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Exploring applications of fractional differential equations with radial basis function

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Abstract
Solving fractional order differential equations is substantially more complex than solving ordinary order differential equations; also, the majority of computing tools do not include built-in functions for this type of issue. In this article, we establish the fundamentals of fractional order differential equations such as Riemann-Liouville, Caputo, Riesz fractional, etc. Also, we provide a brief overview of the fundamental techniques for solving linear fractional differential equations. Due to diverse applications in solving various models based on fractional-order differential equations, the invention and exploration of numerical methods to find their solutions. Radial basis function methods are one of such methods to find the more accurate and convergent solutions compared to the other methods.

Keywords: Fractional calculus, radial basis function, fractional differential equation, riemann-liouville

Introduction
A mathematical model based on fractional calculus is the demanding area of today's research. Therefore, practical problems based on differential equations with fractional derivatives are currently in vogue, offering a wide range of applications across various fields. Differential equations have an operator that contains integer-order derivatives similar to the fractional integrals. When the order of the integration is ambiguous, the fractional integral can be employed for the better accumulation of a quantity. Similar to how the fractional order derivative is often used to describe damping. It's implication also in the disciplines of viscoelasticity fluid flow, diffusion-like diffusive transport, control theory of dynamical systems, probability, electrical networks, statistics, dynamical phenomena in conscience and hollow formations, ophthalmology, and image analysis. In the applied sciences and engineering over the past ten years, fractional calculus has become an increasingly useful instrument for the description of a wide variety of non-classical events. Many problems in surface and subsurface hydrology [1, 2], finance [3, 4], epidemiology [5, 6], plasma turbulence [7, 8], and ecology [9, 10] have been modeled using fractional-order differential operators.
In the context of nonlinear studies, nonlinear partial differential equations (PDEs) have gained popularity due to their ability to address various issues in the fields of ecology, epidemiology, economic systems, quantum physics, and image processing. Many physical applications, including wave dispersion and propagation, supersonic and turbulent flow, magneto-hydrodynamic movement through pipes, computational fluid dynamics, population modeling, magnetic resonance imaging, medical imaging, electrically signaling nerves, and many more, frequently make use of PDEs [11-13]. Take a look at the source referenced in [14] to learn more. An accurate assessment of COVID patients analyzed in [15, 16] proved the prevalent nature of partial differential equations. As shown in [17], PDE can be used to estimate the resultant form of COVID-19. By this analysis, the FPDE gives more accuracy comparatively to the results of the integer-order PDE for a number of exigent problems in these fields. So this seems to be very crucial for developing computational approaches for finding the results of fractional partial differential equations. Approximate solutions of various FPDEs can be obtained by adopting the concept of symmetry that is widely used to simplify mathematical models in [18, 19].

This is a useful tool to deal with fractional PDEs on a wider scale and can be highly useful in describing the underlying physical processes. Some of the references [20, 21] are mentioned for providing the general idea of FPDEs with their applications. Authors in [22, 23] proposed an all-over review of fractional calculus, FDEs, and their practices in several fields. The fundamental concept of FDEs and their practical implications have been studied in relation to viscoelasticity [24, 25].

Numerical techniques available in the literature with applications

A large number of FDEs are challenging to solve analytically because of the intricacy of fractional orders. Therefore, numerical simulation for the previously discussed FDEs is discovered. Numerous algorithms for numerical computation have been developed for this purpose [26, 27]. Spectrum methods are numerical techniques for approximating the outcome of differential equations. There are several spectrum approaches that have been applied to the solution of FDEs, including the tau, collocation, and Galerkin methods. The tau technique is a spectral approach for approximating the solution of FDEs that makes use of a shifted Legendre polynomial basis. This collocation method approximates the results of FDEs with collocation regions using a basis of orthogonal polynomials. The efficiency of spectral approaches for solving FDEs has been shown in numerous publications [28-34]. In general, time-fractional differential equations and FDEs can be solved using spectrum techniques like the tau, collocation, and Galerkin methods. These techniques are excellent tools for modeling and simulation in a variety of domains because they deliver precise solutions with high efficiency.

The applications of FPDEs to simulate complicated multi-scale phenomena with overlapped microscopic and macroscopic aspects have become more and more apparent. FPDEs of fractional order can be a function of time as well as space or even a distribution respect to integer-order PDEs. Numerous approaches have been addressed for finding numerical results of FPDEs as time-fractional phi-four equations [35], the reduced differential transform method for coupled time fractional nonlinear evolution equations [36], the Yang transform decomposition method for fractional-order diffusion equations [37] and the natural transform decomposition method for the solution of fractional Caudrey-Dodd-Gibbon equations [38], fractional Kuramoto-Sivashinsky equations [39], fractional homotopy analysis method for solving the fractional epidemic model [40] and fractional KdV-Burgers-Kuramoto equation [41], homotopy perturbation transform method for solving fractional Noyes-Field model [42] and time-fractional Fisher's equation [43], variational iteration transform method for fractional-order Newell-Whitehead-Segel equations [44] and for fractional-order Boussinesq equation [45], approximate analytical method for the solution of time-fractional telegraph equations [46] and the Adams-Bashforth method to study the time-fractional Tricomi equation with nonlocal and nonsingular kernel [47] and many more [48-54].

This paper focuses on the review of fractional differential equations and their applicability with radial basis functions. Therefore, mesh-free radial function based global techniques are provided in our study for the numerical simulation of a class of fractional differential equations. This article follows the pattern as: Section 2 gives some basic calculus theory, like formulas for fractional integrals and derivatives, which is useful for the derivation of FDEs. Section 3 presents the methods available in literature for solving FDEs and also focuses on the methodology using RBF for solving the FDE. The whole article is concluded in the last section.

Basics of Fractional Calculus

Differential equations are solved by some rules that are based on some special functions. Here, we mentioned some basic functions for fractional calculus.

Gamma functions

The base function for fractional calculus is Euler's gamma function defined as:

i) For natural numbers as $\Gamma(n) = (n-1)!$

ii) For a complex no. X, $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$ is absolutely convergent and also known as Euler's integral equation of second kind. Also $\Gamma(\mathfrak{z}+1) = \mathfrak{z}\Gamma(\mathfrak{z})$

iii) For $-r < Re(\mathfrak{z}) \le -r+1$, $\Gamma(\mathfrak{z}) = \frac{\Gamma(\mathfrak{z}+r)}{\mathfrak{z}(\mathfrak{z}+1)\ldots(\mathfrak{z}+r-1)}$

Derivative calculated as

$$\frac{d^n}{dx^n}\Gamma(x) = \int_0^\infty e^{-t}\,(\ln t)^n\, t^{x-1}\, dt, x > 0$$

Beta function

$$\mathcal{B}(\mathrm{x},\mathrm{m}) = \int_0^1 (1-t)^{m-1} t^{x-1}\, dt = \frac{(m-1)!\,(x-1)!}{(m+x-1)!}$$

$= \mathcal{B}(\mathrm{m},\mathrm{x});\ x, m \in \mathbb{R}^+$

Relation of Beta function and Gamma function defined by $\mathcal{B}(x,m) = \frac{\Gamma(\mathrm{x})\Gamma(\mathrm{m})}{\Gamma(\mathrm{x+m})}$

Mittag Lefler (ML) Function

Exponential function has a vital role in the field of fractional calculus. So one parameter and two parameter ML function are totally based on exponential function (e^x *or* $\exp(x)$) which is defined by

$$E_\beta(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\beta r+1)}, \beta > 0$$

And

$E_{\beta,\gamma}(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\beta r+\gamma)};\ \beta > 0, \gamma > 0$ respectively.

Caputo Fractional Derivative (CFD)

CFD of a function with an order $\beta > 0$ on the interval [a,b] is defined with $m \in Z^+; m-1 < \beta \le m$

Left hand sided CFD

$${}_{C}D_{a^+}^{\beta} f(x) = \frac{1}{\Gamma(m-\beta)} \int_0^x (x-t)^{m-\beta-1} f^m(t)dt$$

Right hand sided CFD

$${}_{C}D_{b^-}^{\beta} f(x) = \frac{1}{\Gamma(m-\beta)} \int_0^x (t-x)^{m-\beta-1} f^m(t)dt,$$ respectively.

Interpolation in the Caputo operator is analyzed by

$$\lim_{\beta\to m} D_a^\beta f(x) = f^m(x)$$

$$\lim_{\beta\to m-1} D_a^\beta f(x) = f^{m-1}(x) - f^{m-1}(a)$$

The Caputo operator satisfied the condition of linearity but not the commutative law defined as

$$D_a^\beta[pf(x)+qg(x)] = pD_a^\beta f(x) + qD_a^\beta f(x)$$

And

$D_a^\beta D^m f(x) = D_a^{m+\beta} f(x) \neq D^m D_a^\beta$; respectively.

Riemann-Liouville (RL) operators

Riemann-Liouville Fractional Integral (RLFI)

RLFI of the left hand side and right hand side are defined below of order β on the interval [a, b], respectively.

$\left(I_{a^+}^\beta f\right)(x) = \frac{1}{\Gamma(\beta)}\int_a^x (x-t)^{\beta-1} f(t)dt$ and $\left(I_{b^-}^\beta f\right)(x) = \frac{1}{\Gamma(\beta)}\int_a^x (t-x)^{\beta-1} f(t)dt$

Following properties satisfied by the fractional integration:

$I_{a^+}^\beta I_{a^+}^\gamma f = I_{a^+}^{\beta+\gamma} f,\ I_{b^-}^\beta I_{b^-}^\gamma f = I_{b^-}^{\beta+\gamma} f;\ \beta > 0, \gamma > 0$

$I_{a^+}^\beta[pf(x)+qg(x)] = pI_{a^+}^\beta f(x) + qI_{a^+}^\beta g(x)$

$$I_{b^-}^\beta[pf(x)+qg(x)] = pI_{b^-}^\beta f(x) + qI_{b^-}^\beta g(x)$$

Riemann-Liouville Fractional Derivative (RLFD)

The left and right hand side RL fractional derivative (RLFD) of a function f of order β is defined on [a, b] as

$D_{a^+}^\beta f(x) = \frac{1}{\Gamma(m-\beta)}\frac{d^m}{dx^m}\int_a^x (x-t)^{m-\beta-1} f(t)dt$ and

$$D_{b^-}^\beta f(x) = \frac{(-1)^m}{\Gamma(m-\beta)}\frac{d^m}{dx^m}\int_x^b (t-x)^{m-\beta-1} f(t)dt$$

Or we can say that

$$D_{a^+}^\beta f = D^m I_a^{m-\beta} f$$

Interpolation in RLFD is defined by

$$\lim_{\beta\to m} D_a^\beta f(x) = D^m I_a^{m-\beta} f$$

RLFD satisfies the property of linearity but does not hold commutative.

Following properties satisfied by the fractional integration

$D_{a^+}^\beta I_{a^+}^\beta f(x) = f(x)\ and\ D_{b^-}^\beta I_{b^-}^\beta f(x) = f(x);\ \beta > 0$

$D^m D_{a^+}^\beta f(x) = D_{a^+}^{m+\beta} f(x)\ and\ D^m D_{b^-}^\beta f(x)$

$= D_{b^-}^{m+\beta} f(x)\ (if\ FDs\ exist)$

Relation between RLFD and CFD

${}_C D_{a^+}^\beta f(x) = D_{a^+}^\beta f(x) - \sum_{r=0}^{m-1}\frac{f^r(a)}{\Gamma(r+1-\beta)}(x-a)^{r-\beta}$

${}_C D_{b^-}^\beta f(x) = D_{b^-}^\beta f(x) - \sum_{r=0}^{m-1}\frac{(-1)^r f^r(b)}{\Gamma(r+1-\beta)}(b-x)^{r-\beta}$

Riesz-space fractional derivative

Riesz space fractional derivative of order β of function f(x,t) on [a, b] is defined by $\frac{\partial^\beta}{\partial|x|^\beta} f(x,t) = -C_\beta(left\ sided\ RLFD + Right\ sided\ RLFD)$, where $C_\beta = \frac{1}{2\cos\left(\frac{\pi\beta}{2}\right)}, \beta \neq odd\ numbers$

The entire operators (defined above) are applicable on functions with some properties (according to the requirement) that are defined below.

$D_{a^+}^\beta f(x) = D^m[\ I_{a^+}^{m-\beta} f(x)]$

$D_{b^-}^\beta f(x) = (-1)^m D^m[\ I_{b^-}^{m-\beta} f(x)]$

Left sided CFD $= I_{a^+}^{m-\beta}[f^m(x)]$

Right sided CFD $= (-1)^m I_{b^-}^{m-\beta}[f^m(x)]$

Mesh free Approaches for Fractional Equations

Since fractional derivatives are clearly non-local operators, even straightforward material models or unified theories have a high computational cost in terms of storage, processing time, and general complexity of numerical techniques. As a result, these local schemes lose their attractiveness as the coefficient matrices are no longer sparse. Therefore, due to their high precision and use of fewer discretization points, global approaches seem to offer certain advantages in numerical simulation of the fractional derivative models. Even though many of these approaches are precise and have higher orders of convergence, their expansion to high dimension systems can be time-consuming and can necessitate proper method modification. There are two different categories of numerical approaches for FPDE models. The first is the mesh-free approach, and the second is the mesh grid method. Researchers are currently paying more attention to mesh-free approaches to find numerical solutions to FPDEs, particularly in fractional orders.

Meshfree methods just need evenly distributed nodes in the domain because meshing is not necessary for them. Meshfree methods are more accurate than mesh grid methods and take less time to complete. Several well-known mesh-free techniques include homotopy analysis [55], variational iteration method [56], spectral method [57], adomian decomposition method [58], and radial basis function approach [59, 60]. With starting and boundary value issues, it is simple to implement, and there are a variety of generating functions available. Both local and global forms of the radial basis function (RBF) approach are employed. The generating functions used to implement the RBF method include Gaussian, multi-quadric, inverse multi-quadric, quadric, and so on. The primary challenge with these functions is selecting the shape parameter, which is up for debate and essential for the best numerical solution. Finding the numerical answers can usually be difficult due to this in a high dimensional model. The selection of the shape parameter is not particularly difficult for Gaussian, though.

Researchers proposed various mesh-less approaches in the literature focused on certain techniques with RBF in differences and structured meshes. One can derive the numerical solutions to various FPDEs by the collocation and RBF approaches over unevenly distributed data. Also, time-space dependent PDE [61] of fraction order is solved by the mesh-less RBF method with a Gaussian function and has various applications in physical sciences and chemical engineering. Authors prove this approach highly accurate in high dimensional models using derivatives in the Caputo sense for time dependent derivatives and Riemann-Liouville, Grunwald-Letnikov, and Riesz derivatives for space dependent derivatives.

A numerical approach with RBFs was proposed in [62] using Coimbra theory of fractional derivatives. The authors in [63] revealed a scheme for solving time fractional diffusion equations numerically by the mesh-free method locally with RBF using the Laplace transform. If the error decreases or stays constant over the course of computations, a time-stepping technique is stable. The approach proposed in [64] also confronts higher computing expenses because the best results are only attained for a relatively short time step.

Radial Basis Function (RBF)

A function $\Phi: \mathbb{R}^t \to \mathbb{R}$ is called radial if there exist a function of one variable $\Phi: [0,\infty) \to \mathbb{R}$ such that $\Phi(\mathrm{x}) = \varphi(\|\mathrm{x}\|)$, here the Euclidean norm$\|.\|$ is used. $\Phi(r)$ is a univariate continuous real valued radial basis function whose value is based upon a distance value that is measured from any fixed center point or the origin. From the definition, it is clear that Φ is a special function that is radially symmetric and only depends on the distance between points. The application of RBF to the high dimensional problem is easy, as the interpolation problem is insensitive to the space dimension. In all space dimensions, one can work with the function φ that is univariate instead of using a multivariate function Φ. RBFs are distinguished by the smoothness-piecewise smooth RBFs, which are free from shape parameter λ and infinitely differentiable, which have parameters called the shape parameterλ. Some of the piecewise smooth and infinitely smooth functions are the Cubic function, linear radial function, monomial and Gaussian function, multi quadric, inverse multi quadric etc. respectively.

RBF Discretization by Fractional Operators

Interpolation of RBF in 1-dim on the center node points$\mathbb{x}_j$, j varies from 1 to m, of the form is

$$S_{\sigma}(x) = \sum_{j=1}^{m} \gamma_j \varphi\left(\frac{|\mathbb{x}-\mathbb{x}_j|}{\epsilon}\right) \quad (1)$$

With interpolation coefficients γ_j. Here, $\varphi$is one of the available RBFs and ϵ is the shape parameter connected to RBF. These interpolation coefficients $\gamma_j$are unknown, and their value can be found by interpolating the following conditions at the collocation points $\mathbb{x}_\mathrm{i}$.

$$S_{\sigma}(\mathbb{x}_i) = \zeta_i, i = 1,2,\dots,m \quad (2)$$

This leads to a representation of a system of linear equations as follows:

$$\mathbb{A}\mathcal{X} = \mathfrak{B}; \text{ Where } \mathbb{A} \text{ is non-singular} \quad (3)$$

$$\mathbb{A} = \left(\varphi\left(\frac{|\mathbb{x}_i - \mathbb{x}_j|}{\epsilon}\right)\right)_{1\le i,j\le m}$$

$$\mathcal{X} = [\gamma_1, \gamma_2, \dots, \gamma_m]^T$$

$$\mathfrak{B} = [\zeta_1, \zeta_2, \dots, \zeta_m]^T$$

By taking a differential operator D^{α}(one of the operators defined above) whose discretization is done with expansion of radial basis function. Resultant is

$$\mathbb{g}_i = \sum_{j=1}^{m} \gamma_j \left(D^{\alpha}\varphi\left(\frac{|\mathbb{x}-\mathbb{x}_j|}{\epsilon}\right)\right)(\mathbb{x}_i), 1 \le i \le m \quad (4)$$

Where $\mathbb{g}_i$ are fractional operators of function at each point. After applyingD^{α}, the obtained results give a new system of equations as follows:

$$\mathbb{A}'\mathcal{X} = \mathfrak{B}' \quad (5)$$

Where $\mathbb{A}' = \left(D^{\alpha}\varphi\left(\frac{|\mathbb{x}-\mathbb{x}_j|}{\epsilon}\right)\right)$

$\mathfrak{B}' = [\mathbb{g}_1, \mathbb{g}_2, \dots, \mathbb{g}_m]^T$.

Equation (3) gives $\mathcal{X} = \mathbb{A}^{-1}\mathcal{B}$ by the non singularity of $\mathbb{A}$. So, eliminating $\mathcal{X}$ from equation (5), we get

$$\mathfrak{B}' = \mathbb{A}'\mathbb{A}^{-1}\mathfrak{B} \quad (6)$$

Here, $\mathbb{A}'\mathbb{A}^{-1}$ is a matrix which discretizesD^{α}.

Hence, for finding the $\left(D^{\alpha}\varphi\left(\frac{|.-\mathbb{x}_t|}{\epsilon}\right)\right)$; Define a function for an arbitrary no. t, $\varphi_t: \mathbb{R} \to \mathbb{R}$ such that $\varphi_t(\mathrm{x}) = \varphi\left(\frac{|\mathbb{x}-t|}{\epsilon}\right)$

Then evaluate $(D^{\alpha}\varphi_t)(\mathrm{x})$ for getting desired results.

Conclusion

This work provides a depth analysis of the fundamentals of fractional differential equations with RBFs. Also, discretization of fractional operators with RBF is proposed and numerical methods and their computational approach with RBF are also defined in this article for the successful implementation of scientific and engineering applications in the real world.

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