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Study of W_9: Curvature Tensors on Lorentzian Para-Kenmotsu Manifolds

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Abstract
This paper presents a study of the W_9-curvature tensor on Lorentzian Para-Kenmotsu manifolds, expanding upon existing research to explore areas including W_9 flatness, $\xi - W_9$ flatness, $\phi - W_9$ flatness, and W_9 -semi-symmetric on LP-Kenmotsu manifolds. Lorentzian Para-Kenmotsu manifolds satisfying the conditions;

$$Q.W_9 = 0, W_9.Q = 0, R.W_9 = 0, \text{ and } W_9.W_9 = 0.$$

The findings enhance knowledge of the geometric properties of these manifolds and offer fresh insights into the behavior of the W_9-curvature tensor within this context.

Keywords: Para-contact metric manifold, Lorentzian almost paracontact manifold, Lorentzian Para-Kenmotsu manifold, Einstein manifold, η −Einstein manifold, *W_9-curvature tensor, W_9 flat, $\xi - W_9$ flat, $\phi - W_9$ flat, and W_9 -semi-symmetric.*

Introduction

In 1989, [1] K. Matsumoto introduced Lorentzian paracontact manifolds, specifically LP-Sasakian manifolds, which have since been extensively studied by various geometers. Subsequent research by Matsumoto, Mihai, Rosca, Shaikh, De, Venkatesha, Pradeep Kumar, and Bagewadi focused on these manifolds with significant results [3]. In 1995, [4] Sinha and Sai Prasad defined para-Kenmotsu and special para-Kenmotsu manifolds, akin to P-Sasakian and SP-Sasakian manifolds. Abdul Haseeb and Rajendra Prasad in 2018 introduced Lorentzian Para-Kenmotsu (briefly LP-Kenmotsu) manifolds, studying ϕ-semisymmetric LP-Kenmotsu manifolds with a quarter-symmetric non-metric connection admitting Ricci solitons [6]. Njori et al. did several studies on $W_8 -$ curvature tensors on various types of manifolds, including Kenmotsu manifolds [2, 8].

In 1970, [5] Pokhariyal and Mishra introduced new tensor fields on a Riemannian manifold, called the Weyl-projective curvature tensor of type (1, 3) and the tensor field E. The Weyl-projective curvature tensor W_9 with respect to a Riemannian connection on a manifold M is defined as:

$$W_9(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(X,Y)Z - g(Y,Z)QX] \quad (2.1)$$

where $QX = (n-1)X$, which plays an important role in the theory of the projective transformations of connections.

Preliminaries

An n −dimensional differentiable manifold M admitting a (1,1) tensor field ϕ, contravariant vector field ξ, a 1 −form η and the Lorentzian metric g is called Lorentzian almost Paracontact manifold [7] if it satisfies:

$$i)\ \phi^2 X = X + \eta(X)\xi, \quad (3.1)$$

$ii)\ g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y)$ (3.2)

and

$i)\ \eta(\xi) = -1, (ii)\ \phi(\xi) = 0, \eta(\phi X) = 0, (iii)\ g(X,\xi) = \eta(X), (iv)\ rank\phi = n-1$ (3.3)

In a Lorentzian almost Paracontact manifold, we have

$\Phi(X,Y) = \Phi(Y,X)$ (3.4)

where $\Phi(X,Y) = g(X, \phi Y)$

A Lorentzian almost Paracontact manifold M is called Lorentzian Para-Kenmotsu (briefly LP-Kenmotsu) manifold [9] if

$(\nabla_X \phi)Y = -g(\phi X, Y) - \eta(Y)\phi X,$ (3.5)

for any vector fields X, and Y on M and ∇ is the operator of covariant differentiation with respect to the Lorentzian metric g.
In the LP-Kenmotsu manifold, the following relations hold:

$\nabla_X \xi = -\phi^2 X = -X - \eta(X)\xi$ (3.6)

$(\nabla_X \eta)Y = -g(X,Y) - \eta(X)\eta(Y)$ (3.7)

$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y)$ (3.8)

$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X$ (3.9)

$R(X,Y)\xi = \eta(Y)X - \eta(X)Y; when\ X\ is\ orthogonal\ to\ \xi,$ (3.10)

$R(\xi,X)\xi = X + \eta(X)\xi,$ (3.11)

$S(X,\xi) = (n-1)\eta(X),$ (3.12)

$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(\mathrm{Y})$ (3.13)

$Q\xi = (n-1)\xi$ (3.14)

Where S is Ricci Tensor and Q, Ricci operator.
A Lorentzian Para-Kenmotsu manifold M is said to be an $\eta -$ Einstein manifold if its Ricci-tensor $S(X,Y)$ is of the form

$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$

where a and b are scalar functions on M. In particular, if b = 0, then the manifold is said to be an Einstein manifold.

4.0. A $W_9 -$ flat Lorentzian para-Kenmotsu manifold
Definition 4.1: An n-dimensional LP-Kenmotsu manifold is said to be W_9-flat if
W_9 -curvature tensors satisfies the following condition

$W_9(X,Y)Z = 0\ or\ W(X,Y,Z,U) = 0$

Where $(g(W(X,Y)Z,U) = {'W}(X,Y,Z,U)$

Theorem 4.1: A W_9-flat LP-Kenmotsu manifold is an $\eta -$ Einstein manifold.
Proof.
Suppose the LP-Kenmotsu manifold is W_9 $-$flat, then the following hold

$W_9(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(X,Y)Z - g(Y,Z)QX] = 0$ (4.1)
Transforming W_9 $curvature\ tensor$ from (1, 3) to (0,4) tensor yields.

${'W}(X,Y,Z,U) = {'R}(X,Y,Z,U) + \frac{1}{n-1}[S(X,Y)g(Z,U) - g(Y,Z)S(X,U)] = 0$ (4.2)

${'R}(X,Y,Z,U) = \frac{1}{n-1}[g(Y,Z)S(X,U) - S(X,Y)g(Z,U)]$ (4.3)

Let $\{ei\}$ be an orthonormal basis of the tangent space at any point. Putting $\{Y = Z = e_i\}$ in the above equation and summing over $i, 1 \leq i \leq n-1$, we get

$$S(X,U) - g(X,U) - \eta(X)\eta(U) = \frac{1}{n-1}[(n-1)S(X,U) - \{S(X,U) - (n-1)\eta(X)\eta(U)\}]$$

$$S(X,U) - g(X,U) - \eta(X)\eta(U) = S(X,U) - \frac{S(X,U)}{n-1} + \eta(X)\eta(U)$$

$$S(X,U) = (n-1)g(X,U) + 2(n-1)\eta(X)\eta(U) \tag{4.4}$$

Hence, the proof that W_9 −flat LP-Kenmotsu manifold is an $\eta -$ *Einstein manifold.*
Therefore, a $W_9 - flat$ LP-Kenmotsu manifold is an $\eta -$ Einstein manifold.

5 0. A $\xi - W_9$-flat Lorentzian Para Kenmotsu manifold

Definition 5.1: An n-dimensional LP-Kenmotsu manifold is said to be $\xi - W_9$-flat if its W_9 -curvature tensor satisfies the following condition.

$$W_9(X,Y)\xi = 0 \tag{5.1}$$

Theorem: 5.1, *A $\xi - W_9$-flat LP-Kenmotsu manifold is a special type of $\eta -$ Einstein manifold.*

Proof:

Suppose the LP-Kenmotsu manifold is $\xi - W_9$ −flat, then the following hold

$$W_9(X,Y)\xi = R(X,Y)\xi + \frac{1}{n-1}[S(X,Y)\xi - g(Y,\xi)QX] = 0$$

$$0 = \eta(Y)X - \eta(X)Y + g(X,Y)\xi - \eta(Y)X$$

$$0 = g(X,Y)\xi - \eta(X)Y$$

$$R(X,Y)\xi = \frac{1}{n-1}[g(Y,\xi)QX - S(X,Y)\xi]$$

$$\eta(Y)X - \eta(X)Y = \eta(Y)X - \frac{1}{n-1}S(X,Y)\xi$$

$$S(X,Y)\xi = (n-1)\eta(X)Y \tag{5.2}$$

Contracting (5.2) with respect to ξ we get

$$S(X,Y) = -(n-1)\eta(Y)\eta(X) \tag{5.3}$$

Thus the proof that an $\xi - W_9$-flat LP-Kenmotsu manifold is a special type of $\eta -$ Einstein manifold.

6.0. A $\phi - W_9$-LP-Kenmotsu manifold

Definition 6.1: An n-dimensional LP-Kenmotsu manifold is said to be $\phi - W_9$-flat if W_9 -curvature tensor satisfies the following condition

$$W_9(X,Y).\phi)Z = W_9(X,Y)\phi Z - \phi(W_9(X,Y)Z = 0$$

Theorem: 6.1: A W_9 -LP-Kenmotsu –manifold is a $\phi - W_9\ flat$ manifold

Proof.

Consider $\phi - W_9$ LP-Kenmotsu manifold, then the following hold

$$W_9(X,Y)\phi Z - \phi(W_9(X,Y)Z$$

$$= R(X,Y)\phi Z - g(Y,Z)\phi X + g(X,Z)\phi Y + g(X,Y)\phi Z - g(Y,\phi Z)X - g(X,Y)\phi Z + g(Y,Z)\phi X$$

$$= R(X,Y)\phi Z + g(X,Z)\phi Y - g(Y,\phi Z)X$$

$$R(X,Y)\phi Z = g(Y,\phi Z)X - g(X,Z)\phi Y$$

$$= g(Y,\phi Z)X - g(X,\phi Z)Y - g(Y,Z)\phi X + g(X,Z)\phi Y + g(X,Y)\phi Z - g(Y,\phi Z)X - g(X,Y)\phi Z + g(Y,Z)\phi X$$

$= -g(X, \phi Z)Y + g(X, Z)\phi Y$ (6.1)

Putting $Y = Z = \xi$ in (8) yields
$W_9(X, \xi).\phi)\xi = 0$ (6.2)

This completes the proof.

7.0 W_9-semisymmetric LP-Kenmotsu manifold
Definition 7.1: An n-dimensional para-Kenmotsu manifold is called W_9-semi-symmetric if its W_9 -curvature tensor satisfies the following condition: $R(U, V) \cdot W_9 = 0$
where R is the Riemannian curvature tensor and considered to be a derivation of tensor algebra at each point of the manifold for tangent vectors U and V.

Theorem 7.1: W_9 $-$semi-symmetric LP-Kenmotsu manifold is a special type of η $-$Einstein manifold.
Proof.
Suppose that an LP-Kenmotsu manifold is W_9 $-$semisymmetric. Then, we have

$(R(U, V) \cdot W_9)(X, Y)Z = 0$ (7.1)

The above equation can be written as follows:

$R(U, V)W_9(X, Y)Z - W_9(R(U, V)X, Y)Z - W_9(X, R(U, V)Y)Z$

$-W_9(X, Y)R(U, V)Z = 0$ (7.2)

Putting $U = \xi$ in (7.2) we get

$R(\xi, V)W_9(X, Y)Z - W_9(R(\xi, V)X, Y)Z - W_9(X, R(\xi, V)Y)Z - W_9(X, Y)R(\xi, V)Z = 0$ (7.3)

In view of (10), the above equation reduces to

$[g(V, W_9(X, Y)Z)\xi - \eta(V)W_9(X, Y)Z] - [W_9(X, g(V, Y)\xi - \eta(Y)V)Z] - [W_9(X, Y)((g(V, Z)\xi - \eta(Z)V)] = 0$ (7.4)

Equation (7.4) reduces to

$[g(V, W_9(X, Y)Z)\xi - \eta(V)W_9(X, Y)Z] - [g(V, Y)W_9(X, \xi)Z - \eta(Y)W_9(X, V)Z] - [\ g(V, Z)W_9(X, Y)\xi - \eta(Z)W_9(X, Y)V] = 0$
(7.5)

Now, taking the inner product of the above equation with ξ and using (7(i)) and (7 (iii)), we get

$[\eta(g(W_9(X, Y)Z)\xi) - \eta(V)\eta(W_9(X, Y)Z)] - [g(V, Y)\eta(W_9(X, \xi)Z) - \eta(Y)\eta(W_9(X, V)Z)]$

$-[\ g(V, Z)\eta(W_9(X, Y)\xi) - \eta(Z)\eta(W_9(X, Y)V)] = 0$ 7.6)

Part I

$[g(V, W_9(X, Y)Z)\xi - \eta(V)W_9(X, Y)Z]$

$\eta(g(V, W_9(X, Y)Z)\xi) = \eta(W_9(X, Y)g(V, Z)\xi) + \frac{1}{n-1}[-(n-1)g(X, Y)g(V, Z) - (n-1)g(Y, g(V, Z)\xi)\eta(X)]$

$\eta(g(V, W_9(X, Y)Z)\xi) = \eta(W_9(X, Y)g(V, Z)\xi) - g(X, Y)g(V, Z) - g(Y, g(V, Z)\xi)\eta(X)$

$\eta(g(V, W_9(X, Y)Z)\xi)$

$= \eta(g(Y, g(V, Z)\xi)X - g(X, g(V, Z)\xi)Y) - g(X, Y)g(V, Z) - g(V, Z)\eta(X)\eta(Y)$

$= \eta(g(V, Z)g(Y, \xi)X - g(V, Z)g(X, \xi)Y) - g(X, Y)g(V, Z) - g(V, Z)\eta(X)\eta(Y)$

$= g(V, Z)\eta(Y)\eta(X) - g(V, Z)\eta(X)\eta(Y) - g(X, Y)g(V, Z) - g(V, Z)\eta(X)\eta(Y)$

$= -g(X, Y)g(V, Z) - g(V, Z)\eta(X)\eta(Y)$ (7.7)

$\eta(V)\eta(W_9(X, Y)Z) = g(Y, Z)\eta(X)\eta(V) - g(X, Z)\eta(Y)\eta(V) + g(X, Y)\eta(Z)\eta(V) - g(Y, Z)\eta X)\eta(V)$

$\eta(V)\eta(W_9(X,Y)Z) = g(X,Y)\eta(Z)\eta(V) - g(X,Z)\eta(Y)\eta(V)$ (7.8)

Part ii

$[g(V,Y)\eta(W_9(X,\xi)Z) - \eta(Y)\eta(W_9(X,V)Z)]$

$$g(V,Y)\eta(W_9(X,\xi)Z) = g(V,Y)\eta(Z)\eta(X) + g(V,Y)g(X,Z) + g(V,Y)\eta(X)\eta(Z) - g(V,Y)\eta(Z)\eta(X)$$
$$g(V,Y)\eta(W_9(X,\xi)Z) = g(V,Y)\eta(Z)\eta(X) + g(V,Y)g(X,Z) \quad (7.9)$$

$$\eta(Y)\eta(W_9(X,V)Z) = g(V,Z)\eta(X)\eta(Y) - g(X,Z)\eta(V)\eta(Y) + g(X,V)\eta(Z)\eta(Y) - g(V,Z)\eta(X)\eta(Y)$$
$$\eta(Y)\eta(W_9(X,V)Z) = -g(X,Z)\eta(V)\eta(Y) + g(X,V)\eta(Z)\eta(Y) \quad (7.10)$$

Part iii
$[g(V,Z)\eta(W_9(X,Y)\xi) - \eta(Z)\eta(W_9(X,Y)V)]$

$$g(V,Z)\eta(W_9(X,Y)\xi) = g(V,Z)\eta(Y)\eta(X) - g(V,Z)\eta(X)\eta(Y) - g(V,Z)(g(X,Y) - g(V,Z)\eta(Y)\eta(X)$$

$$g(V,Z)\eta(W_9(X,Y)\xi) = -g(V,Z)\eta(X)\eta(Y) - g(V,Z)(g(X,Y) \quad (7.11)$$

$$\eta(Z)\eta(W_9(X,Y)V) = g(Y,V)\eta(X)\eta(Z) - g(X,V)\eta(Y)\eta(Z) + g(X,Y)\eta(V)\eta(Z) - g(Y,V)\eta(X)\eta(Z)$$

$$\eta(Z)\eta(W_9(X,Y)V) = -g(X,V)\eta(Y)\eta(Z) + g(X,Y)\eta(V)\eta(Z) \quad (7.12)$$

Summing up (7.7), (7.8), (7.9), (7.10), (7.11) and (7.12) gives

$$[-g(X,Y)g(V,Z) - g(V,Z)\eta(X)\eta(Y)] - [g(X,Y)\eta(Z)\eta(V) - g(X,Z)\eta(Y)\eta(V)]$$

$$-[g(V,Y)\eta(Z)\eta(X) + g(V,Y)g(X,Z)] - [-g(X,Z)\eta(V)\eta(Y) + g(X,V)\eta(Z)\eta(Y)]$$

$$-[-g(V,Z)\eta(X)\eta(Y) - g(V,Z)(g(X,Y)] - [-g(X,V)\eta(Y)\eta(Z) + g(X,Y)\eta(V)\eta(Z)] = 0$$

$$-g(X,Y)g(V,Z) - g(V,Z)\eta(X)\eta(Y) - g(X,Y)\eta(Z)\eta(V) + g(X,Z)\eta(Y)\eta(V)$$

$$-g(V,Y)\eta(Z)\eta(X) - g(V,Y)g(X,Z) - g(X,Z)\eta(V)\eta(Y) + g(X,V)\eta(Z)\eta(Y)$$

$$+g(V,Z)\eta(X)\eta(Y) + g(V,Z)(g(X,Y) - g(X,V)\eta(Y)\eta(Z) + g(X,Y)\eta(V)\eta(Z) = 0$$

$$\therefore\ -g(V,Y)\eta(Z)\eta(X) - g(V,Y)g(X,Z) = 0 \quad (7.13)$$
$$0 = -g(X,Z) - \eta(Z)\eta(X) \quad (7.14)$$

$$\therefore S(X,Y) = -(n-1)\eta(X)\eta(Y) \quad (7.15)$$

Equation (7.15) is a special type of η −Einstein manifold. Thus, the proof.

8.0. A W_9 −Lorentzian-para-Kenmotsu manifold satisfying the condition $W_9.R = 0$
Definition 8.1: A W_9 −Lorentzian-para-Kenmotsu manifold is said to satisfy $W_9.R = 0$ condition if

$$(W_9(U,V)\cdot R)(X,Y)Z = 0 \quad (8.1)$$

Theorem 4.2. A W_9 −Lorentzian-Para-Kenmotsu manifold satisfying the condition $W_9.R = 0$ is a special type of η −Einstein manifold
The above equation (10) can be written as follows:

$$W_9(U,V)R(X,Y)Z - R(W_9(U,V)X,Y)Z - R(X,W_9(U,V)Y)Z$$

$$-R(X,Y)W_9(U,V)Z = 0 \quad (8.2)$$

Putting $U = \xi$ in (9) we get

$$W_9(\xi,V)R(X,Y)Z - R(W_9(\xi,V)X,Y)Z - R(X,W_9(\xi,V)Y)Z - R(X,Y)W_9(\xi,V)Z = 0 \quad (8.3)$$

But

$$W_9(\xi,V)W = g(V,W)\xi - \eta(W)V + \eta(V)W - g(V,W)\xi$$

$W_9(\xi,V)W = \eta(V)W - \eta(W)V$ (8.4)

Computing the four terms in (8.3) separately gives

First term: $W_9(\xi,V)R(X,Y)Z$

$W_9(\xi,V)R(X,Y)Z = \eta(V)R(X,Y)Z - \eta(R(X,Y)Z)V$

$= \eta(V)g(Y,Z)X - \eta(V)g(X,Z)Y - \eta(X)g(Y,Z)V + \eta(Y)g(X,Z)V$ (8.5)

Second Term: $R(W_9(\xi,V)X,Y)Z$

Using (8.3) we get

$R(W_9(\xi,V)X,Y)Z = g(Y,Z)[\eta(V)X - \eta(X)V] - g([\eta(V)X - \eta(X)V],Z)Y$

$= \eta(V)g(Y,Z)X - \eta(X)g(Y,Z)V - \eta(V)g(X,Z)Y + \eta(X)g(V,Z)Y$ (8.6)

Third Term: $R(X,W_9(\xi,V)Y)Z$

$R(X,W_9(\xi,V)Y)Z = g([\eta(V)Y - \eta(Y)V],Z)X - g(X,Z)[\eta(V)Y - \eta(Y)V]$

$= \eta(V)g(Y,Z)X - \eta(Y)g(V,Z)X - \eta(V)g(X,Z)Y + \eta(Y)g(X,Z)V$ (8.7)

Fourth Term: $R(X,Y)W_9(\xi,V)Z$

$R(X,Y)W_9(\xi,V)Z = g(Y,[\eta(V)Z - \eta(Z)V])X - g(X,[\eta(V)Z - \eta(Z)V])Y$

$= \eta(V)g(Y,Z)X - \eta(Z)g(Y,V)X - \eta(V)g(X,Z)Y + \eta(Z)g(X,V)Y$ (8.8)

Simplifying (8.5), (8.6), (8.7) and (8.8) together reduces equation (8.3) to

: $W_9(\xi,V)R(X,Y)Z = \eta(V)g(Y,Z)X - \eta(V)g(X,Z)Y - \eta(X)g(Y,Z)V + \eta(Y)g(X,Z)V$

: $-R(W_9(\xi,V)X,Y)Z = -\eta(V)g(Y,Z)X + \eta(X)g(Y,Z)V + \eta(V)g(X,Z)Y - \eta(X)g(V,Z)Y$

$-R(X,W_9(\xi,V)Y)Z = -\eta(V)g(Y,Z)X + \eta(Y)g(V,Z)X + \eta(V)g(X,Z)Y - \eta(Y)g(X,Z)V$

$-R(X,Y)W_9(\xi,V)Z = -\eta(V)g(Y,Z)X + \eta(Z)g(Y,V)X + \eta(V)g(X,Z)Y - \eta(Z)g(X,V)Y$

$\Rightarrow -2\eta(V)g(Y,Z)X + 2\eta(V)g(X,Z)Y + \eta(Y)g(V,Z)X + \eta(Z)g(Y,V)X - \eta(Z)g(X,V)Y - \eta(X)g(V,Z)Y = 0$ (8.9)

Putting $Z = \xi$ in equation (8.9) and taking inner product with ξ yields

$-\eta(X)g(Y,V) + \eta(Y)g(X,V) = 0$ (8.10)

Putting $Y = \xi$ in equation (8.10) gives

$S(X,V) = -(n-1)\eta(X)\eta(V)$ (8.11)

Equation (8.11) completes the proof that an LP-Kenmotsu manifold satisfying the condition $R.W_9 = 0$ is a special type of η −Einstein manifold.
This completes the proof.

9.0 A $W_9.Q$ – Lorentzian Para-Kenmotsu manifold

Definition 9.1: A W_9 −Lorentzian Para-Kenmotsu manifold is such that $W_9.Q = 0$

Theorem: 9.1: An n-dimensional Lorentzian Para-Kenmotsu manifold is such that $W_9.Q = 0$

Proof;

Consider and proof that $W_9.Q = 0$

$\therefore (W(X,Y).Q)Z = W_9(X,Y)QZ - Q(W_9(X,Y)Z$

$= R(X,Y)QZ - Q(R(X,Y)Z) + \frac{1}{n-1}[S(X,Y)QZ - (g(Y,QZ)QX)]$

$$-\frac{1}{n-1}[S(X,Y)QZ - Q(g(Y,Z)QX)]$$

$$= R(X,Y)QZ - Q(R(X,Y)Z) + \frac{1}{n-1}[S(X,Y)QZ - S(X,Y)QZ - (g(Y,QZ)QX + Q(g(Y,Z)QX]$$

$$= [g(Y,QZ)X - g(X,QZ)Y] - [g(Y,Z)QX - g(X,Z)QY] + \frac{1}{n-1}[Q(g(Y,Z)QX) - (g(Y,QZ)QX)]$$

$$= [g(Y,QZ)X - g(X,QZ)Y] - [g(Y,Z)QX - g(X,Z)QY] + [Q(g(Y,Z)X) - g(Y,QZ)X]$$

$$= [-g(X,QZ)Y] - [g(Y,Z)QX - g(X,Z)QY] + [g(Y,Z)QX]$$

$$= [-g(X,QZ)Y] - [-g(X,Z)QY]$$

$$= g(X,Z)QY - g(X,QZ)Y$$

$$= (n-1)g(X,Z)Y - (n-1)g(X,Z)Y$$

$$= 0$$

This completes the proof.

10.0. An n-dimensional Lorentzian-Para-Kenmotsu manifold satisfying $Q.W_9 = 0$

Definition 10.1: An n-dimensional Lorentzian-Para-Kenmotsu manifold is said to satisfy the condition $Q.W_9 = 0$ if $(Q.W_9)(X,Y)Z = 0$.

Theorem 10.1: An LP-Kenmotsu manifold satisfying the condition $Q.W_9 = 0$ is a special type of η −Einstein manifold

Proof.

Let us consider an LP −Kenmotsu manifold which satisfies the condition

$$(Q.W_9)(X,Y)Z = 0,$$

$$\therefore Q(W_9(X,Y)Z) - W_9(QX,Y)Z - W_9(X,QY)Z - W_9(X,Y)QZ = 0, \tag{10.1}$$

Computing the four terms separately gives

First term: $Q(W_9(X,Y)Z)$

$$Q(W_9(X,Y)Z) = g(Y,Z)QX - g(X,Z)QY + \frac{1}{n-1}[S(X,Y)QZ] - g(Y,Z)QX$$

$$= g(Y,Z)QX - g(X,Z)QY + [g(X,Y)QZ] - g(Y,Z)QX$$

$$= g(X,Y)QZ - g(X,Z)QY \tag{10.2}$$

Second Term: $W_9(QX,Y)Z$

$$W_9(QX,Y)Z = g(Y,Z)QX - g(QX,Z)Y + \frac{1}{n-1}[S(QX,Y)Z] - g(Y,Z)QX$$

$= g(QX,Y)Z - g(QX,Z)Y$ (10.3)

Third Term: $W_9(X,QY)Z$

$$W_9(X,QY)Z = g(QY,Z)X - g(X,Z)QY + g(X,QY)Z - g(QY,Z)X$$

$$= g(X,QY)Z - g(X,Z)QY \tag{10.4}$$

Fourth term: $W_9(X,Y)QZ$

$$W_9(X,Y)QZ = g(Y,QZ)X - g(X,QZ)Y + g(X,Y)QZ - g(Y,QZ)X$$

$$= g(X,Y)QZ - g(X,QZ)Y \tag{10.5}$$

Putting (10.2), (10.3), (10.4) and (10.5) in (10.1) yields

$g(X,Y)QZ - g(X,Z)QY \; - g(QX,Y)Z + g(QX,Z)Y - g(X,QY)Z + g(X,Z)QY - g(X,Y)QZ + \; g(X,QZ)Y = 0$
$\Rightarrow - \; g(QX,Y)Z + g(QX,Z)Y - g(X,QY)Z + g(X,QZ)Y = 0$

$$\Rightarrow - \; g(X,Y)Z + g(X,Z)Y = 0 \qquad (10.6)$$

Putting $Z = \xi$ in (10.6) above and taking inner products of ξ yields
$S(X,Y) = -(n-1)\eta(X)\eta(Y)$ (10.7)
This completes the proof that an LP-Kenmotsu manifold satisfying the condition $Q.W_9 \; = \; 0$ is a special type of η −Einstein manifold

11.0 . An n-dimensional LP-Kenmotsu manifold satisfying the condition $W_9.W_9 = 0$
Definition 11.1 An n-dimensional LP-Kenmotsu manifold is said to satisfy the condition $W_9.W_9 = 0$ if

$$W_9.W_9 = W_9(U,V).W_9)(X,Y)Z = 0$$

Theorem 11.1: An LP-Kenmotsu manifold satisfying the condition $W_9.W_9 = 0$ is a special type of η −Einstein manifold
Proof:
Consider $W_9.W_9 = W_9(U,V).W_9)(X,Y)Z = 0$
where,

$W_9(U,V)W_9(X,Y)Z = (W_9(U,V).W_9)(X,Y)Z + \; W_9(W_9(U,V)X,Y)Z + W_9(X,W_9(U,V)Y)Z \; W_9(X,Y)W_9(U,V)Z$
$W_9(U,V)W_9(X,Y)Z - \; W_9(W_9(U,V)X,Y)Z - W_9(X,W_9(U,V)Y)Z - W_9(X,Y)W_9(U,V)Z = 0$ (11.1)

Putting $V = \xi \; in$ (11.1) above we get

$W_9(U,\xi)W_9(X,Y)Z - \; W_9(W_9(U,\xi)X,Y)Z - W_9(X,W_9(U,\xi)Y)Z - W_9(X,Y)W_9(U,\xi)Z = 0$ (11.2)

Simplifying each term in (11.2) separately gives

Term 1: $W_9(U,\xi)W_9(X,Y)Z$.
Let $W_9(X,Y)Z = W$

$$W_9(U,\xi)W \; = g(\xi,W)U - g(U,W)\xi + \frac{1}{n-1}[S(U,\xi)W - g(\xi,W)QU]$$

$$W_9(U,\xi)W \; = \eta(W)U - g(U,W)\xi + \eta(U)W - \eta(W)U$$

$W_9(U,\xi)W \; = \eta(U)W - g(U,W)\xi$ (11.3)

From (11.3) we have
$W_9(U,\xi)W_9(X,Y)Z \; = \eta(U)W_9(X,Y)Z \; - g(U,W_9(X,Y)Z\,)\xi$

$$W_9(U,\xi)W_9(X,Y)Z \; = \eta(U)W_9(X,Y)Z \; - [g(Y,Z)g(U,X)\xi - g(X,Z)g(U,Y)\xi + g(X,Y)g(U,Z)\xi - \; g(Y,Z)g(X,U)\xi]$$

$$W_9(U,\xi)W_9(X,Y)Z \; = \eta(U)W_9(X,Y)Z \; - [-g(X,Z)g(U,Y)\xi + \; g(X,Y)g(U,Z)\xi]$$

$W_9(U,\xi)W_9(X,Y)Z \; = \eta(U)W_9(X,Y)Z + g(X,Z)g(U,Y)\xi - \; g(X,Y)g(U,Z)\xi]$ (11.4)

Term II: $W_9(W_9(U,\xi)X,Y)Z$
Applying the definition of W_9 on above, we get

$$W_9(W_9(U,\xi)X,Y)Z = g(Y,Z)W_9(U,\xi)X - g(W_9(U,\xi)X,Z)X + g(W_9(U,\xi)X,Y)Z - g(Y,Z)W_9(U,\xi)X$$

$$W_9(W_9(U,\xi)X,Y)Z = g(Y,Z)W_9(U,\xi)X - g(W_9(U,\xi)X,Z)Y + \tfrac{1}{n-1}[S(W_9(U,\xi)X,Y)Z - \; g(Y,Z)Q(W_9(U,\xi)X)]$$

$$= g(Y,Z)[\eta(U)X - g(U,X)\xi] - g([\eta(U)X - g(U,X)\xi],Z)Y) + g([\eta(U)X - g(U,X)\xi],Y)Z - \tfrac{1}{n-1}g\,(Y,Z)Q([\eta(U)X - g(U,X)\xi])\,)\,]$$

$$= \eta(U)g(Y,Z)X - g(Y,Z))g(U,X)\xi - \eta(U)g(X,Z)Y + \eta(Z)g(U,X)Y + \eta(U)g(X,Y)Z - \; \eta(Y)g(U,X)Z - \eta(U)g(Y,Z)X + g(Y,Z)g(U,X)\xi$$

$$= \eta(U)g(Y,Z)X - g(Y,Z))g(U,X)\xi - \eta(U)g(X,Z)Y + \eta(Z)g(U,X)Y + \eta(U)g(X,Y)Z - \eta(Y)g(U,X)Z - \eta(U)g(Y,Z)X + g(Y,Z)g(U,X)\xi$$

$= -\eta(U)g(X,Z)Y + \eta(Z)g(U,X)Y + \eta(U)g(X,Y)Z - \eta(Y)g(U,X)Z$ (11.5)

Term III: $W_9(X, W_9(U,\xi)Y)Z$

$W_9(X, W_9(U,\xi)Y)Z = g(W_9(U,\xi)Y, Z)X - g(X,Z)\, W_9\,(U,\xi)Y + [g(X, W_9(U,\xi)Y)Z - g(W_9(U,\xi)Y, Z)X]$

$= g(\eta(U)Y - g(U,Y)\xi, Z)X - g(X,Z)\,[\eta(U)Y - g(U,Y)\xi] + [g(X, \eta(U)Y - g(U,Y)\xi)Z - g(\eta(U)Y - g(U,Y)\xi, Z)X]$

$= \eta(U)g(Y,Z)X - \eta(Z)g(U,Y)X - \eta(U)g(X,Z)Y + g(X,Z)g(U,Y)\xi + \eta(U)g(X,Y)Z - \eta(X)g(U,Y)Z - \eta(U)g(Y,Z)X + \eta(Z)g(U,Y)X$

$= g(X,Z)g(U,Y)\xi - \eta(X)g(U,Y)Z$ (11.6)

Term IV: $W_9(X,Y)W_9(U,\xi)Z$

$W_9(X,Y)W_9(U,\xi)Z = g(Y, W_9(U,\xi)Z)X - g(X, W_9(U,\xi)Z)Y + g(X,Y)W_9(U,\xi)Z - g(Y, W_9(U,\xi)Z)X$

$= g(Y, \eta(U)Z - g(U,Z)\xi)X - g(X, \eta(U)Z - g(U,Z)\xi)Y + g(X,Y)\eta(U)Z - g(U,Z)\xi$

$-g(Y, \eta(U)Z - g(U,Z)\xi)X$

$= \eta(U)g(Y,Z)X - \eta(Y)g(U,Z)X - \eta(U)g(X,Z)Y + \eta(X)g(U,Z)Y + \eta(U)g(X,Y)Z - g(X,Y)g(U,Z)\xi - \eta(U)g(Y,Z)X + \eta(Y)g(U,Z)X$

$= -\eta(U)g(X,Z)Y + \eta(X)g(U,Z)Y + \eta(U)g(X,Y)Z - g(X,Y)g(U,Z)\xi$ (11.7)

Solving for equation (11.2) through (11.4), (11.5), (11.6) and (11.7) giving

$W_9(U,\xi)W_9(X,Y)Z = \eta(U)W_9(X,Y)Z + g(X,Z)g(U,Y)\xi - g(X,Y)g(U,Z)\xi]$ (11.8)

$-W_9(W_9(U,\xi)X, Y)Z = \eta(U)g(X,Z)Y - \eta(Z)g(U,X)Y - \eta(U)g(X,Y)Z + \eta(Y)g(U,X)Z$ *(11.9)*

$-W_9(X, W_9(U,\xi)Y)Z = -g(X,Z)g(U,Y)\xi + \eta(X)g(U,Y)Z$ (11.10)

$-W_9(X,Y)W_9(U,\xi)Z = \eta(U)g(X,Z)Y - \eta(X)g(U,Z)Y - \eta(U)g(X,Y)Z + g(X,Y)g(U,Z)\xi$ (11.11)

$W_9(U,\xi)W_9(X,Y)Z = \eta(U)g(Y,Z)X - \eta(U)g(X,Z)Y + \eta(U)g(X,Y)Z - \eta(U)g(Y,Z)X + g(X,Z)g(U,Y)\xi - g(X,Y)g(U,Z)\xi]$ (11.12)

$-W_9(W_9(U,\xi)X, Y)Z = \eta(U)g(X,Z)Y - \eta(Z)g(U,X)Y - \eta(U)g(X,Y)Z + \eta(Y)g(U,X)Z$ (11.13)

$-W_9(X, W_9(U,\xi)Y)Z = -g(X,Z)g(U,Y)\xi + \eta(X)g(U,Y)Z$ (11.14)

$-W_9(X,Y)W_9(U,\xi)Z = \eta(U)g(X,Z)Y - \eta(X)g(U,Z)Y - \eta(U)g(X,Y)Z + g(X,Y)g(U,Z)\xi$ (11.15)

Equation (11.2) reduces therefore to the given form

$\therefore \eta(U)g(X,Z)Y - \eta(Z)g(U,X)Y - \eta(U)g(X,Y)Z + \eta(Y)g(U,X)Z + \eta(X)g(U,Y)Z - \eta(X)g(U,Z)Y = 0$ (11.16)

Putting $X = Z = \xi$ in equation (11.16) above yields

$-\eta(U)Y + \eta(U)Y - \eta(U)\eta(Y)\xi + \eta(U)\eta(Y)\xi - g(U,Y)\xi + \eta(U)Y = 0$

$\Rightarrow -g(U,Y)\xi + \eta(U)Y = 0$ (11.17)

Taking inner product of (9) with ξ, we get

$S(U,Y) = -(n-1)\eta(U)\eta(Y)$ (11.18)

This completes proof of the theorem that an LP-Kenmotsu manifold satisfying $W_9.W_9 = 0$ is a special type of an η −Einstein manifold.

References

1. Tripathi MM, De UC. Lorentzian almost para-contact manifolds and their submanifolds. The Pure and Applied Mathematics. 2001;8(2):101-125.
2. Njori PW, Moindi SK, Pokhariyal GP. A study on W_6-curvature tensors and W_8-curvature tensors in Kenmotsu manifolds admitting semi-symmetric metric connection. 2021;1:41-57.
3. Njori PW. Curvature Tensors on Semi-Riemannian and Generalized Sasakian Space Forms Admitting Semi-symmetric Metric Connection. [dissertation]. Nairobi: University of Nairobi; 2018.
4. Satyanarayana T, Prasad KS. On semi-symmetric Para Kenmotsu Manifolds. Turkish Journal of Analysis and Number Theory. 2015;3(6):145-148.
5. Pokhariyal GP. Relativistic significance of curvature tensors. International Journal of Mathematics and Mathematical Sciences. 1982;5:133-139.
6. Prasad RAJENDRA, Haseeb ABDUL, Gautam UK. On φ-semisymmetric LP-Kenmotsu manifolds with a qsnm-connection admitting Ricci solitons. Kragujevac Journal of Mathematics. 2021;45(5):815-827.
7. Njori P, Mutinda F. Study of Some New Curvature Tensors on Lorentzian Para Sasakian Manifolds and Other Related Manifolds. 2021.
8. Yoldas H, Yasar E. Some notes on Kenmotsu manifolds. Facta Universitatis, Series: Mathematics and Informatics. 2021;1:949-961.
9. Devi SS, Prasad KS, Satyanarayana T. Certain curvature conditions on Lorentzian Para-Kenmotsu manifolds. Reliability: Theory & Applications. 2022;17(2(68)):413-421.
10. Mert T, Atçeken M. On some important characterizations of Lorentz para-Kenmotsu manifolds on some special curvature tensors. Asian-European Journal of Mathematics. 2024;17(01):2350246.
11. Bilal M, Raghuwanshi T, Pandey MK, Haseeb A, Goyal A. On Generalized Weyl Conformal Curvature Tensor in Para-Kenmotsu Manifolds. Results in Nonlinear Analysis. 2024;7(3):55-64.