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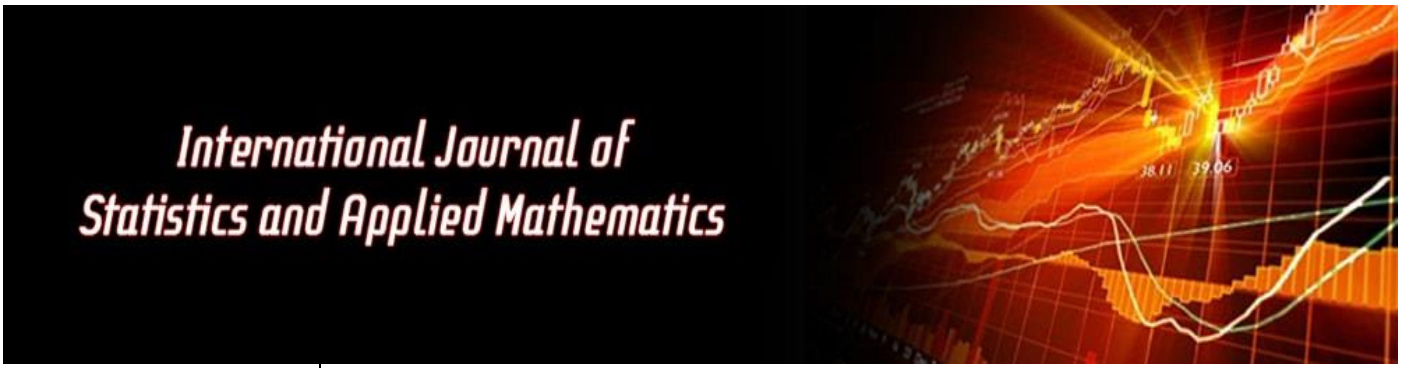

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Generalization of convergence of a class of iterative sequences with variable coefficients

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Abstract

This paper studies the sequence $\{x_n\}$ defined by $x_{n+1} = x_n + \frac{a_n}{x_n^{p_n}}$. First, A necessary and sufficient condition of convergence of $\{x_n\}$ is proved. Second, The corresponding equivalent quantity is given. Finally, we give some examples as application.

Keywords: Iterative sequence; equivalent quantity; convergence order; monotonicity

1. Introduction

Zhu [1, Example 1.7.8] discussed the convergence of the sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n + \frac{1}{n^\alpha x_n}. \tag{1.1}$$

Wu, He and Huang [2, Example 3] explored equivalent infinity for $0 < \alpha \leq 1$ for $\{x_n\}$. Huang and Lei [3] considered the convergence and equivalent quantity of $\{x_n\}$, which is given by

$$x_{n+1} = x_n + \frac{a_n}{x_n{}^p}, x_1 > 0, p > 0,$$

by extending $\frac{1}{n^\alpha}$ in (1.1) to a positive series $\{a_n\}$. In the present paper, we generalize the constant p to a convergent sequence $\{p_n\}$ and investigate the convergence and equivalent quantity of the sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n + \frac{a_n}{x_n{}^{p_n}}, x_1 > 0, \tag{1.2}$$

where,
$a_n > 0,\ n \in N^*$, $\lim\limits_{n\to\infty} p_n = p$, $p > 0$.

2 The main results

2.1 Criterion of convergence of the sequence $\{x_n\}$

Theorem 1 The sequence $\{x_n\}$ defined by (1.2) is convergent if and only if the positive series $\sum a_n$ is convergent.

Proof Necessity. Let $\lambda = \lim\limits_{n\to\infty} x_n$. The positive series $\sum(x_{n+1} - x_n)$ converges since $\{x_n\}$ is convergent. By (1.2) we know that $x_{n+1} - x_n = \frac{a_n}{x_n{}^{p_n}} \sim \frac{a_n}{\lambda^p}$

According to the comparison test $\frac{\sum a_n}{\lambda^p}$ converges and thus $\sum a_n$ converges.

Sufficiency. Let $\sum a_n$ converge. Since $\lim_{n\to\infty} p_n = p$, it follows that $\forall 0 < \varepsilon < p$, $\exists N > 0$,

such that if $n > N$, then $p - \varepsilon < p_n < p + \varepsilon$. Thus we have if $x_n < 1$, $\forall n \in N^*$, then

$\frac{a_n}{{x_n}^{p_n}} \le \frac{a_n}{{x_1}^{p+\varepsilon}}$, $\forall n \in N^*$;

if $\exists n_0 > 0$ such that $x_{n_0} \ge 1$, then

$\frac{a_n}{{x_n}^{p_n}} \le \frac{a_n}{{x_{n_0}}^{p-\varepsilon}}$, $\forall n > n_0$.

since $0 \le x_{n+1} - x_n = \frac{a_n}{{x_n}^{p_n}}$, the comparison test leads to $\sum (x_{n+1} - x_n)$ convergent, i.e., $\{x_n\}$ converges.

2.2 Equivalent quantity for the convergent series

First we prove the lemma which occurred in [3] but is not verified there.

Lemma 1 Let $a_i, b_i > 0$, $i = 1,2,\cdots,m$, and $\frac{a_1}{b_1} \le \frac{a_2}{b_2} \le \cdots \frac{a_m}{b_m}$. Then

$\frac{a_1}{b_1} \le \frac{a_1+a_2+\cdots a_m}{b_1+b_2+\cdots b_m} \le \frac{a_m}{b_m}$.

Proof Let $\frac{a_1}{b_1} = P$, $\frac{a_m}{b_m} = Q$. Thus by assumption we have $a_2 \ge P b_2$, …, $a_m \ge P b_m$ and thus

$\frac{a_1+a_2+\cdots a_m}{b_1+b_2+\cdots b_m} \ge \frac{Pb_1+Pb_2+\cdots Pb_m}{b_1+b_2+\cdots b_m} = P = \frac{a_1}{b_1}$.

Similarly,

$\frac{a_1+a_2+\cdots a_m}{b_1+b_2+\cdots b_m} \le \frac{Qb_1+Qb_2+\cdots Qb_m}{b_1+b_2+\cdots b_m} = Q = \frac{a_m}{b_m}$.

Theorem 2 Let the positive series $\sum a_n$ converges,$\lambda = \lim_{n\to\infty} x_n$, where the series$\{x_n\}$ is defined by (1.2). If one of the following two conditions is satisfied:

1) $x_1 \ge 1$ or $a_1 \ge 1$, and the sequence $\{p_n\}$ is increasing;
2) $\sum_{n=1}^{\infty} a_n = A$, $\frac{A}{{x_1}^{p_1}} + x_1 \le 1$, and the sequence $\{p_n\}$ in decreasing,

then $\lambda - x_n \sim \frac{1}{\lambda^p} \sum_{k=n}^{\infty} a_k$.

Proof (1) Due to $\lim_{n\to\infty} p_n = p$, there holds that $\forall \varepsilon_1 > 0$, $\exists N_1 > 0$, such that if $n \ge N_1$, then

$$p - \varepsilon_1 < p_n < p + \varepsilon_1. \quad (2.1)$$

Since $x_1 \ge 1$ or $a_1 \ge 1$, we have that $\forall n \in N^*$,

$$x_{n+1} = x_n + \frac{a_n}{{x_n}^{p_n}} \ge 1. \quad (2.2)$$

Note that $\{x_n\}$ is increasing. By (2.1) 和 (2.2),

$x_{n+1} - x_n = \frac{a_n}{{x_n}^{p_n}} \ge \frac{a_n}{\lambda^{p+\varepsilon_1}} > 0, n > N_1$.

Therefore there holds that $\forall m, n > N_1$,

$x_{n+m+1} - x_n = \sum_{k=1}^{m+1}(x_{n+k} - x_{n+k-1}) \geq \frac{1}{\lambda^{p+\varepsilon_1}}\sum_{k=1}^{m+1} a_{n+k-1} = \frac{1}{\lambda^{p+\varepsilon_1}}\sum_{k=n}^{n+m} a_k.$ (2.3)

On the other hand, the monotonicity of $\{x_n\}$, $\{p_n\}$ and (2.2) yield that $\{x_n{}^{p_n}\}$ is increasing and thus

$\frac{x_{n+1}-x_n}{a_n} = \frac{1}{x_n{}^{p_n}}$

with respect to n is decreasing. So
$\frac{x_{n+1}-x_n}{a_n} \geq \frac{x_{n+2}-x_{n+1}}{a_{n+1}} \geq \cdots \geq \frac{x_{n+m+1}-x_{n+m}}{a_{n+m}}, \forall m, n \in N^*$.

By Lemma 1,

$\frac{x_{n+1}-x_n}{a_n} \geq \frac{x_{n+m-1}-x_n}{a_n+a_{n+1}+\cdots+a_{n+m}} > 0, \forall m, n \in N^*.$ (2.4)

Since $x_{n+1} - x_n = \frac{a_n}{x_n{}^{p_n}}$, it follows that $\forall \varepsilon_2 > 0$, $\exists N_2 > 0$, such that if $n > N_2$, then

$\frac{x_{n+1}-x_n}{a_n} < (1+\varepsilon_2)\frac{1}{\lambda^p}.$

Therefore by (2.4) we know that when $n > N_2$ and $m \in N^*$,

$\frac{x_{n+m-1}-x_n}{a_n+a_{n+1}+\cdots+a_{n+m}} < (1+\varepsilon_2)\frac{1}{\lambda^p}.$ (2.5)

Thus by (2.3) and (2.5) it follows that if $n > max(N_1, N_2)$ and $m \in N^*$, then

$$\frac{1}{\lambda^{p+\varepsilon_1}}\sum_{k=n}^{n+m} a_k \leq x_{n+m+1} - x_n < (1+\varepsilon_2)\frac{1}{\lambda^p}\sum_{k=n}^{n+m} a_k$$

Letting $m \to \infty$, we have

$\frac{1}{\lambda^{p+\varepsilon_1}}\sum_{k=n}^{\infty} a_k \leq \lambda - x_n < (1+\varepsilon_2)\frac{1}{\lambda^p}\sum_{k=n}^{\infty} a_k, \forall n > max(N_1, N_2),$

and thus

$\lambda - x_n \sim \frac{1}{\lambda^p}\sum_{k=n}^{\infty} a_k.$

(2) Since $\{x_n\}$ is increasing, $\{p_n\}$ is decreasing and $x_1 \leq 1$, we get that

$x_n{}^{p_n} \geq x_1{}^{p_1}$

and thus

$x_{n+1} - x_n = \frac{a_n}{x_n{}^{p_n}} \leq \frac{a_n}{x_1{}^{p_1}}, \forall n \in N^*.$

So

$x_{n+1} - x_1 \leq \frac{\sum_{i=1}^{n} a_i}{x_1{}^{p_1}} \leq \frac{A}{x_1{}^{p_1}}.$

In the light of $\frac{A}{x_1{}^{p_1}} + x_1 \leq 1$, we have that

$x_n \leq 1, \ \forall n \in N^*.$

Hence $\{x_n{}^{p_n}\}$ is increasing and thus $\frac{x_{n+1}-x_n}{a_n} = \frac{1}{x_n{}^{p_n}}$ is decreasing with respect to n. On the other hand, by (2.1),

$x_{n+1} - x_n = \frac{a_n}{x_n{}^{p_n}} \geq \frac{a_n}{\lambda^{p-\varepsilon_1}} > 0, n > N_1.$

Using the same method as in (1) we obtain

$\frac{1}{\lambda^{p-\varepsilon_1}}\sum_{k=n}^{\infty} a_k \leq \lambda - x_n \leq (1+\varepsilon_2)\frac{1}{\lambda^p}\sum_{k=n}^{\infty} a_k, \forall n > max(N_1, N_2),$

which implies that

$\lambda - x_n \sim \frac{1}{\lambda^p}\sum_{k=n}^{\infty} a_k.$

3 Applications

Example 1 Let $x_{n+1} = x_n + \frac{\frac{1}{4n(n+1)(n+2)}}{x^{1+\sqrt{1+\frac{1}{n^2}}}}$, $x_1 = \frac{1}{2}$. Please judge the convergence of $\{x_n\}$. Give the equivalent quantity of $\lambda - x_n$ if $\{x_n\}$ converges where $\lambda = \lim_{n\to\infty} x_n$.

Solution Let

$a_n = \frac{1}{4n(n+1)(n+2)}, \quad p_n = 1 + \sqrt{1+\frac{1}{n^2}}.$

Then

$s_n = \sum_{i=1}^{n} a_n = \sum_{k=1}^{n}\frac{1}{8}\left[\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\right] = \frac{1}{8}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right].$

Thus

$A = \sum_{n=1}^{\infty} a_n = \frac{1}{16}.$

Hence $\{x_n\}$ converges by Theorem 1. Note that

$\sum_{k=n}^{\infty} a_k = \sum_{k=n}^{\infty}\frac{1}{8}\left[\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\right] = \frac{1}{8}\frac{1}{n(n+1)},$

$\{p_n\}$ is decreasing, $\lim_{n\to\infty} p_n = 2$, and

$\frac{A}{x_1{}^{p_1}} + x_1 = \frac{1}{16} \div \left(\frac{1}{2}\right)^{1+\sqrt{2}} + \frac{1}{2} < 1.$

Thus by Theorem 2 (2)

$\lambda - x_n \sim \frac{1}{8\lambda^2 n(n+1)} \sim \frac{1}{8\lambda^2 n^2}.$

Sometimes the estimate of the sum of series is not easy to get and under certain condition the sum of series can transfer to infinite integral.

Lemma 2 [3] Let non-negative function $f(x)$ be non-increasing in $[a, +\infty)$, $a > 0$, $a_n = f(n)$ and $\sum_{n=1}^{\infty} a_n$ converge. If $\lim_{n\to\infty}\frac{a_n}{\int_n^{+\infty} f(x)dx} = 0$ or $\lim_{n\to\infty}\frac{a_n}{\sum_{k=n}^{\infty} a_k} = 0$, then

$\sum_{k=n}^{\infty} a_k \sim \int_n^{+\infty} f(x)dx.$

Example 2 Let $x_{n+1} = x_n + \frac{1}{n^2 x_n^{\frac{n}{n+1}}}$, $x_1 > 0$. Please judge the convergence of $\{x_n\}$. Give the equivalent quantity of $\lambda - x_n$ if $\{x_n\}$ converges where $\lambda = \lim_{n\to\infty} x_n$.

Solution Let $a_n = \frac{1}{n^2}$, $p_n = \frac{n}{n+1}$. Obviously $\sum a_n$ converges . Thus $\{x_n\}$ converges by Theorem 1. Since $\{p_n\}$ is increasing, $\lim_{n\to\infty} p_n = 1 > 0$ and $a_1 = 1$, by Theorem 2 (1) we get that

$\lambda - x_n \sim \frac{1}{\lambda^p}\sum_{k=n}^{\infty} a_k = \frac{1}{\lambda}\sum_{k=n}^{\infty}\frac{1}{k^2}.$

Furthermore,

$\int_{n}^{+\infty} \frac{dx}{x^2} = \frac{n^{1-2}}{2-1} = \frac{1}{n}$,

Thus by Lemma 2,

$\lambda - x_n \sim \frac{1}{n\lambda}$.

4. Conclusion
This paper examines the convergence of iterative sequences, providing a comprehensive criterion and equivalent quantities. We demonstrate that the convergence of a sequence, $\{x_n\}$, depends on the convergence of the series $\sum a_n$, as shown in Theorem 1. Additionally, we introduce the equivalent quantity for the convergent sequence, utilizing both theoretical proofs and practical applications to solidify the findings. Through examples, we confirm that the convergence of sequences is not only defined by the nature of the terms but also by the rate at which they approach infinity. This study provides valuable insights for analyzing and predicting the behavior of complex iterative sequences.

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