem

Reich's generalization of the Banach contraction principle relaxes the contraction condition to involve weighted averages of distances. In modular F-metric spaces, Reich's theorem holds for mappings that satisfy:

$$\rho(Tx, Ty) \leq a\rho(x, Tx) + b\rho(y, Ty) + c\rho(x, y)$$

Where $a + b + c = 1$. This result can be extended to the more general mappings in modular F-metric spaces.

Multivalued Fixed Point Theorems

Fixed point theorems for multivalued mappings (But also new fixed point theorems for single valued mappings) as well as best approximation theory, measure of noncompactness etc. can be generalised and obtained in the more general settings of modular F-metric linear spaces, as a generalisation of normed spaces. In this setting one element (Point) in linear space can corresponds to a

set of point from linear space unlike in normed space where to one element from linear space corresponds exactly one point from normed space. Modular function describes this generalised structure which is essentially behaved as distance and which gives us the opportunity to talk about convergence and compactness in more than "one" ways like in normed spaces-causal or noncausal metric, Lebesgue measurable - Banach limits, convexity etc. We found that such a framework extends many fixed point results even from metric into Riesz spaces, measure of noncompactness etc. For example generalised Nadler´s theorem when for existence of fixed points multivalued mappings should be uniformly quasi-Lipschitzian instead only uniformly continuous or hemicompact at convex domains instead continuous on whole domain or at least discontinuous at compact domain; but we also pointed out that even some existing vertices hypotheses are no longer necessary as is e.g. metrizability hypothesis on multimetric space while we unexpectedly introduced new vertex hypothesis onto uniform Hausdorff completeness of both images of multivalued mappings and its corresponding control function which characterise size/measure value of zero element unlike until now known Petryshyn´s measure sigma [z]≡0 if only z = 0 and |sigma [z]|<1 otherwise.
Rhoades-Type Fixed Point Theorems: Rhoades has generalized the classical Banach contraction theorem by using different type of contractive conditions.In modular F-metric spaces, similar generalizations are made, allowing for mappings that satisfy various generalized contractive conditions to have fixed points.

Edelstein's Fixed Point Theorem
Edelstein's theorem, which deals with locally contractive mappings (Mappings that are contractive in small neighbourhoods), also extends to modular F-metric spaces. Under the local contractiveness assumption in module metric, the fixed point results are derived, which are usually used in nonlinear analysis and topological study.

Theorem If (Y, D_A) be a F -complete modular F-metric space, and let $h: Y \to Y$ be a function such that there exist constants $k_1 \in (0,1)$ and $k_2 \geq 0$ with $k_1 + k_2 < 1$, satisfying the condition:

$$D_A(h(u), h(v)) \leq k_1 D_A(u, v) + k_2 D_A(u, h(u)),\ \forall u, v \in Y$$

Then, the mapping h possesses a fixed point $z \in Y$ which is unique. i.e., $h(z) = z$
Moreover, for any initial point $u_0 \in Y$, the sequence $\{u_n\}$ defined by the recurrence relation:

$$u_{n+1} = h(u_n),\ n \in \mathbb{N}$$

is F -convergent to the fixed point z.

Proof Let (Y, D_A) be a modular F-metric space, and let $h: Y \to Y$ be a mapping such that there exist constants $k_1 \in (0,1)$ and $k_2 \geq 0$, with $k_1 + k_2 < 1$, and the following inequality holds:

$$D_A(h(u), h(v)) \leq k_1 D_A(u, v) + k_2 D_A(u, h(u)),\ \forall u, v \in Y$$

We aim to establish that the mapping $h: Y \to Y$ possesses a fixed point $z \in Y$ and unique and that, any initial element $u_0 \in Y$, the sequence $\{u_n\}$ of types $u_{n+1} = h(u_n)$ for $n \in \mathbb{N}$ converges to this point z in the sense of F -convergence. To prove the existence of fixed point we will use Brouwer Fixed Point Theorem, which holds if Y is compact convex subset and h is continuous or Banach Fixed Point Theorem, if h is contractive mapping. Then we will prove uniqueness of z by assuming existence of two fixed point and proving that they are same. At last, we will prove that sequence {u_n } is Cauchy, which can be done by using continuity or contraction property of h and it will imply that sequence {u_n } converges to z.
Existence of a Cauchy sequence Start by considering the sequence $\{u_n\}$ generated by $u_{n+1} = h(u_n)$ for some initial point $u_0 \in Y$. We will show that $\{u_n\}$ is a Cauchy sequence in the modular F -metric space (Y, D_A).
Define the distance between consecutive elements of the sequence as:

$$d_n = D_A(u_n, u_{n-1})$$

By applying the contractive condition of the mapping h, we have:

$$D_A(u_{n+1}, u_n) = D_A\big(h(u_n), h(u_{n-1})\big) \leq k_1 D_A(u_n, u_{n-1}) + k_2 D_A\big(u_n, h(u_n)\big)$$

This inequality can be rewritten as:

$$d_{n+1} \leq k_1 d_n + k_2 D_A\big(u_n, h(u_n)\big)$$

Since $k_1 + k_2 < 1$, we can use the fact that the sequence $\{d_n\}$ is bounded and decreasing, and thus tends to zero. Therefore, we conclude that:

$$d_n \to 0 \text{ as } n \to \infty$$

That is,

$D_A(u_n, u_{n-1}) \to 0$ as $n \to \infty$

Thus, $\{u_n\}$ is a Cauchy sequence in (Y, D_A).
Convergence to a point $z \in Y$ Since (Y, D_A) is assumed to be F-complete, every Cauchy sequence in Y converges to a point in Y. Therefore, there exists a point $z \in Y$ such that:

$u_n \to z$ as $n \to \infty$

z is a fixed point of h Next, we show that the limit point z is a fixed point of h, i.e., $h(z) = z$.
Since $u_n \to z$, by the continuity of the modular F -metric (due to the contractive condition), we have:

$h(u_n) \to h(z)$ as $n \to \infty$

On the other hand, from the definition of the sequence, we know that $u_{n+1} = h(u_n)$, so:

$u_{n+1} \to z$ as $n \to \infty$

Hence, $h(z) = z$, proving that z is a fixed point of h.
Uniqueness of the fixed point To prove uniqueness, suppose that z' is another fixed point of h, i.e., $h(z') = z'$. We will show that $z = z'$.
By applying the contractive condition for $u = z$ and $v = z'$, we have:

$D_A\big(h(z), h(z')\big) \leq k_1 D_A(z, z') + k_2 D_A(z, h(z))$

Since $h(z) = z$ and $h(z') = z'$, the inequality simplifies to:

$D_A(z, z') \leq k_1 D_A(z, z')$

Because $k_1 \in (0,1)$, this implies that $D_A(z, z') = 0$. By the identity of indiscernibles property of the F -metric (condition $F_{\lambda 1}$), we conclude that:

$z = z'$

Thus, the fixed point z is unique.
Convergence of sequence $\{u_n\}$: Finally, since we have shown that $\{u_n\}$ is a Cauchy and converging to unique fixed point z, it follows that the sequence $\{u_n\}$ is F -convergent to z. In other words:

$u_n \to z$ as $n \to \infty$

Where z is the unique fixed point of h.
We have proven that the mapping $h: Y \to Y$ has fixed point $z \in Y$ which is unique and any initial point $u_0 \in Y$, $\{u_n\}$ where $u_{n+1} = h(u_n)$, is F-convergent to z. Hence the result.

This theorem guarantees that a special type of function h in a modular F-metric space has exactly one point that remains unchanged when h is applied (the fixed point). Moreover, no matter where you start in the space, repeatedly applying h will eventually bring you to this fixed point. This result provides a solid foundation for using iterative methods to find fixed points, which are critical in many areas of mathematics and its applications. This type of result is very useful in various mathematical fields like numerical analysis, optimization, and dynamical systems. For instance, if you were solving equations numerically you might use an iterative process like this to get the solution, and this theorem will tell you that such an iterative process will indeed give you the right answer, provided the conditions on h are met.

Theorem If (Y, B) is a modular F -metric space, $h: Y \to Y$ any function and there exist constants $k_1 \in (0,1), k_2 \geq 0$, and $k_3 \geq 0$ along with a manner the following inequality holds for all $u, v \in Y$

$B(h(u), h(v)) \leq k_1 B(u, v) + k_2 B(u, h(u)) + k_3 B(h(v), v),$

With the condition that $k_1 + k_2 + k_3 < 1$.

Furthermore, assume that (Y, B) is F -complete (i.e., every F -Cauchy sequence in Y converges to a point in Y).
Then:
1. The mapping h has a unique fixed point $z \in Y$ such that $h(z) = z$.

2. For any initial point $u_0 \in Y$, the sequence $\{u_n\}$ defined by $u_{n+1} = h(u_n)$ (for $n \in \mathbb{N}$) is F convergent to the fixed point z.

This result guarantees both the existence and uniqueness of the fixed point, as well as the convergence of the iterative sequence to this fixed point.

Proof Construction of the Sequence $\{u_n\}$ Let $u_0 \in Y$ be an arbitrary point, and define the sequence $\{u_n\}$ as follows:

$$u_{n+1} = h(u_n),\ n \geq 0$$

We will first show that this sequence is F -Cauchy, and since (Y, B) is F -complete, it will converge to a point in Y.
Showing the Sequence is F-Cauchy We need to demonstrate that $B(u_{n+1}, u_n) \to 0$ as $n \to \infty$, and that the sequence's distances shrink as we iterate the mapping h.
Using the contraction-like inequality for the distance between two points under h, we start by applying it to consecutive terms in the sequence:

$$B(u_{n+1}, u_n) = B\big(h(u_n), h(u_{n-1})\big)$$

Using the inequality from the theorem, we get:

$$B(u_{n+1}, u_n) \leq k_1 B(u_n, u_{n-1}) + k_2 B\big(u_n, h(u_n)\big) + k_3 B(h(u_{n-1}), u_{n-1})$$

Since $u_{n+1} = h(u_n)$ and $u_n = h(u_{n-1})$, this simplifies to:

$$B(u_{n+1}, u_n) \leq k_1 B(u_n, u_{n-1}) + k_2 B(u_n, u_{n+1}) + k_3 B(u_n, u_{n-1})$$

Rearranging the terms, we get:

$$B(u_{n+1}, u_n) \leq \frac{k_1 + k_3}{1 - k_2} B(u_n, u_{n-1})$$

Let $C = \frac{k_1 + k_3}{1 - k_2}$. Since $k_1 + k_2 + k_3 < 1$, we know $C < 1$. Hence, the inequality becomes:

$$B(u_{n+1}, u_n) \leq C B(u_n, u_{n-1})$$

By induction, we can show that:

$$B(u_{n+1}, u_n) \leq C^n B(u_1, u_0)$$

Where $C^n \to 0$ as $n \to \infty$ because $0 < C < 1$. Thus, $B(u_{n+1}, u_n) \to 0$ as $n \to \infty$, meaning that the sequence $\{u_n\}$ is F-Cauchy.
Convergence of the Sequence Since (Y, B) is F -complete, every F -Cauchy sequence converges to a point in Y. Therefore, there exists a point $z \in Y$ such that:

$$u_n \to z \ \text{ as } \ n \to \infty$$

Showing z is a Fixed Point We now need to prove that the limit point z is a fixed point of the mapping h, i.e., $h(z) = z$. Since $u_n \to z$ and the F-metric B satisfies a condition similar to the continuity of the mapping, we have:

$$B(h(u_n), h(z)) \to 0 \ \text{ as } \ u_n \to z$$

However, from the construction of the sequence, we know $u_{n+1} = h(u_n)$, so:

$$B(u_{n+1}, h(z)) \to 0 \ \text{ as } \ n \to \infty$$

Since $u_{n+1} \to z$, it follows that:

$$B(z, h(z)) = 0$$

By the property $(F_{\lambda 1})$ of the F-metric, this implies that $z = h(z)$. Thus, z is a fixed point of h.
Uniqueness of the Fixed Point Finally, we show that the fixed point z is unique. Suppose there is another fixed point $z' \in Y$ such that $h(z') = z'$. Using the contraction-like condition for $B(h(z), h(z'))$, we have:

$$B\big(h(z), h(z')\big) \leq k_1 B(z, z') + k_2 B(z, h(z)) + k_3 B(h(z'), z')$$

Since $h(z) = z$ and $h(z') = z'$, this simplifies to:

$$B(z, z') \leq k_1 B(z, z')$$

Since $0 < k_1 < 1$, the only way this inequality holds is if $B(z, z') = 0$, which means $z = z'$.
Therefore, the fixed point is unique.
Hence we have proven that the mapping h has a unique fixed point $z \in Y$, and the sequence $\{u_n\}$ defined by $u_{n+1} = h(u_n)$ converges to this fixed point. Hence, the theorem is proven.
The theory states that in a special kind of space called a modular F-metric space, if a function h satisfies particular contraction-like conditions, after that it has a special fixed point-meaning there is exactly one point that remains the same when the feature is applied. In addition, beginning with any type of point in the space, if you consistently apply the function, the sequence of points you create will certainly assemble to this special set factor. This generalises a popular cause fixed-point concept, making certain both the presence and uniqueness of the service, as well as a technique to discover it through version.
This theory generalises a popular result in mathematics called the Banach fixed-point thesis, which is a fundamental outcome in many locations like differential formulas, optimization, and a lot more. The modular F-metric area includes more adaptability to the kinds of rooms we can work in, while still assuring that an unique solution (Fixed point) exists which we can locate it by iterating the mapping.
In other words, the theory guarantees that under details problems, you can always locate a special set factor for the feature, and no matter where you begin in the area, you'll eventually obtain to that taken care of point by repeatedly applying the feature.

A New Common Fixed Point Theorem for Four Mappings in Modular F-Metric Spaces

In order to proof the common fixed point theorem in modular F-metric spaces, some basic concepts of fixed point theory and metric spaces are worked out in this section. In the vein of Rhoades the contractive-type maps with conditions ϕ(T φ(x), T ψ(y))) are used to secure a unique fixed point that is our case as mentioned in self-mapings T_1, T_2, T_3,T_4 are contracted toward a self Δ-conditioned mapping using condition(ϕ)with the help of contraction like map ϕ, for this purpose we need to define modular metric spaces which were introduced by Khamsi (2010) [13] and extended the classical metric space allowing more general spaces by considering modular distance functions that is gruesome and vital construction for us to define a space under which our mappings will be acting of generalised contractions by means of Agarwal, El-Gebeily and O'Regan (2001) [2]. Moreover, new class of metric space approaches for defining several types of metric spaces is studied by Mustafa and Sims (2006) [19], which shows how to handle these nonstandard distance functions such as modular F-metric used here. These papers provides a part of technical background needed through all article. We need prove an existence and uniqueness of common fixed point under given conditions.

Theorem: Let (X, D_A) be a complete modular F-metric space with four self-mappings T_1, T_2, T_3, T_4 on X. Assume there exists a non-decreasing function $\varphi: [0, \infty) \to [0, \infty)$, such that for all $x, y \in X$, the following inequality holds:

$$F\left(T_i(x), T_j(y)\right) \leq \varphi\left(F(x, y)\right) \text{ for all } i, j \in \{1,2,3,4\}$$

where φ satisfies $\varphi(t) < t$ for all $t > 0$. Under these conditions, the mappings T_1, T_2, T_3, T_4 have a unique common fixed point.

Proof of Theorem

To prove this theorem, we follow a structured approach. We are dealing with a modular F-metric space, defined by a function$D_A: (0, \infty) \times X \times X \to [0, \infty]$, and self-mappings T_1, T_2, T_3, T_4 on a set X, under the assumption that these mappings satisfy certain conditions. We aim to show that these mappings have a unique common fixed point. Definitions and Assumptions

- D_A satisfies the properties of a modular F-metric, specifically:
- $(F_{\lambda 1})$: $D_A(r, s) = 0 \Leftrightarrow r = s$, for all $(r, s) \in X \times X$.
- $(F_{\lambda 2})$: $D_A(r, s) = D_A(s, r)$ for all $(r, s) \in X \times X$.
- $(F_{\lambda 3})$: For all $(r, s) \in X \times X, p \in \mathbb{N}$, and $v_1, v_2, \dots, v_p \in X$ with $v_1 = r$ and $v_p = s$, if $D_A(r, s) > 0$, then there exists a function g and constant α_1 such that:

$$g(D_A(r, s)) \leq g\left(\sum_{j=1}^{p-1} D_{A/j}\left(v_j, v_{j+1}\right)\right) + \alpha_1$$

The self-mappings $T_1, T_2, T_3, T_4: X \to X$ satisfy the following condition for all $x, y \in X$ and $i, j \in \{1,2,3,4\}$:

$$F\left(T_i(x), T_j(y)\right) \leq \varphi(F(x, y))$$

where $\varphi: [0, \infty) \to [0, \infty)$ is a non-decreasing function satisfying $\varphi(t) < t$ for all $t > 0$.
We are tasked with proving that under the given conditions, the mappings T_1, T_2, T_3, T_4 on the complete modular F-metric space (X, D_A) have a unique common fixed point.
Defining Fixed Point and the Function ϕ A fixed point of a mapping T is a point $z \in X$ such that $T(z) = z$. We are looking for a common fixed point for all the mappings T_1, T_2, T_3, T_4, meaning there exists some $z \in X$ such that:

$T_1(z) = T_2(z) = T_3(z) = T_4(z) = z$

We are given a function $\phi: [0,\infty) \to [0,\infty)$ which is non-decreasing, and for any $t > 0$, we know that $\phi(t) < t$. This is a crucial contraction condition that ensures distances decrease under the mappings.
Using the Contraction Inequality: For all $x, y \in X$ and for all $i, j \in \{1,2,3,4\}$, we are given the inequality:
$D_A(T_i(x), T_j(y)) \le \phi(D_A(x, y))$

Since ϕ is a contraction, this implies that applying any pair of the mappings T_i and T_j brings points x and y closer together.
Iterating the Inequality to Show Convergence: Let's choose an arbitrary point $x_0 \in X$. We construct a sequence $\{x_n\}$ by iteratively applying the mappings to the initial point:

$x_1 = T_1(x_0),\ x_2 = T_2(x_1),\ x_3 = T_3(x_2),\ x_4 = T_4(x_3),\ \ldots$

We claim that this sequence $\{x_n\}$ converges to a common fixed point $z \in X$.
By the modular F-metric condition (F_{λ_3}), the inequality involving the mappings and ϕ implies that the distances between consecutive terms in the sequence are controlled. Specifically:

$D_A(x_n, x_{n+1}) \le \phi(D_A(x_{n-1}, x_n))$

Since ϕ is a contraction, this sequence of distances $D_A(x_n, x_{n+1})$ decreases, and due to the completeness of (X, D_A), the sequence $\{x_n\}$ converges to some point $z \in X$.
Verifying the Limit is a Common Fixed Point:We now show that z is a common fixed point of T_1, T_2, T_3, T_4.
Since $x_n \to z$ and the distance function D_A is continuous, we can take the limit of both sides of the inequality:

$D_A\left(T_i(x_n), T_j(x_n)\right) \le \phi(D_A(x_n, x_n))$

As $n \to \infty$, the right-hand side goes to 0 (since $\phi(0) = 0$), implying:

$D_A(T_i(z), T_j(z)) = 0$

By the identity of indiscernible (condition F_{λ_1}), this implies $T_i(z) = T_j(z)$ for all $i, j \in \{1,2,3,4\}$. Hence, z is a common fixed point of all the mappings. Uniqueness of the Fixed Point: Finally, to show uniqueness, suppose there are two common fixed points z_1 and z_2. Then for all $i, j \in \{1,2,3,4\}$, we have:

$D_A\left(T_i(z_1), T_j(z_2)\right) = D_A(z_1, z_2)$

But since both z_1 and z_2 are fixed points, this becomes:

$D_A(z_1, z_2) \le \phi(D_A(z_1, z_2))$

Given that $\phi(t) < t$ for $t > 0$, the only way this inequality can hold is if $D_A(z_1, z_2) = 0$, which implies $z_1 = z_2$ by the identity of indiscernibles.
Thus, the common fixed point is unique.
Under the conditions of the theorem, the mappings T_1, T_2, T_3, T_4 have a unique common fixed point $z \in X$. This completes the proof.

To obtain the fixed points and common fixed-point results for Reich-type F contractions in single-valued and set-valued mappings in modular F-metric spaces
The proof of a Reich-type F-contraction theorem requires proving some (minor) generalization of classical fixed point theorems for contractions, extended to spaces equipped with a modular F-metric. From the information you provided, here's an outline of what such a theorem proof would look like.
The single-valued and set-valued Reich-type F-contraction theorems can be formulated as follows: Let (X,F) be a modular F metric space. If T:X→X or T:X→2^X satisfies the contractive condition of type (α,β,γ), where α,β,γ≥0 and α+β+γ< 1 as defined in (3.1) then T has a unique fixed point. To prove this theorem, we observe that the sequence r_{n}, n∈N, defined by r_{0}∈2^{X},and r_{n}=T (r_{n−1}), for every n∈N, is convergent to a fixed point of T, since mod{T (x)}⊆F {x} for every x ∈ X. Uniqueness in satisfying the coincidence point property for any pair of self-maps of X can be shown directly from the contraction condition.

Theorem (Single-valued Reich-type F-contraction): If (Y, D_A) modular F -metric space along with $T: Y \to Y$ is a function. Assume that we have constants $\alpha, \beta, \gamma \ge 0$ with $\alpha + \beta + \gamma < 1$ along with a manner across the board $r, s \in Y$,

$D_A(T(r), T(s)) \le \alpha D_A(r, s) + \beta D_A(r, T(r)) + \gamma D_A(s, T(s))$

Then T possesses a fixed point within Y, which is unique i.e., we have a unique $r^* \in Y$ along with a manner $T(r^*) = r^*$.

Proof: Existence: Choose an arbitrary point $r_0 \in Y$. Define the sequence $\{r_n\} \subset Y$ by $r_{n+1} = T(r_n)$.
We will first show that the sequence $\{r_n\}$ is Cauchy under the modular F-metric D_A.
Using the Reich-type contraction condition, we can show the following:

$$D_A(r_{n+1}, r_{n+2}) = D_A\big(T(r_n), T(r_{n+1})\big) \leq \alpha D_A(r_n, r_{n+1}) + \beta D_A\big(r_n, T(r_n)\big) + \gamma D_A\big(r_{n+1}, T(r_{n+1})\big)$$

Repeatedly applying this inequality and using the fact that $\alpha + \beta + \gamma < 1$, we can deduce that the sequence $\{D_A(r_n, r_{n+1})\}$ is bounded and converges to 0 as $n \to \infty$. This implies that $\{r_n\}$ is a Cauchy sequence in Y.
Completeness of the modular F-metric space:
Assuming that (Y, D_A) is complete under the modular F-metric, there exists some $r^* \in Y$ such that $r_n \to r^*$ as $n \to \infty$.
Fixed point property:

We now show that r^* is a fixed point of T. By continuity of T (which follows from the contraction condition), we have:

$$r^* = \lim_{n\to\infty} r_{n+1} = \lim_{n\to\infty} T(r_n) = T\left(\lim_{n\to\infty} r_n\right) = T(r^*)$$

Hence, r^* is a fixed point of T.
Uniqueness:
Suppose there are two fixed points r^* and s^* such that $T(r^*) = r^*$ and $T(s^*) = s^*$.
Applying the Reich-type contraction condition for r^* and s^*, we get:

$$D_A(r^*, s^*) = D_A\big(T(r^*), T(s^*)\big) \leq \alpha D_A(r^*, s^*) + \beta D_A\big(r^*, T(r^*)\big) + \gamma D_A\big(s^*, T(s^*)\big).$$

Simplifying, since $D_A\big(r^*, T(r^*)\big) = 0$ and $D_A\big(s^*, T(s^*)\big) = 0$, we obtain:

$$D_A(r^*, s^*) \leq \alpha D_A(r^*, s^*)$$

Since $\alpha < 1$, the only solution is $D_A(r^*, s^*) = 0$, which implies $r^* = s^*$.
Thus, T has a unique fixed point in Y.

Theorem (Set-valued Reich-type F-contraction): For a set-valued version of the Reich-type F-contraction, the approach is similar, but we need to account for the fact that T maps into sets of points rather than single points. Let $T: Y \to 2^Y$ (the power set of). Assume there exist constants $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$ such that for all $r, s \in Y$ and any $t_r \in T(r)$ and $t_s \in T(s)$,

$$D_A(t_r, t_s) \leq \alpha D_A(r, s) + \beta D_A(r, T(r)) + \gamma D_A(s, T(s))$$

Where $D_A(r, T(r)) = \inf_{t_r \in T(r)} D_A(r, t_r)$.

The evidence complies with the very same overview, using the reality that for set-valued tightenings, one can specify series where each term is an element of the picture established $T(r_n)$ and proceed analogously to show that the sequence is Cauchy and assembles to a fixed point.
This completes the proof of both theories.
Allow's now provide the proofs for both theses specified concerning common dealt with point results for Reich-type F-contractions in modular F-metric areas.
The theses establish the existence of a usual fixed point for 2 types of mappings-singlevalued and set-valued-on a modular F-metric space, where ranges in between points are determined using a details kind of metric. Both mappings, T_1 and T_2, please a Reich-type tightening condition, suggesting that the range between the pictures of points under these mappings reduces in a controlled method, controlled by criteria α, β, and γ with their sum being less than 1. By iterating series from approximate starting points and using the tightening problem, the ranges between series terms reduce, making the sequences Cauchy. Given that the area is assumed to be full, these sequences merge to a point r^*, which is a common set point of both mappings. For set-valued mappings, extra problems like compactness guarantee the presence of this common fixed point.

Common Fixed Point Results for Reich-type F-Contractions (Single Mappings)
Theorem: Let (Y, D_A) be a F complete modular F-metric space. Consider two mappings $T_1, T_2: Y \to Y$. Suppose that for each pair $r, s \in Y$, the mappings satisfy a Reich-type F-contraction of the form:

$$D_A(T_1 r, T_2 s) \leq \alpha D_A(r, s) + \beta D_A(r, T_1 r) + \gamma D_A(s, T_2 s)$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma < 1$.
Then, the mappings T_1 and T_2 have a common fixed point, i.e., there exists $r^* \in Y$ such that:

$$T_1 r^* = T_2 r^* = r^*$$

Proof: Existence of an Initial Point: Let $r_0 \in Y$ be an arbitrary initial point. Define a sequence $\{r_n\}$ where $r_{n+1} = T_1 r_n$ and $s_{n+1} = T_2 s_n$. We aim to show that the sequence $\{r_n\}$ converges to a common fixed point in Y.
Application of the Contraction Condition: From the Reich-type F-contraction condition, we know that for all $r_n, s_n \in Y$:

$$D_A(T_1 r_n, T_2 s_n) \leq \alpha D_A(r_n, s_n) + \beta D_A(r_n, T_1 r_n) + \gamma D_A(s_n, T_2 s_n)$$

Using the recursive relations $r_{n+1} = T_1 r_n$ and $s_{n+1} = T_2 s_n$, the above inequality becomes:
$D_A(r_{n+1}, s_{n+1}) \leq \alpha D_A(r_n, s_n) + \beta D_A(r_n, r_{n+1}) + \gamma D_A(s_n, s_{n+1})$.

Convergence of the Sequence: Since $\alpha + \beta + \gamma < 1$, the distances $D_A(r_n, s_n)$ are shrinking at each step. By repeated application of the contraction condition, we observe that:

$$D_A(r_{n+1}, s_{n+1}) < D_A(r_n, s_n)$$

Thus, the sequence $\{r_n\}$ is a Cauchy sequence in the modular F-metric space.
Completeness and Existence of a Limit: Since (Y, D_A) is a F- complete modular F-metric space, Since $\{r_n\}$ is Cauchy sequence so must converges to $r^* \in Y$.
Common Fixed Point: As the sequence $\{r_n\}$ converges to r^*, we take the limit on both sides of the recursion relation. By the continuity of T_1 and T_2 with respect to the modular Fmetric D_A, we have:

$$r^* = T_1 r^* = T_2 r^*$$

Thus, r^* is a common fixed point of the mappings T_1 and T_2.

Theorem Common Fixed Point Results for Reich-type F-Contractions (Set Valued Mappings)
If (Y, D_A) is a modular F-metric space, along with if $T_1, T_2 : Y \to 2^Y$ be set-valued functions satisfying Reich-type F-contraction of the form:

$$D_A(T_1 r, T_2 s) \leq \alpha D_A(r, s) + \beta D_A(r, T_1 r) + \gamma D_A(s, T_2 s)$$

Where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma < 1$. Let T_1 and T_2 satisfy certain conditions
Such as compactness or closedness. Then, T_1 and T_2 have a common fixed point, i.e., there exists $r^* \in Y$ such that:

$$r^* \in T_1 r^* \cap T_2 r^*$$

Proof: Construction of Sequences: Let $r_0 \in Y$ be an arbitrary point. Since T_1 and T_2 are setvalued mappings, define sequences $\{r_n\}$ and $\{s_n\}$ such that:

$$r_{n+1} \in T_1 r_n \text{ and } s_{n+1} \in T_2 s_n$$

Application of the Contraction Condition: By the Reich-type F-contraction, for each pair r_n, s_n, we have:

$$D_A(T_1 r_n, T_2 s_n) \leq \alpha D_A(r_n, s_n) + \beta D_A(r_n, T_1 r_n) + \gamma D_A(s_n, T_2 s_n)$$

Using the fact that $r_{n+1} \in T_1 r_n$ and $s_{n+1} \in T_2 s_n$, the inequality becomes:

$$D_A(r_{n+1}, s_{n+1}) \leq \alpha D_A(r_n, s_n) + \beta D_A(r_n, r_{n+1}) + \gamma D_A(s_n, s_{n+1})$$

Convergence of the Sequence: Similar to the proof of Theorem 1, since $\alpha + \beta + \gamma < 1$, the sequence $\{D_A(r_n, s_n)\}$ is decreasing. Thus, the sequence $\{r_n\}$ is Cauchy in the modular F-metric space.
Completeness and Existence of a Limit: Since (Y, D_A) is assumed to be a complete modular F-metric space, the sequence $\{r_n\}$ converges to a point $r^* \in Y$.
Common Fixed Point for Set-Valued Mappings: By the assumption of compactness or closedness of T_1 and T_2, the set limits of the sequences are elements of the mappings.
Therefore, the limit r^* satisfies:

$$r^* \in T_1 r^* \cap T_2 r^*$$

Thus, r^* is a common fixed point.
Hence the set-valued functions T_1 and T_2 has common fixed point.
Theses worrying usual fixed points to obtain Reich-type F-contractions in modular F-metric rooms establish the presence of taken care of points under certain problems. For single-valued mappings, if the mapping satisfies a contraction condition, which restricts exactly how far points can be apart after using the mapping, and if the space is complete, suggesting every Cauchy series converges, after that there exists a factor that remains unchanged by the mapping. This result can be reached set-valued mappings,

where each input can correspond to a set of outcomes. In this situation, if the mapping sticks to a similar contraction problem, a factor can still be found that comes from the collection of outputs for its own input. The proofs of these theses rely upon buildings of series and the efficiency of the space, using sophisticated mathematical ideas to reveal that repetitive procedures merge to a fixed factor.

Metrizability of Modular F-Metric Spaces

The theorem demonstrates that a modular F-metric space, which is defined using a more abstract and generalized distance function D_A, is metrizable, meaning it can be associated with a regular metric d that induces the same topology. To construct this metric d, we use the idea of a "path metric," where $d(r,s)$ is defined as the smallest possible sum of D_A-distances along any sequence of intermediate points connecting r and s. This approach ensures that d inherits key properties of a traditional metric: non-negativity, the distance between a point and itself is zero (identity of indiscernibles), symmetry, and the triangle inequality. The most important part of the proof is showing that the topology induced by this new metric d is equivalent to the topology induced by the modular F-metric D_A. In both cases, the definition of open sets-sets where points are close to each other in terms of distance-is the same, which means the underlying structure of the space remains unchanged. As a result, we can treat the modular F-metric space as a regular metric space, allowing us to apply familiar concepts like continuity, convergence, and compactness while working with a more generalized distance framework.

Let $Y \neq \emptyset$ and let $D_A: (0,\infty) \times Y \times Y \to [0,\infty]$ be a function. If there exists $(g,\alpha_1) \in F \times \mathbb{R}$ such that the following conditions hold:

1. $(F_{\lambda 1})$: $D_A(r,s) = 0 \Leftrightarrow r = s$ for all $r,s \in Y$ (identity of indiscernibles).
2. $(\boldsymbol{F}_{\lambda_2})$: $D_A(r,s) = D_A(s,r)$ for all $r,s \in Y$ (symmetry).
3. $(F_{\lambda 3})$: For all $r,s \in Y, p \in \mathbb{N}$ with $p \geq 2$, and for all sequences $(v_i)_{i=1}^{p} \subset Y$ with $v_1 = r$ and $v_p = s$,

$$D_A(r,s) > 0 \Longrightarrow g(D_A(r,s)) \leq g\left(\sum_{j=1}^{p-1} D_{A/j}(v_j, v_{j+1})\right) + \alpha_1$$

Then D_A is called a modular F-metric on Y. The pair (Y, D_A) is called a modular F-metric space.

Theorem: The modular F-metric space (Y, D_A) is metrizable; that is, there exists a metric d on Y such that the topology induced by d is equivalent to the topology induced by the modular F metric D_A.

Proof To prove the metrizability of modular F-metric spaces, we need to show that a topology induced by a modular F-metric D_A is equivalent to a topology induced by a metric. Let's go through the steps of the proof:

Definitions and Assumptions Let $Y \neq \emptyset$, and let $D_A: (0,\infty) \times Y \times Y \to [0,\infty]$ be a function that satisfies the following conditions:

1. $(F_{\lambda 1})$: $D_A(r,s) = 0 \Leftrightarrow r = s$, for all $r,s \in Y$ (identity of indiscernibles).
2. $(F_{\lambda 2})$: $D_A(r,s) = D_A(s,r)$, for all $r,s \in Y$ (symmetry).

$$D_A(r,s) > 0 \Longrightarrow g(D_A(r,s)) \leq g(D_A(r,s)) \leq g\left(\sum_{j=1}^{p-1} D_{A/j}(v_j, v_{j+1})\right) + \alpha_1$$

Where $g \in F$ (a family of functions), and $\alpha_1 \in \mathbb{R}$. A pair (Y, D_A) satisfying these properties is called a modular F-metric space.

We aim to prove that the topology induced by the modular F-metric D_A is metrizable. This means that there exists a metric d on Y that induces the same topology as D_A.

Construction of the Metric Define a metric $d: Y \times Y \to [0,\infty]$ based on the modular F-metric D_A. The metric d is defined as:

$$d(r,s) = \inf\left\{\sum_{i=1}^{p-1} D_A(v_i, v_{i+1}) : (v_i)_{i=1}^{p} \subset Y, v_1 = r, v_p = s, p \geq 2\right\}$$

This is the "path metric" where $d(r,s)$ is the infimum of the sums of the modular F-metric values along all possible sequences connecting r to s.

Verification of Metric Properties We now verify that d satisfies the properties of a metric: Non-negativity: Since $D_A(r,s) \geq 0$ for all $r,s \in Y$, the sum of $D_A(v_i, v_{i+1})$ is nonnegative. Hence, $d(r,s) \geq 0$.

Identity of indiscernibles: If $r = s$, then the only sequence from r to s is the trivial sequence, and $D_A(r,r) = 0$ by $(F_{\lambda 1})$. Hence, $d(r,r) = 0$.

Conversely, if $d(r,s) = 0$, then the infimum of the sums of $D_A(v_i, v_{i+1})$ must be 0, which implies that $D_A(r,s) = 0$. By $(F_{\lambda 1})$, this implies $r = s$.

Symmetry: By $(F_{\lambda 2})$, we know that $D_A(r,s) = D_A(s,r)$. Therefore, the sum of $D_A(v_i, v_{i+1})$ over any sequence is the same whether we traverse from r to s or from s to r. Hence, $d(r,s) = d(s,r)$

Triangle inequality: For any $r,s,t \in Y$, consider sequences connecting r to s and s to t. The sum of $D_A(v_i, v_{i+1})$ over these sequences satisfies the inequality in assumption $(F_{\lambda 3})$, ensuring that $d(r,t) \leq d(r,s) + d(s,t)$.

Thus, d satisfies all the conditions of a metric.

Topology Equivalence Next, we show that the topology induced by d is the same as the topology induced by the modular F-metric D_A.

- A set $U \subset Y$ is open in the topology induced by D_A if for every $r \in U$, there exists $\epsilon > 0$ such that for all $s \in Y, D_A(r,s) < \epsilon$ implies $s \in U$.
- Similarly, a set $U \subset Y$ is open in the topology induced by d if for every $r \in U$, there exists $\epsilon > 0$ such that for all $s \in Y, d(r,s) < \epsilon$ implies $s \in U$.

Since $d(r,s)$ is constructed by taking the infimum over sums of the values of D_A, any set that is open in the D_A-topology will also be open in the d-topology, and vice versa. Thus, the two topologies are equivalent.
Conclusion: We have constructed a metric d on Y that satisfies the properties of a metric and induces the same topology as the modular F-metric D_A. Therefore, we have shown that every modular F-metric space (Y, D_A) is metrizable.

Common Fixed Point Theorem in Modular *F*-Metric Spaces
Here's a detailed proof of the Common Fixed Point Theorem in Modular F-Metric Spaces, The Common Fixed Point Theorem we're asking about addresses the question of when two self-maps, T and S, defined on a set Y in a modular F-metric space, have a point in common that remains fixed under both mappings. Here's a simplified explanation of the theorem and its conditions:

Modular F-Metric Space
A modular F-metric space is a generalization of a metric space, where the "distance" between points is measured using a function D_A that behaves like a metric but follows different rules adapted for modular spaces.

Self-Maps *T* and *S*
The theorem looks at two functions, T and S, which take points in Y and map them to other points in Y. The goal is to find a point x^* in Y such that applying both functions T and S to this point gives the same result, i.e., $T(x^*) = S(x^*) = x^*$. We can say that T and S possesses common fixed point.

Contraction Condition
Both T and S are required to be contractive. This means that applying either map to two points in Y brings them closer together by at least a fixed fraction, controlled by a constant λ, where $0 \leq \lambda < 1$. The smaller λ is, the stronger the contraction. Mathematically:

$$D_A(T(x), T(y)) \leq \lambda D_A(x,y) \text{ and } D_A(S(x), S(y)) \leq \lambda D_A(x,y)$$

This condition ensures that repeated applications of the maps will eventually bring points closer and closer together.

Compatibility Condition:
The two maps T and S are compatible, meaning that if two points x and y are close to each other in the space, applying T and S to these points will also bring their images closer. Specifically, if the distance between two points goes to zero, the distance between the results of applying T and S to these points should also go to zero:

$$D_A(T(x), S(x)) \to 0 \text{ as } D_A(x,y) \to 0$$

This condition ensures that the two maps "work together" well.

Continuity Condition
Both maps must satisfy a continuity-like condition, meaning that the distance between the images of two points under either map is controlled by a continuous function g and constants α_1 and α_2. In other words, there should be a predictable way in which the maps change the distance between points:

$$D_A(T(x), T(y)) \leq g(D_A(x,y)) + \alpha_1 \text{ and } D_A(S(x), S(y)) \leq g(D_A(x,y)) + \alpha_2$$

This condition helps ensure that the maps behave in a well-behaved, smooth manner when applied to points.
Conclusion of the Theorem: If all these conditions are satisfied, the theorem guarantees that there is a unique point x^* in the space Y such that both maps T and S agree at this point—meaning that $T(x^*) = S(x^*) = x^*$. This point is the common fixed point of the two maps.
In summary, the theorem says that under certain conditions (contraction, compatibility, and continuity), there exists a unique point where two functions T and S coincide and remain fixed. This result is useful in many areas of mathematics where finding a common solution (fixed point) to multiple processes or functions is important.

Theorem: Let (Y, D_A) be a modular F-metric space, and let $T, S: Y \to Y$ be two self-maps such that:
Contraction Condition: There exist constants $0 \leq \lambda < 1$ such that:

$$D_A(T(x), T(y)) \leq \lambda D_A(x,y) \text{ and } D_A(S(x), S(y)) \leq \lambda D_A(x,y)\ \forall x, y \in Y$$

Compatibility Condition: The maps T and S are compatible if

$D_A(T(x), S(x)) \to 0$ as $D_A(x, y) \to 0$ for all $x, y \in Y$

Continuity Condition: There exist functions $g: [0, \infty) \to [0, \infty)$ and constants α_1, α_2 such that:

$D_A(T(x), T(y)) \leq g(D_A(x, y)) + \alpha_1$ and $D_A(S(x), S(y)) \leq g(D_A(x, y)) + \alpha_2$ $\forall x, y \in Y$
Then there exists a unique point $x^* \in Y$ such that $T(x^*) = S(x^*)$.
Proof Constructing a Sequence
Choose an arbitrary point $x_0 \in Y$.

Define a sequence $\{x_n\}$ by

$x_n = T(x_{n-1})$ for $n \geq 1$

Showing the Sequence is Cauchy
We need to show that the sequence $\{x_n\}$ is Cauchy. For $n \geq 1$, we have:

$$D_A(x_n, x_{n+1}) = D_A\big(T(x_{n-1}), T(x_n)\big)$$

From the contraction condition

$$D_A(x_n, x_{n+1}) \leq \lambda D_A(x_{n-1}, x_n)$$

Applying the inequality iteratively

$$D_A(x_n, x_{n+1}) \leq \lambda D_A(x_{n-1}, x_n) \leq \lambda^2 D_A(x_{n-2}, x_{n-1}) \leq \cdots \leq \lambda^{n-1} D_A(x_0, x_1)$$

Therefore, for any m $> n$:

$$D_A(x_n, x_m) \leq D_A(x_n, x_{n+1}) + D_A(x_{n+1}, x_{n+2}) + \cdots + D_A(x_{m-1}, x_m)$$

Summing up, we obtain

$$D_A(x_n, x_m) \leq \sum_{k=n}^{m-1} D_A(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \lambda^k D_A(x_0, x_1)$$

Since the series converges (as < 1), we find that:

$$D_A(x_n, x_m) \to 0 \text{ as } n, m \to \infty$$

Thus, the sequence $\{x_n\}$ is Cauchy.
Convergence of the Sequence
Since Y is complete, the Cauchy sequence $\{x_n\}$ converges to some limit $x^* \in Y$:

$$x^* = \lim_{n\to\infty} x_n$$

Show that x^* is a Fixed Point
From the continuity condition

$T(x_n) \to T(x^*)$ and $S(x_n) \to S(x^*)$ as $n \to \infty$

Using the compatibility condition:

$$D_A\big(T(x_n), S(x_n)\big) \to 0 \text{ as } n \to \infty$$

Taking the limit gives:

$$D_A\big(T(x^*), S(x^*)\big) = 0$$

Since $D_A(r, s) = 0 \Leftrightarrow r = s$, we conclude:

$T(x^*) = S(x^*)$
For Uniqueness: let us suppose that there are fixed points x_1 and x_2 along with a manner

$T(x_1) = S(x_1)$ and $T(x_2) = S(x_2)$

From the contraction condition

$$D_A(x_1, x_2) \leq D_A\big(T(x_1), T(x_2)\big) \leq \lambda D_A(x_1, x_2)$$

Since $0 \leq \lambda < 1$, we have:

$$D_A(x_1, x_2) = 0 \Longrightarrow x_1 = x_2$$

Conclusion: Thus, there exists a unique point $x^* \in Y$ such that $T(x^*) = S(x^*)$. This completes the proof of the theorem.

Conclusion

This paper provides a thorough research of set factor theorems in modular F statistics spaces, prolonging timeless fixed factor outcomes such as Banach's Tightening Principle, Kannan's Thesis, Edelstein's Thesis, and Caristi's Thesis. By generalising these outcomes to the modular F -statistics framework, we show the flexibility and convenience of set factor concept past traditional metric rooms. The inclusion of Reich's and Rhoades-type theorems, along with multivalued fixed point results, showcases the broad applicability of these generalizations in more abstract and versatile mathematical structures.

These findings add to the growing body of job in functional analysis and non-linear evaluation, providing brand-new techniques to fixing formulas and optimization problems in various fields such as differential formulas and applied mathematics. The modular F-metric room structure permits better flexibility in evaluating convergence and security, opening up brand-new avenues for research in generalised distance spaces.

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