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A study of the bi-shadowing property on topologically conjugate dynamical systems on group space

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Abstract
In this paper, we studied the metric group space by addressing the concept of topological conjugation of actions and studying the transfer of the bi-shadowing property between two uniformly topologically conjugate actions under some secondary conditions. This study was done using the concept of bi-shadowing in its two types, parameterized and non-parameterized, as well as when the parameters are fixed and different.

Keywords: Bi-shadowing, group space, dynamical systems, topologically conjugate, uniformly conjugate

Introduction
Shadowing theory is a major theory in dynamical systems. It has an important role in studying asymptotic behavior and stability in systems, see also [1, 2]. With it, computer calculations of the dynamic system can be confirmed and the existence of the true-orbit of the system near the pseudo-orbit, see [3, 4].
The first to develop the concept of shadowing was Walters P, [5], see more [6-8] and the concept of inverse shadowing and the concept of bi-shadowing were developed by other researchers, see [9-11]. In [12, 13] the research are studied the topologically conjugate systems of the shadowing, inverse shadowing and bi-shadowing in the dynamical system. Later, these and other properties of dynamical systems in group space were studied and good results were obtained. See [14-19]. As in paper [19], we will mention in section two of this paper the principles of the metric group space and some basic concepts of the shadowing theory such as the definitions of (α, β)-bi-shadowing, bi-shadowing, the concept of topological conjugate and topologically conjugate actions.
Later in this paper, we will prove some theorems that show the transfer of the bi-shadowing property in its two types and under some conditions from action in dynamical system to another action in different dynamical system between which there is a topological conjugate.

Fundamental Definitions
Let $\mathbb{G}$ be a group, χ be a Hausdorff topological space, we called the map $\Phi: \mathbb{G} \times \chi \to \chi$ which satisfies the following conditions is an action:

1. $\Phi(\mathbb{g}_1, .)$ is a homeomorphism of χ, 2- $\Phi(e, x) = x$, 3- $\Phi\left(\mathbb{g}_1, \Phi\left(\mathbb{g}_2, x\right)\right) = \Phi\left(\mathbb{g}_1\mathbb{g}_2, x\right)$.
For any $\mathbb{g}_1, \mathbb{g}_2 \in \mathbb{G}$, and any $x \in \chi$, and e is the identity of the group $\mathbb{G}$.

In the definitions, theories and remarks that will be mentioned in this paper we will suppose that the action $\Phi: \mathbb{G} \times \chi \to \chi$ of a finitely generated group $\mathbb{G}$ with respect to the finite symmetric generating set $\mathbb{S}$, ($\mathbb{S}$ is any finite symmetric generating set of $\mathbb{G}$) on a metric group space $\left(\chi, \mathbb{d}_\chi\right)$. And the image of x by Φ is $\Phi(\mathbb{s}, x)$, and the inverse image is $\Phi^{-1}(\mathbb{s}, x) = \Phi(\mathbb{s}^{-1}, x)$ for $\mathbb{s} \in \mathbb{S}$.

We need to introduce the true-orbit and pseudo-orbit definitions

Definition 2.1. [18] We called the sequence $\{x_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ is true-orbit of the action Φ if $\Phi(\mathbb{s}, x_{\mathbb{g}}) = x_{\mathbb{g}\mathbb{s}}$ for $\mathbb{s} \in \mathbb{S}, \mathbb{g} \in \mathbb{G}$.

For $\rho > 0$, we called the sequence $\{y_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ is ρ-pseudo-orbit of an action Φ if $\mathbb{d}_\chi(\Phi(\mathbb{s}, y_{\mathbb{g}}), y_{\mathbb{g}\mathbb{s}}) < \rho$, for $\mathbb{s} \in \mathbb{S}, \mathbb{g} \in \mathbb{G}$.

Below we mention the bi-shadowing definition, therefore, we need to provide the definition of distance between two actions. For Φ and Ψ are actions, the distance between them is given by:

$$\mathbb{d}_{\chi_0}(\Phi, \Psi) = \sup_{x \in \chi}\{\mathbb{d}_\chi(\Phi(\mathbb{s}, x), \Psi(\mathbb{s}, x))\} \text{ for } \mathbb{s} \in \mathbb{S}.$$

Definition 2.2. [17] Let $(\chi, \mathbb{d}_\chi)$ be metric group space, we called the action $\Phi: \mathbb{G} \times \chi \to \chi$ is bi-shadowing with positive parameters α and β if there exists $0 < \rho \leq \beta$ such that for any ρ-pseudo-orbit $\{y_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ of Φ and any action $\psi: \mathbb{G} \times \chi \to \chi$ satisfying

$$\mathbb{d}_{\chi_0}(\Phi, \psi) \leq \beta - \rho, \tag{1}$$

Then there exists a true-orbit $\{x_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ of ψ such that

$$\mathbb{d}_\chi(x_{\mathbb{g}}, y_{\mathbb{g}}) \leq \alpha\left(\rho + \mathbb{d}_{\chi_0}(\Phi, \Psi)\right) \leq \alpha\beta, \text{ for all } \mathbb{g} \in \mathbb{G}.$$

We called the set of all actions $\{\psi_1, \psi_2, \dots\}$ which satisfies (1) is the comparison class $\mathbb{A}(\chi)$, and if the action Φ is bi-shadowing with positive parameters α and β with respect to $\{\psi_1, \psi_2, \dots\}$ then for simply we called the action Φ is $(\alpha, \beta, \mathbb{A}(\chi))$-bi-shadowing.

Definition 2.3. [19] Let $(\chi, \mathbb{d}_\chi)$ be metric group space, we called the action $\Phi: \mathbb{G} \times \chi \to \chi$ is bi-shadowing if for all $\beta > 0$ there exists $\alpha > 0$ and $0 < \rho \leq {}^{\beta}/_{\alpha}$ such that for any ρ-pseudo-orbit $\{y_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ of Φ and any action $\psi: \mathbb{G} \times \chi \to \chi$ satisfying $\mathbb{d}_{\chi_0}(\Phi, \psi) \leq \beta - \rho$, then there exists a true-orbit $\{x_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ of ψ such that $\mathbb{d}_\chi(x_{\mathbb{g}}, y_{\mathbb{g}}) \leq \alpha\left(\rho + \mathbb{d}_{\chi_0}(\Phi, \Psi)\right) \leq \beta$, for all $\mathbb{g} \in \mathbb{G}$. And if the action Φ is bi-shadowing with respect to the comparison class $\mathbb{A}(\chi) = \{\psi_1, \psi_2, \dots\}$ then for simply we called the action Φ is $\mathbb{A}(\chi)$-bi-shadowing.

Definition 2.4. If $(\chi, \mathbb{d}_\chi)$ and $(\Upsilon, \mathbb{d}_\Upsilon)$ are two metric group spaces and $\Phi: \mathbb{G} \times \chi \to \chi$ and $\Psi: \mathbb{G} \times \Upsilon \to \Upsilon$ are actions on χ and Υ, respectively, then we called Φ and Ψ are topologically conjugate if there exists an action $\mathbb{T}: \mathbb{G} \times \chi \to \Upsilon$ such that $\mathbb{T} \circ \Phi = \Psi \circ \mathbb{T}$. If, in addition, $\mathbb{T}$ is uniformly continuous, then Φ and Ψ are called uniformly topologically conjugate, or simply uniformly conjugate. The two classes $\mathbb{A}(\chi)$ and $\mathbb{A}(\Upsilon)$ are called $(\mathbb{A}(\chi), \mathbb{A}(\Upsilon))$-topologically conjugate by $\mathbb{T}$ in the sense that individual actions in one class are topologically conjugate to an action in the other class by $\mathbb{T}$.

1. Topologically Conjugate Systems

For the invariance of (α, β)-bi-shadowing for topologically conjugate systems, we conclude the following:

Theorem 3.1. *Let* $(\chi, \mathbb{d}_\chi)$ and $(\Upsilon, \mathbb{d}_\Upsilon)$ *be metric group spaces and let* $\Phi: \mathbb{G} \times \chi \to \chi$ *and* $\Psi: \mathbb{G} \times \Upsilon \to \Upsilon$ *be actions that are topologically conjugate by* $\mathbb{T}: \mathbb{G} \times \chi \to \Upsilon$. *Assume also that* $\mathbb{A}(\chi)$ *and* $\mathbb{A}(\Upsilon)$ *are* $(\mathbb{A}(\chi), \mathbb{A}(\Upsilon))$-*topologically conjugate. If there exists* $\lambda \geq 1$ *such that*

$$\mathbb{d}_\chi(x', x'') \leq \mathbb{d}_\Upsilon(\mathbb{T}(\mathbb{s}, x'), \mathbb{T}(\mathbb{s}, x'')) \leq \lambda \mathbb{d}_\chi(x', x''), \text{ for } \mathbb{s} \in \mathbb{S} \text{ and for all } x', x'' \in \chi, \tag{2}$$

Then we have

- If Φ is $(\alpha, \beta, \mathbb{A}(\chi))$-bi-shadowing, then Ψ is $(\lambda\alpha, \beta, \mathbb{A}(\Upsilon))$-bi-shadowing.
- If Ψ has the $(\alpha, \lambda\beta, \mathbb{A}(\Upsilon))$-bi-shadowing, then Φ is $(\lambda\alpha, \beta, \mathbb{A}(\chi))$-bi-shadowing.

Proof. a) Let $\{y_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ be a ρ-pseudo-orbit of an action Ψ, with $0 \leq \rho \leq \beta$, which implies that $\mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y_{\mathbb{g}}), y_{\mathbb{g}\mathbb{s}}) < \rho$, and let $\varphi \in \mathbb{A}(\Upsilon)$ satisfy$\mathbb{d}_{\Upsilon_0}(\Psi, \varphi) = \sup_{y \in \Upsilon} \mathbb{d}_\Upsilon(\Psi(y), \varphi(y)) \leq \beta - \rho$.

$$\rho + \sup_{y \in \Upsilon} \mathbb{d}_\Upsilon(\Psi(y), \varphi(y)) \leq \beta. \tag{3}$$

Note that condition (2) is equivalent to the following condition:

$$\mathbb{d}_\chi(\mathbb{T}(\mathbb{s}^{-1}, y'), \mathbb{T}(\mathbb{s}^{-1}, y'')) \leq \mathbb{d}_\Upsilon(y', y'') \leq \lambda \mathbb{d}_\chi(\mathbb{T}(\mathbb{s}^{-1}, y'), \mathbb{T}(\mathbb{s}^{-1}, y'')),$$

For $\mathbb{s} \in \mathbb{S}$ and for all $y', y'' \in \Upsilon$. (4)

Now,

$$\mathbb{d}_\chi\left(\Phi\left(\mathbb{s},\left(\mathbb{T}\left(\mathbb{s}^{-1}, y_{\mathbb{g}}\right)\right)\right), \mathbb{T}\left(\mathbb{s}^{-1}, y_{\mathbb{g}\mathbb{s}}\right)\right) = \mathbb{d}_\chi\left(\mathbb{T}\left(\mathbb{s}^{-1}, \Psi\left(y_{\mathbb{g}}\right)\right), \mathbb{T}\left(\mathbb{s}^{-1}, y_{\mathbb{g}\mathbb{s}}\right)\right) \le \mathbb{d}_\Upsilon\left(\Psi\left(y_{\mathbb{g}}\right), y_{\mathbb{g}\mathbb{s}}\right) < \rho.$$

Hence $\{x_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\} = \{\mathbb{T}(\mathbb{s}^{-1}, y_{\mathbb{g}}): \mathbb{g} \in \mathbb{G}\}$ is a ρ-pseudo-orbit of Φ. Let $x \in \chi$ and $\psi \in \mathbb{A}(\chi)$, then using (2) and the conjugacy $\mathbb{T}$, we obtain

$$\mathbb{d}_\chi(\Phi(\mathbb{s}, x), \psi(\mathbb{s}, x)) \le \mathbb{d}_\Upsilon\left(\mathbb{T}(\mathbb{s}, \Phi(\mathbb{s}, x)), \mathbb{T}(\mathbb{s}, \psi(\mathbb{s}, x))\right) = \mathbb{d}_\Upsilon\left(\Psi(\mathbb{s}, \mathbb{T}(\mathbb{s}, x)), \varphi(\mathbb{s}, \mathbb{T}(\mathbb{s}, x))\right) = \mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y), \varphi(\mathbb{s}, y)),$$

Wherem,$y = \mathbb{T}(\mathbb{s}, x)$. This implies that

$$\sup_{x\in\chi} \mathbb{d}_\chi(\varphi(\mathbb{s}, x), \psi(\mathbb{s}, x)) \le \sup_{y\in\Upsilon} \mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y), \varphi(\mathbb{s}, y)) \quad (5)$$

From (3) and (5) and for any $\psi \in \mathbb{A}(\chi)$, where $\psi = \mathbb{T}^{-1} \circ \varphi \circ \mathbb{T}$, we have $\rho + \sup_{x\in\chi} \mathbb{d}_\chi(\Psi(\mathbb{s}, x), \psi(\mathbb{s}, x)) \le \beta$.

Since Φ is (α, β)-bi-shadowing then there exists a true-orbit $\{w_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ of ψ such that

$$\mathbb{d}_\chi(x_{\mathbb{g}}, w_{\mathbb{g}}) \le \alpha\left(\rho + \sup_{x\in\chi} \mathbb{d}_\chi(\Phi(\mathbb{s}, x), \psi(\mathbb{s}, x))\right) \le \alpha\left(\rho + \sup_{y\in\Upsilon} \mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y), \varphi(\mathbb{s}, y))\right).$$

Thus, using the second part of (2), we obtain

$$\mathbb{d}_\Upsilon\left(y_{\mathbb{g}}, \mathbb{T}(\mathbb{s}, w_{\mathbb{g}})\right) = \mathbb{d}_\Upsilon\left(\mathbb{T}(\mathbb{s}, x_{\mathbb{g}}), \mathbb{T}(\mathbb{s}, w_{\mathbb{g}})\right) \le \lambda \mathbb{d}_\chi(x_{\mathbb{g}}, w_{\mathbb{g}}) = \lambda\alpha\left(\rho + \sup_{y\in\Upsilon} \mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y), \varphi(\mathbb{s}, y))\right).$$

Note that $\{\mathbb{T}(\mathbb{s}, w_{\mathbb{g}}): \mathbb{g} \in \mathbb{G}\} \coloneqq \{a_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ is a true-orbit of φ since

$$\varphi(\mathbb{s}, a_{\mathbb{g}}) = \varphi\left(\mathbb{s}, \mathbb{T}(\mathbb{s}, w_{\mathbb{g}})\right) = \mathbb{T}\left(\mathbb{s}, \psi(\mathbb{s}, w_{\mathbb{g}})\right) = \mathbb{T}(\mathbb{s}, w_{\mathbb{g}\mathbb{s}}) = a_{\mathbb{g}\mathbb{s}}$$

This ends the proof that Ψ is $(\lambda\alpha, \beta, \mathbb{A}(\Upsilon))$-bi-shadowing. ■

Proof. b) Let $\{x_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ be a ρ-pseudo-orbit of an action Φ with $0 \le \rho \le \beta$, which implies that

$\mathbb{d}_\chi(\Phi(\mathbb{s}, x_{\mathbb{g}}), x_{\mathbb{g}\mathbb{s}}) < \rho$, and let $\psi \in \mathbb{A}(\chi)$ satisfy$\mathbb{d}_{\chi_0}(\Phi, \psi) = \sup_{x\in\chi} \mathbb{d}_\chi(\Phi(x), \psi(x)) \le \beta - \rho$.

$$\rho + \sup_{x\in\chi} \mathbb{d}_\chi(\Phi(x), \psi(x)) \le \beta \quad (6)$$

Now, $\mathbb{d}_\Upsilon\left(\Psi\left(\mathbb{s}, \mathbb{T}(\mathbb{s}, x_{\mathbb{g}})\right), \mathbb{T}(\mathbb{s}, x_{\mathbb{g}\mathbb{s}})\right) = \mathbb{d}_\Upsilon\left(\mathrm{h}\left(\mathbb{s}, \Psi(\mathbb{s}, x_{\mathbb{g}})\right), \mathbb{T}(\mathbb{s}, x_{\mathbb{g}\mathbb{s}})\right) \le \lambda \mathbb{d}_\chi(\Phi(\mathbb{s}, x_{\mathbb{g}}), x_{\mathbb{g}\mathbb{s}})$, hence $\mathbb{d}_\Upsilon\left(\Psi\left(\mathbb{s}, \mathbb{T}(\mathbb{s}, x_{\mathbb{g}})\right), \mathbb{T}(\mathbb{s}, x_{\mathbb{g}\mathbb{s}})\right) < \lambda\rho$.

It follows that $\{y_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\} = \{\mathbb{T}(\mathbb{s}, x_{\mathbb{g}}): \mathbb{g} \in \mathbb{G}\}$ is a $\lambda\rho$-pseudo-orbit of Ψ. Now let $y \in \Upsilon$ and $\varphi \in \mathbb{A}(\Upsilon)$, then using (4) we have$\mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y), \varphi(\mathbb{s}, y)) \le \lambda \mathbb{d}_\chi\left(\mathbb{T}(\mathbb{s}^{-1}, \Psi(\mathbb{s}, y)), \mathbb{T}(\mathbb{s}^{-1}, \varphi(\mathbb{s}, y))\right) = \lambda \mathbb{d}_\chi\left(\Phi(\mathbb{s}, \mathbb{T}(\mathbb{s}^{-1}, y)), \psi(\mathbb{s}, \mathbb{T}(\mathbb{s}^{-1}, y))\right) = \lambda \mathbb{d}_\chi(\Phi(\mathbb{s}, x), \psi(\mathbb{s}, x))$,

Where $x = \mathbb{T}(\mathbb{s}^{-1}, y)$. Therefore $\mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y), \varphi(\mathbb{s}, y)) \le \lambda \mathbb{d}_\chi(\Phi(\mathbb{s}, x), \psi(\mathbb{s}, x))$ for every $y \in \Upsilon$ and $x = \mathbb{T}(\mathbb{s}^{-1}, y)$, hence

$$\sup_{y\in\Upsilon} \mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y), \varphi(\mathbb{s}, y)) \le \lambda \sup_{x\in\chi} \mathbb{d}_\chi(\Phi(\mathbb{s}, x), \psi(\mathbb{s}, x)) \quad (7)$$

From (6) and (7) we get for any $\varphi \in \mathbb{A}(\Upsilon)$, where $\varphi = \mathbb{T} \circ \psi \circ \mathbb{T}^{-1}$, that $\lambda\rho + \sup_{y\in\Upsilon} \mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y), \varphi(\mathbb{s}, y)) \le \lambda\beta$.

Since Ψ has the $(\alpha, \lambda\beta)$-bi-shadowing, there exists a true-orbit $\{z_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ of φ such that

$$\mathbb{d}_\Upsilon(y_{\mathbb{g}}, z_{\mathbb{g}}) \le \alpha\left(\lambda\rho + \sup_{y\in\Upsilon} \mathbb{d}_\Upsilon(\mathbb{s}, \Psi(y), \varphi(\mathbb{s}, y))\right) \le \alpha\left(\lambda\rho + \lambda \sup_{x\in\chi} \mathbb{d}_\chi(\Phi(\mathbb{s}, x), \psi(\mathbb{s}, x))\right)$$

$$= \lambda\alpha\left(\rho + \sup_{x\in\chi} \mathbb{d}_\chi(\Phi(\mathbb{s},x),\psi(\mathbb{s},x))\right).$$

Thus, using (3) we obtain

$$\mathbb{d}_\chi\left(x_{\mathbb{g}},\mathbb{T}(\mathbb{s}^{-1},z_{\mathbb{g}})\right) = \mathbb{d}_\chi\left(\mathbb{T}(\mathbb{s}^{-1},y_{\mathbb{g}}),\mathbb{T}(\mathbb{s}^{-1},z_{\mathbb{g}})\right) \le \lambda\alpha\left(\rho + \sup_{x\in\chi} \mathbb{d}_\chi(\Phi(\mathbb{s},x),\psi(\mathbb{s},x))\right),$$

Note that $\{\mathbb{T}(\mathbb{s}^{-1},z_{\mathbb{g}}):\mathbb{g}\in\mathbb{G}\} = \{b_{\mathbb{g}}:\mathbb{g}\in\mathbb{G}\}$ is a true-orbit of ψ since

$$\psi(\mathbb{s},b_{\mathbb{g}}) = \psi\left(\mathbb{s},\mathbb{T}(\mathbb{s}^{-1},z_{\mathbb{g}})\right) = \mathbb{T}\left(\mathbb{s}^{-1},\varphi(\mathbb{s},z_{\mathbb{g}})\right) = \mathbb{T}(\mathbb{s}^{-1},z_{\mathbb{gs}}) = b_{\mathbb{gs}}$$

This shows that Φ is $(\lambda\alpha,\beta,\mathbb{A}(\chi))$-bi-shadowing. ■

Remark 3.2. In Theorem 3.1, if $\lambda = 1$, then we conclude that the action $\Phi:\mathbb{G}\times\chi\to\chi$ is $(\alpha,\beta,\mathbb{A}(\chi))$-bi-shadowing if and only if $\Psi:\mathbb{G}\times\Upsilon\to\Upsilon$ is $(\alpha,\beta,\mathbb{A}(\Upsilon))$-bi-shadowing.
However, if we assume that the parameter for (α,β)-bi-shadowing is different for topologically conjugate systems, we conclude the following:

Theorem 3.3. Let $(\chi,\mathbb{d}_\chi)$ and $(\Upsilon,\mathbb{d}_\Upsilon)$ be metric group spaces and let $\Phi:\mathbb{G}\times\chi\to\chi$ and $\Psi:\mathbb{G}\times\Upsilon\to\Upsilon$ be actions that are topologically conjugate by $\mathbb{T}:\mathbb{G}\times\chi\to\Upsilon$. Assume also that $\mathbb{A}(\chi)$ and $\mathbb{A}(\Upsilon)$ are $(\mathbb{A}(\chi),\mathbb{A}(\Upsilon))$-topologically conjugate. If there exists $\lambda \ge 1$ such that equation (2) is valid then we have

a) If Φ is $(\alpha,\beta,\mathbb{A}(\chi))$-bi-shadowing, then Ψ is $(\lambda\alpha,\beta,\mathbb{A}(\Upsilon))$-bi-shadowing for $\alpha' \ge \alpha$ and $\beta' \le \beta$. (8)
b) If Ψ has the $(\alpha,\lambda\beta\mathbb{A}(\Upsilon))$-bi-shadowing, then Φ is $(\lambda\alpha',\beta',\mathbb{A}(\chi))$-bi-shadowing for $\alpha' \le \alpha$ and $\beta' \ge \beta$. (9)

Proof. a) Let $\{y_{\mathbb{g}}:\mathbb{g}\in\mathbb{G}\}$ be a ρ-pseudo-orbit of an action Ψ, with $0 \le \rho' \le \beta'$, which implies that $\mathbb{d}_\Upsilon(\Psi(\mathbb{s},y_{\mathbb{g}}),y_{\mathbb{gs}}) < \rho'$, and let $\varphi\in\mathbb{A}(\Upsilon)$ satisfy$\mathbb{d}_{\Upsilon_0}(\Psi,\varphi) = \sup_{y\in\Upsilon}\mathbb{d}_\Upsilon\left(\Psi(y),\varphi(y)\right) \le \beta' - \rho'$.

$$\rho' + \sup_{y\in\Upsilon}\mathbb{d}_\Upsilon\left(\Psi(y),\varphi(y)\right) \le \beta'. \tag{10}$$

Note that condition (2) is equivalent to the following condition

$$\mathbb{d}_\chi(\mathbb{T}(\mathbb{s}^{-1},y'),\mathbb{T}(\mathbb{s}^{-1},y'')) \le \mathbb{d}_\Upsilon(y',y'') \le \lambda\mathbb{d}_\chi(\mathbb{T}(\mathbb{s}^{-1},y'),\mathbb{T}(\mathbb{s}^{-1},y'')),\ \text{for } \mathbb{s}\in\mathbb{S} \text{ and for all } y',y''\in\Upsilon. \tag{11}$$

Now,

$$\mathbb{d}_\chi\left(\Phi\left(\mathbb{s},\left(\mathbb{T}(\mathbb{s}^{-1},y_{\mathbb{g}})\right)\right),\mathbb{T}(\mathbb{s}^{-1},y_{\mathbb{gs}})\right) = \mathbb{d}_\chi\left(\mathbb{T}\left(\mathbb{s}^{-1},\Psi(y_{\mathbb{g}})\right),\mathbb{T}(\mathbb{s}^{-1},y_{\mathbb{gs}})\right) \le \mathbb{d}_\Upsilon(\Psi(y_{\mathbb{g}}),y_{\mathbb{gs}}) < \rho'.$$

Hence $\{x_{\mathbb{g}}:\mathbb{g}\in\mathbb{G}\} = \{\mathbb{T}(\mathbb{s}^{-1},y_{\mathbb{g}}):\mathbb{g}\in\mathbb{G}\}$ is a ρ-pseudo-orbit of Φ. Let $x\in\chi$ and $\psi\in\mathbb{A}(\chi)$, Then using (2) and the conjugacy $\mathbb{T}$, we obtain

$$\mathbb{d}_\chi(\Phi(\mathbb{s},x),\psi(\mathbb{s},x)) \le \mathbb{d}_\Upsilon\left(\mathbb{T}(\mathbb{s},\Phi(\mathbb{s},x)),\mathbb{T}(\mathbb{s},\psi(\mathbb{s},x))\right) = \mathbb{d}_\Upsilon\left(\Psi(\mathbb{s},\mathbb{T}(\mathbb{s},x)),\varphi(\mathbb{s},\mathbb{T}(\mathbb{s},x))\right) = \mathbb{d}_\Upsilon(\Psi(\mathbb{s},y),\varphi(\mathbb{s},y)),$$

Where, $y = \mathbb{T}(\mathbb{s},x)$. This implies that

$$\sup_{x\in\chi}\mathbb{d}_\chi(\Phi(\mathbb{s},x),\psi(\mathbb{s},x)) \le \sup_{y\in\Upsilon}\mathbb{d}_\Upsilon(\Psi(\mathbb{s},y),\varphi(\mathbb{s},y)) \tag{12}$$

From (10) and (12) and (8) and for any $\psi\in\mathbb{A}(\chi)$, where $\psi = \mathbb{T}^{-1}\circ\varphi\circ\mathbb{T}$, we have $\rho' + \sup_{x\in\chi}\mathbb{d}_\chi(\Psi(\mathbb{s},x),\psi(\mathbb{s},x)) \le \beta' \le \beta$.
Since Φ is (α,β)-bi-shadowing that is (there exists $0 < \rho = \rho' \le \beta$ such that for any ρ-pseudo-orbit $\{x_{\mathbb{g}}:\mathbb{g}\in\mathbb{G}\}$ of Φ and any action $\psi:\mathbb{G}\times\chi\to\chi$ satisfying $\mathbb{d}_{\chi_0}(\Phi,\psi) \le \beta - \rho$, then there exists a true-orbit $\{w_{\mathbb{g}}:\mathbb{g}\in\mathbb{G}\}$ of ψ such that $\mathbb{d}_\chi(x_{\mathbb{g}},w_{\mathbb{g}}) \le \alpha\left(\rho + \mathbb{d}_{\chi_0}(\Phi,\psi)\right)$).

Then, $\mathbb{d}_\chi(x_{\mathbb{g}},w_{\mathbb{g}}) \le \alpha\left(\rho + \mathbb{d}_{\chi_0}(\Phi,\psi)\right) = \alpha\left(\rho + \sup_{x\in\chi}\mathbb{d}_\chi(\Phi(\mathbb{s},x),\psi(\mathbb{s},x))\right) \le \alpha'\left(\rho' + \sup_{y\in\Upsilon}\mathbb{d}_\Upsilon(\Psi(\mathbb{s},y),\varphi(\mathbb{s},y))\right).$

Thus, using the second part of (2), we obtain

$$\mathbb{d}_\Upsilon\left(y_{\mathbb{g}}, \mathbb{T}(\mathbb{s}, w_{\mathbb{g}})\right) = \mathbb{d}_\Upsilon\left(\mathbb{T}(\mathbb{s}, x_{\mathbb{g}}), \mathbb{T}(\mathbb{s}, w_{\mathbb{g}})\right) \le \lambda \mathbb{d}_\chi(x_{\mathbb{g}}, w_{\mathbb{g}}) = \lambda\alpha'\left(\rho + \sup_{y\in\Upsilon} \mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y), \varphi(\mathbb{s}, y))\right).$$

Note that $\{\mathbb{T}(\mathbb{s}, w_{\mathbb{g}}) : \mathbb{g} \in \mathbb{G}\} \coloneqq \{a_{\mathbb{g}} : \mathbb{g} \in \mathbb{G}\}$ is a true-orbit of φ since

$$\varphi(\mathbb{s}, a_{\mathbb{g}}) = \varphi\left(\mathbb{s}, \mathbb{T}(\mathbb{s}, w_{\mathbb{g}})\right) = \mathbb{T}\left(\mathbb{s}, \psi(\mathbb{s}, w_{\mathbb{g}})\right) = \mathbb{T}(\mathbb{s}, w_{\mathbb{g}\mathbb{s}}) = a_{\mathbb{g}\mathbb{s}}.$$

This ends the proof that Ψ is $(\lambda\alpha', \beta', \mathbb{A}(\Upsilon))$-bi-shadowing. ■

Proof. b) Let $\{x_{\mathbb{g}} : \mathbb{g} \in \mathbb{G}\}$ be a ρ-pseudo-orbit of an action Φ with $0 \le \rho \le \beta$, which implies that $\mathbb{d}_\chi(\Phi(\mathbb{s}, x_{\mathbb{g}}), x_{\mathbb{g}\mathbb{s}}) < \rho$, and let $\psi \in \mathbb{A}(\chi)$ satisfy $\mathbb{d}_{\chi_0}(\Phi, \psi) = \sup_{x\in\chi} \mathbb{d}_\chi\left(\Phi(x), \psi(x)\right) \le \beta - \rho$.

$$\rho + \sup_{x\in\chi} \mathbb{d}_\chi\left(\Phi(x), \psi(x)\right) \le \beta \tag{13}$$

Now, $\mathbb{d}_\Upsilon\left(\Psi\left(\mathbb{s}, \mathbb{T}(\mathbb{s}, x_{\mathbb{g}})\right), \mathbb{T}(\mathbb{s}, x_{\mathbb{g}\mathbb{s}})\right) = \mathbb{d}_\Upsilon\left(\mathrm{h}\left(\mathbb{s}, \Psi(\mathbb{s}, x_{\mathbb{g}})\right), \mathbb{T}(\mathbb{s}, x_{\mathbb{g}\mathbb{s}})\right) \le \lambda \mathbb{d}_\chi(\Phi(\mathbb{s}, x_{\mathbb{g}}), x_{\mathbb{g}\mathbb{s}})$,
Hence $\mathbb{d}_\Upsilon\left(\Psi\left(\mathbb{s}, \mathbb{T}(\mathbb{s}, x_{\mathbb{g}})\right), \mathbb{T}(\mathbb{s}, x_{\mathbb{g}\mathbb{s}})\right) < \lambda\rho$.

It follows that $\{y_{\mathbb{g}} : \mathbb{g} \in \mathbb{G}\} = \{\mathbb{T}(\mathbb{s}, x_{\mathbb{g}}) : \mathbb{g} \in \mathbb{G}\}$ is a $\lambda\rho$-pseudo-orbit of Ψ. Now let $y \in \Upsilon$ and $\varphi \in \mathbb{A}(\Upsilon)$, then using (11) we have $\mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y), \varphi(\mathbb{s}, y)) \le \lambda \mathbb{d}_\chi\left(\mathbb{T}(\mathbb{s}^{-1}, \Psi(\mathbb{s}, y)), \mathbb{T}(\mathbb{s}^{-1}, \varphi(\mathbb{s}, y))\right) = \lambda \mathbb{d}_\chi\left(\Phi(\mathbb{s}, \mathbb{T}(\mathbb{s}^{-1}, y)), \psi(\mathbb{s}, \mathbb{T}(\mathbb{s}^{-1}, y))\right) = \lambda \mathbb{d}_\chi(\Phi(\mathbb{s}, x), \psi(\mathbb{s}, x))$,

Where $x = \mathbb{T}(\mathbb{s}^{-1}, y)$. Therefore $\mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y), \varphi(\mathbb{s}, y)) \le \lambda \mathbb{d}_\chi(\Phi(\mathbb{s}, x), \psi(\mathbb{s}, x))$ for every $y \in \Upsilon$ and $x = \mathbb{T}(\mathbb{s}^{-1}, y)$, hence

$$\sup_{y\in\Upsilon} \mathbb{d}_\Upsilon\left(\Psi(\mathbb{s}, y), \varphi(\mathbb{s}, y)\right) \le \lambda \sup_{x\in\chi} \mathbb{d}_\chi(\Phi(\mathbb{s}, x), \psi(\mathbb{s}, x)). \tag{14}$$

From (13) and (14) and (9) we get for any $\varphi \in \mathbb{A}(\Upsilon)$, where $\varphi = \mathbb{T} \circ \psi \circ \mathbb{T}^{-1}$, that $\lambda\rho + \sup_{y\in\Upsilon} \mathbb{d}_\Upsilon\left(\Psi(\mathbb{s}, y), \varphi(\mathbb{s}, y)\right) \le \lambda\beta \le \lambda\beta'$. Since Ψ is $(\alpha', \lambda\beta')$-bi-shadowing that is (there exists $0 < \rho' = \lambda\rho \le \lambda\beta'$ such that for any ρ'-pseudo-orbit $\{y_{\mathbb{g}} : \mathbb{g} \in \mathbb{G}\}$ of Ψ and any action $\varphi: \mathbb{G} \times \Upsilon \to \Upsilon$ satisfying $\mathbb{d}_{\Upsilon_0}(\Phi, \varphi) \le \beta' - \rho'$, then there exists a true-orbit $\{z_{\mathbb{g}} : \mathbb{g} \in \mathbb{G}\}$ of φ such that $\mathbb{d}_\Upsilon(y_{\mathbb{g}}, z_{\mathbb{g}}) \le \alpha'\left(\rho' + \mathbb{d}_{\Upsilon_0}(\Psi, \varphi)\right)$).

Then,

$$\mathbb{d}_\Upsilon(y_{\mathbb{g}}, z_{\mathbb{g}}) \le \alpha'\left(\rho' + \mathbb{d}_{\Upsilon_0}(\Psi, \varphi)\right) = \alpha'\left(\lambda\rho + \sup_{y\in\Upsilon} \mathbb{d}_\Upsilon(\mathbb{s}, \Psi(y), \varphi(\mathbb{s}, y))\right) \le \alpha\left(\lambda\rho + \lambda \sup_{x\in\chi} \mathbb{d}_\chi(\Phi(\mathbb{s}, x), \psi(\mathbb{s}, x))\right)$$
$$= \lambda\alpha\left(\rho + \sup_{x\in\chi} \mathbb{d}_\chi(\Phi(\mathbb{s}, x), \psi(\mathbb{s}, x))\right).$$

Thus, using (10) we obtain

$$\mathbb{d}_\chi\left(x_{\mathbb{g}}, \mathbb{T}(\mathbb{s}^{-1}, z_{\mathbb{g}})\right) = \mathbb{d}_\chi\left(\mathbb{T}(\mathbb{s}^{-1}, y_{\mathbb{g}}), \mathbb{T}(\mathbb{s}^{-1}, z_{\mathbb{g}})\right) \le \lambda\alpha\left(\rho + \sup_{x\in\chi} \mathbb{d}_\chi(\Phi(\mathbb{s}, x), \psi(\mathbb{s}, x))\right),$$

Note that $\{\mathbb{T}(\mathbb{s}^{-1}, z_{\mathbb{g}}) : \mathbb{g} \in \mathbb{G}\} = \{b_{\mathbb{g}} : \mathbb{g} \in \mathbb{G}\}$ is a true-orbit of ψ since

$$\psi(\mathbb{s}, b_{\mathbb{g}}) = \psi\left(\mathbb{s}, \mathbb{T}(\mathbb{s}^{-1}, z_{\mathbb{g}})\right) = \mathbb{T}\left(\mathbb{s}^{-1}, \varphi(\mathbb{s}, z_{\mathbb{g}})\right) = \mathbb{T}(\mathbb{s}^{-1}, z_{\mathbb{g}\mathbb{s}}) = b_{\mathbb{g}\mathbb{s}}.$$

This shows that Φ is $(\lambda\alpha, \beta, \mathbb{A}(\chi))$-bi-shadowing. ■

Remark 3.4. In Theorem 3.3, if $\lambda = 1$, then we conclude that the action $\Phi: \mathbb{G} \times \chi \to \chi$ is $(\alpha, \beta, \mathbb{A}(\chi))$-bi-shadowing if and only if $\Psi: \mathbb{G} \times \Upsilon \to \Upsilon$ is $(\alpha', \beta', \mathbb{A}(\Upsilon))$-bi-shadowing.

As for the non-parametric bi-shadowing for topologically conjugate systems, we conclude the following:

Theorem 3.5. Let $(\chi, \mathbb{d}_\chi)$ and $(\Upsilon, \mathbb{d}_\Upsilon)$ be metric group spaces and let $\Phi: \mathbb{G} \times \chi \to \chi$ and $\Psi: \mathbb{G} \times \Upsilon \to \Upsilon$ be actions that are topologically conjugate by $\mathbb{T}: \mathbb{G} \times \chi \to \Upsilon$. Assume also that $\mathbb{A}(\chi)$ and $\mathbb{A}(\Upsilon)$ are $(\mathbb{A}(\chi), \mathbb{A}(\Upsilon))$-topologically conjugate. If there exists $\lambda \ge 1$ such that equation (2) is valid then we have

If Φ is bi-shadowing with respect to the comparison class $\mathbb{A}(\chi)$, then Ψ is $\mathbb{A}(\Upsilon)$-bi-shadowing.

Proof. Assume that for any $\beta' > 0$ there exists $\alpha' > 0$ and $0 \leq \rho' \leq \beta'/\alpha'$, let $\{y_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ be a ρ'-pseudo-orbit of an action Ψ, which implies that $\mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y_{\mathbb{g}}), y_{\mathbb{gs}}) < \rho'$, and let $\varphi \in \mathbb{A}(\Upsilon)$ satisfy $\mathbb{d}_{\Upsilon_0}(\Psi, \varphi) = \sup_{y\in\Upsilon} \mathbb{d}_\Upsilon(\Psi(y), \varphi(y)) \leq \beta' - \rho'$.

$$\rho' + \sup_{y\in\Upsilon} \mathbb{d}_\Upsilon(\Psi(y), \varphi(y)) \leq \beta'. \tag{15}$$

Note that condition (2) is equivalent to the following condition:

$\mathbb{d}_\chi(\mathbb{T}(\mathbb{s}^{-1}, y'), \mathbb{T}(\mathbb{s}^{-1}, y'')) \leq \mathbb{d}_\Upsilon(y', y'') \leq \lambda \mathbb{d}_\chi(\mathbb{T}(\mathbb{s}^{-1}, y'), \mathbb{T}(\mathbb{s}^{-1}, y''))$, for $\mathbb{s} \in \mathbb{S}$ and for all $y', y'' \in \Upsilon$.

Now,

$$\mathbb{d}_\chi\left(\Phi\left(\mathbb{s}, \left(\mathbb{T}(\mathbb{s}^{-1}, y_{\mathbb{g}})\right)\right), \mathbb{T}(\mathbb{s}^{-1}, y_{\mathbb{gs}})\right) = \mathbb{d}_\chi\left(\mathbb{T}\left(\mathbb{s}^{-1}, \Psi(y_{\mathbb{g}})\right), \mathbb{T}(\mathbb{s}^{-1}, y_{\mathbb{gs}})\right) \leq \mathbb{d}_\Upsilon(\Psi(y_{\mathbb{g}}), y_{\mathbb{gs}}) < \rho'.$$

Hence $\{x_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\} = \{\mathbb{T}(\mathbb{s}^{-1}, y_{\mathbb{g}}): \mathbb{g} \in \mathbb{G}\}$ is a ρ'-pseudo-orbit of Φ. Let $x \in \chi$ and $\psi \in \mathbb{A}(\chi)$, then using (2) and the conjugacy $\mathbb{T}$, we obtain

$$\mathbb{d}_\chi(\Phi(\mathbb{s}, x), \psi(\mathbb{s}, x)) \leq \mathbb{d}_\Upsilon\left(\mathbb{T}(\mathbb{s}, \Phi(\mathbb{s}, x)), \mathbb{T}(\mathbb{s}, \psi(\mathbb{s}, x))\right) = \mathbb{d}_\Upsilon\left(\Psi(\mathbb{s}, \mathbb{T}(\mathbb{s}, x)), \varphi(\mathbb{s}, \mathbb{T}(\mathbb{s}, x))\right) = \mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y), \varphi(\mathbb{s}, y)),$$

Where $y = \mathbb{T}(\mathbb{s}, x)$. This implies that

$$\sup_{x\in\chi} \mathbb{d}_\chi(\Phi(\mathbb{s}, x), \psi(\mathbb{s}, x)) \leq \sup_{y\in\Upsilon} \mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y), \varphi(\mathbb{s}, y)). \tag{16}$$

From (15) and (16) and for any $\psi \in \mathbb{A}(\chi)$, where $\psi = \mathbb{T}^{-1} \circ \varphi \circ \mathbb{T}$, we have $\rho' + \sup_{x\in\chi} \mathbb{d}_\chi(\Psi(\mathbb{s}, x), \psi(\mathbb{s}, x)) \leq \beta'$.

Since Φ is bi-shadowing that is (for any $\beta > 0$ there exists $\alpha = \alpha'/\lambda > 0$ and $0 < \rho = \rho' \leq \beta/\alpha$ such that for any ρ-pseudo-orbit $\{x_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ of Φ and any action $\psi: \mathbb{G} \times \chi \to \chi$ satisfying $\mathbb{d}_{\chi_0}(\Phi, \psi) \leq \beta - \rho$, then there exists a true-orbit $\{w_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ of ψ such that $\mathbb{d}_\chi(x_{\mathbb{g}}, w_{\mathbb{g}}) \leq \alpha\left(\rho + \mathbb{d}_{\chi_0}(\Phi, \psi)\right)$).

Then,

$$\mathbb{d}_\chi(x_{\mathbb{g}}, w_{\mathbb{g}}) \leq \alpha\left(\rho + \mathbb{d}_{\chi_0}(\Phi, \psi)\right) = \alpha\left(\rho + \sup_{x\in\chi} \mathbb{d}_\chi(\Phi(\mathbb{s}, x), \psi(\mathbb{s}, x))\right) \leq \alpha'/\lambda\left(\rho' + \sup_{y\in\Upsilon} \mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y), \varphi(\mathbb{s}, y))\right).$$

Thus, using the second part of (2), we obtain

$$\mathbb{d}_\Upsilon\left(y_{\mathbb{g}}, \mathbb{T}(\mathbb{s}, w_{\mathbb{g}})\right) = \mathbb{d}_\Upsilon\left(\mathbb{T}(\mathbb{s}, x_{\mathbb{g}}), \mathbb{T}(\mathbb{s}, w_{\mathbb{g}})\right) \leq \lambda \mathbb{d}_\chi(x_{\mathbb{g}}, w_{\mathbb{g}}) = \alpha'\left(\rho' + \sup_{y\in\Upsilon} \mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y), \varphi(\mathbb{s}, y))\right).$$

Note that $\{\mathbb{T}(\mathbb{s}, w_{\mathbb{g}}): \mathbb{g} \in \mathbb{G}\} \coloneqq \{a_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ is a true-orbit of φ since

$$\varphi(\mathbb{s}, a_{\mathbb{g}}) = \varphi\left(\mathbb{s}, \mathbb{T}(\mathbb{s}, w_{\mathbb{g}})\right) = \mathbb{T}\left(\mathbb{s}, \psi(\mathbb{s}, w_{\mathbb{g}})\right) = \mathbb{T}(\mathbb{s}, w_{\mathbb{gs}}) = a_{\mathbb{gs}}.$$

This ends the proof that Ψ is $\mathbb{A}(\Upsilon)$-bi-shadowing. ■

Remark 3.6. In the preceding theorem, if $\lambda = 1$, then we conclude that the action $\Phi: \mathbb{G} \times \chi \to \chi$ is $\mathbb{A}(\chi)$-bi-shadowing if and only if $\Psi: \mathbb{G} \times \Upsilon \to \Upsilon$ is $\mathbb{A}(\Upsilon)$-bi-shadowing.

Conclusion

Shadowing theory plays a crucial role in understanding the behavior and stability of dynamical systems. It allows for the validation of computer-generated pseudo-orbits by confirming the presence of true orbits nearby. Walters P. first introduced the concept of shadowing, and further developments such as inverse shadowing and bi-shadowing have expanded its scope. This paper investigates the principles of bi-shadowing, topological conjugacy, and the conditions under which these properties transfer between dynamical systems. The results obtained illustrate the importance of shadowing theory in both theoretical and practical aspects of dynamical systems, highlighting its continued relevance in mathematical research and applications.

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