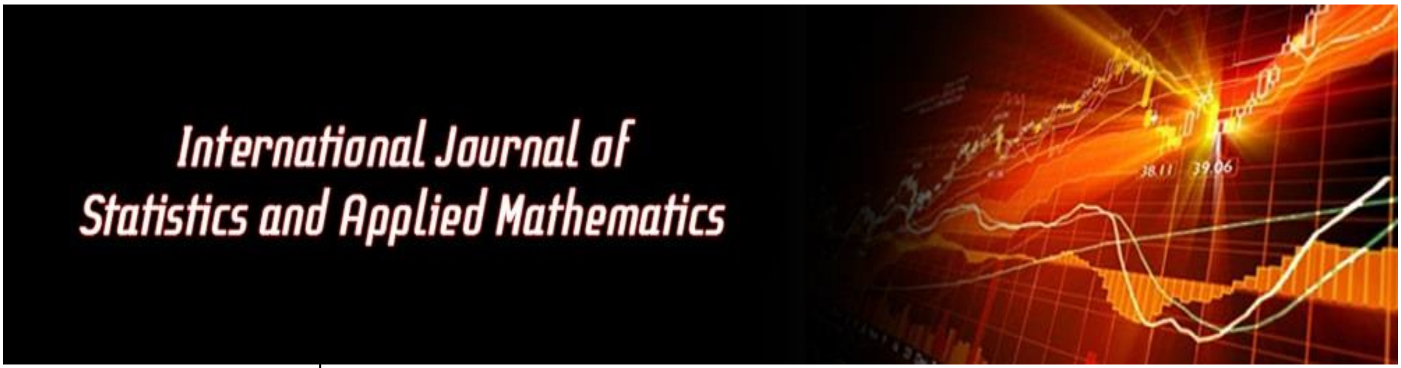

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Mixed structure of approximate Serre co-fibration

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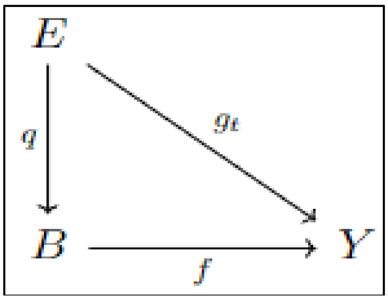
Abstract
The paper aims to give a new structure concept of Mixed structure of approximate Serre co-fibration (MASCof.). Furthermore, mixed approximate Serre cofibration is covered by the same theorems that apply to approximation Serre cofibration. We define this structure and we proved some theorems and properties

Keywords: CW-complex, lowering homotopy property, mixed approximate Serre co-fibration, homotopy extension property, M-criterion

1. Introduction

In this paper there is a description of the mixed approximate Serre co-fibration and approximate homotopy extension property. In [1, 2, 3] mention if $q: E \to B$ is a function has an approximate lowering homotopy property (ALHP) with respect to X provided that given ξ is open cover of X if and only if define a function $f: B \to Y$ and a homotopy $g_t: E \to Y$ satisfying $f \circ q = g_0$, then there exists a homotopy $f_t: B \to Y$ with $f_0 = f$ and $f_t \circ q$ is ξ-closed g_t for all $t \in I$.



Also [1, 4] The map q is co-fibration if it has the approximate lowering homotopy property. In addition to that [1, 5] assume $q: E \to B$ be a continuous function of spaces, q has the approximate lowering homotopy property (ALHP) with respect to a CW- complex space Y.

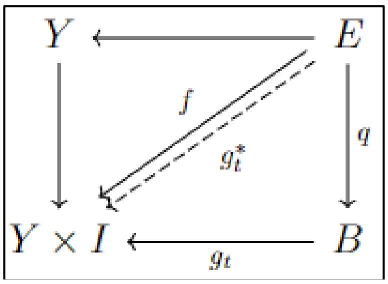


Diagram 1: (L.H.P)

If Y is the space of all CW- complex space or (all finitely triangulable spaces), then $q: E \to B$ is called an approximate Serre co-fibration.

2. Mixed approximate Serre co-fibration

Let $f_1: E_1 \to Z, f_2: E_2 \to Z$ are two fiber space and $\alpha: E_2 \to E_1$ where $f_1 \circ \alpha = f_2$, $let\ E_i = \{E_1, E_2\}$, $f_i = \{f_1, f_2\}$ with Z be a CW-complex $the\ \{E_i, f_i, Z, \alpha\}$, has Mixed approximate lowering homotopy property (MALHP) [2] with respect to a CW-space Y iff given a map $k: Z \to Y$ and a homotopy $g_t: E_1 \to Y$ satisfying $k \circ f_2 = g_0 \circ \alpha$ then there exist a homotopy $k_t: Z \to Y$ with $k_0 = k$ and $k_t \circ f_1 = g_t$ for all $t \in I$. M-fiber space is called M approximate Serre co-fibration.

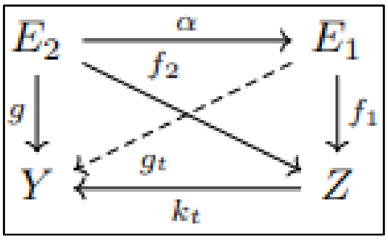


Diagram 2: (M-L.H.P)

Proposition 2.1. Every Serre co-fibration is Mixed Serre cofibration.

Proof: Let $\{E_i, f_i, Z, \alpha\}$ be a M-fiber space such that $E_1 = E_2 = E$, $\alpha =$ identity and $f = f_1 = f_2$. Let $k: Z \to Y$ and a homotopy $g_t: E_1 \to Y$ where $k \circ f_2 = g_0 \circ \alpha$, there exist a homotopy $k_t: Z \to Y$ with $k_0 = k$ and $k_t \circ f_2 = g_t$ for all $t \in I$ then f_i has (MALHP) with respect to Y. Therefor f_i has M approximate Serre co-fibration.

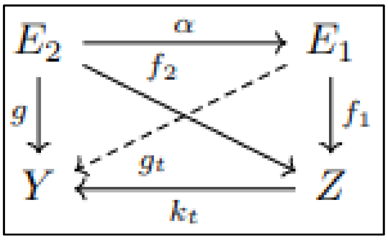


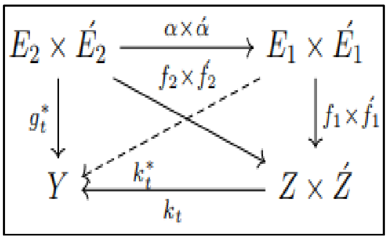
Diagram 3: (M-S co-fibration)

Proposition 2.2. The product of two mixed approximate Serre co-fibration are also mixed approximate Serre cofibration.

Proof: Let $f_i: E_i \to Z$ and $\acute{f_i}: \acute{E_i} \to Z$ be two mixed approximate Serre co-fibration, and let Y be a CW-complex space, to prove $f_i \times \acute{f_i}: E_i \times \acute{E_i} \to Z \times \acute{Z}$ mixed approximate Serre cofibration.

Suppose $k^*: Z \times \acute{Z} \to Y$ be map where $k: Z \to Y$ and $\acute{k}: \acute{Z} \to Y$

Define $g_t^*: E_1 \times \acute{E_1} \to Y$ as $k^* \circ (f_2 \times \acute{f_2}) = g_0^* \circ (\alpha \circ \acute{\alpha})$ such that $\acute{g_t}: \acute{E_1} \to Y$ and $g_t: E_1 \to Y$,since $f_i, \acute{f_i}$ are M approximate Serre cofibration.
There exists a homotopy $k_t: Z \to Y$ with $k_0 = k$ and $k_t \circ f_1 = g_t$ and a homotopy $k_t: \acute{Z} \to Y$ with $\acute{k_0} = \acute{k}$ and $k_t \circ \acute{f_1} = g_t$. Now, for $k_t^*: Z \times \acute{Z} \to Y$ define as $k_t^* \circ (f_1 \times \acute{f_1}) = g_t^*$ and $k_0^* = k^*$. Therefore, $f_i \times \acute{f_i}: E_i \times \acute{E_i} \to Z \times \acute{Z}$ is M-Serre co-fibration.



Proposition 2.3. Let $f_i: E_i \to Z$ and $\acute{f_i}: \acute{E_i} \to Z$ be two mixed approximate Serre co-fibration then mixed pullback of mixed approximate Serre co-fibration is also M approximate Serre co-fibration.

Proof: Let $\acute{k}: \acute{Z} \to Y$ and $k: Z \to Y$. Define a homotopy $g_t: E_1 \to Y$ such tha $k \circ f_2 = g_0 \circ \alpha$, since f_i has M-Serre cofibration then there exists a homotopy $k_t: Z \to Y$ with $k_0 = k$ and $k_t \circ f_1 = g_t$.
Defin $\acute{g_t}: \acute{E_1} \to Y$ such that $\acute{k} \circ \acute{f_2} = \acute{g_0} \circ \alpha$ and $\acute{g_t} = g_t \circ l$ then there exists a homotopy $\acute{k_t}: \acute{Z} \to Y$ with $\acute{k_0} = k$ and $\acute{k_t} \circ \acute{f_1} = g_t$.Therefore $\acute{f_i}: \acute{E_i} \to \acute{Z}$ has mixed approximate Serre cofibration.

3. Homotopy Extension Property

A subset A ⊂ Y is said to have the homotopy extension property (HEP) with respect to Y, iff any partial homotopy:

$\underline{H}: Y \times \{0\} \cup A \times I \rightarrow Z$

Can be extended to a homotopy $H: Y \times I \rightarrow Z$

If $A \subset Y$ has the (HEP) with respect to every Z, then $A \subset Y$ is said to have the absolute homotopy extension property (AHEP).

Definition 3.1. [4] A subset $A \subset Y$ has the approximate Fiber Homotopy Extension Property (AFHEP) iff for any Serre fibration $q: E \rightarrow B$ and map $G: Y \times \{0\} \cup A \times I \rightarrow E$ such that $qG(y,t) = qG(y,0)$ where $y \in Y, 0 \leq t \leq 1$ there is an extension $H: Y \times I \rightarrow E$ of G such that $qG(y,t) = qG(y,0)$.

Definition 3.2. A subset $A_1, A_2 \subset Y$ has the mixed approximate fiber homotopy extension property (MAFHEP) iff for any M-Serre fibration $q_i: E_i \rightarrow B$ where $i = 1,2$, and map $G_1: Y \times \{0\} \cup A_1 \times I \rightarrow E_1$, $G_2: Y \times \{0\} \cup A_2 \times I \rightarrow E_2$ such that $p_1G_1(y_1,t) = p_1G_1(y_1,0)$, $p_2G_2(y_2,t) = p_2G_2(y_2,0)$, where $y_1, y_2 \in Y, 0 \leq t \leq 1$, there is an extention $H_1: Y \times I \rightarrow E_1$, $H_2: Y \times I \rightarrow E_2$ of G_1, G_2 such that $q_1H_1(y_1,t) = q_1H_1(y_1,0)$, $q_2H_2(y_2,t) = q_2H_2(y_2,0)$.

Theorem 3.3. If $q_i: E_i \rightarrow B$ is Mixed approximate Serre co-fibration of a compact Hausdorff E_i into a Hausdorff space B, then q_i must be an imbedding.

Proof: First let us prove that q_i must be injective. This will be proved by contradiction.
Assume that q_i were not one-one. Then, there exist two distinct points u_i and v_i in E_i with $p(u_i) = p(v_i)$. Since E_i is normal, and $\{u_i\}, \{v_i\}$ are closed subsets of E_i, there exists a continuous function

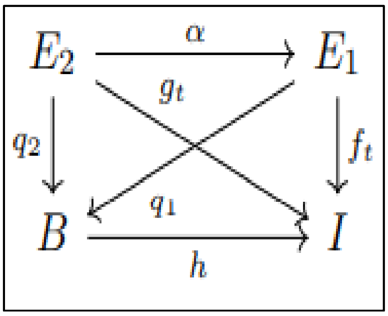
$\lambda_1: E_1 \rightarrow I$

And

$\lambda_2: E_2 \rightarrow I$

where $I = [0,1]$, with $\lambda_1(u_1) = 0, \lambda_1(v_1) = 1$, and $\lambda_2(u_2) = 0, \lambda_2(v_2) = 1$.

Define a homotopy $f_t: E_1 \rightarrow I$ and $g_t: E_2 \rightarrow I$ by $f_t(e_1) = t\lambda_1(e_1)$and $g_t(e_2) = t\lambda_2(e_2)$, $e_1 \in E_1, e_2 \in E_2, t \in I$ and define $h: B \rightarrow I$ by $h(b) = 0$ for all $b \in B$



Then, we have $hq_1 = f_0$ and $hq_2 = g_0$.
But $q_1: E_1 \rightarrow B$ and $q_2: E_2 \rightarrow B$ are a M-Serre co-fibration, hence there exists a homotopy $h_t: B \rightarrow I$ such that $h_0 = h$, and $h_tq_1 = f_t$,

$h_tq_2 = g_t$ for all $t \in I$. Now

$\lambda_1(u_1) = f_1(u_1) = h_1(q_1(u_1))$

$\lambda_1(v_1) = f_1(v_1) = h_1(q_1(v_1))$

And

$\lambda_2(u_2) = g_1(u_2) = h_1(q_2(u_2))$

$\lambda_2(v_2) = g_1(v_2) = h_1(q_2(v_2))$

Since $q_1(u_1) = q_1(v_1)$, therefore $\lambda_1(u_1) = \lambda_1(v_1)$ and $q_2(u_2) = q_2(v_2)$ therefore $\lambda_2(u_2) = \lambda_2(v_2)$ But

$\lambda_1(u_1) = 0 \neq 1 = \lambda_1(v_1)$

And

$\lambda_2(u_2) = 0 \neq 1 = \lambda_2(v_2)$

Therefore q_i must be injective. Since E_i is compact Hausdorff and B is Hausdorff, hence q_i must be an imbedding.
Hence $q_1: E_1 \to B$ and $q_2: E_2 \to B$ are a mixed approximate Serre co-fibration, when $E_i \subset B$ has the HEP with respect to all CW-complex spaces.

Theorem 3.4. Let A_i be a closed subspace of topological space Y_i. Where $i = 1,2$. Then (Y_i, A_i) is a Mixed approximate Serre cofibered pair if and only if there exists:
1) A neighborhood U_i of $A_i$which is deformable in Y_i to A_i $relA_i$ (there exists a homotopy $H_1: U_1 \times I \to Y_1$ such that $H_1(y_1, 0) = y_1, H_1(a_1, t) = a_1$and $H_2: U_2 \times I \to Y_2$ such that
$H_2(y_2, 0) = y_2, H_2(a_2, t) = a_2$and $H_1(y_1, 0) \in A_1, H_2(y_2, 0) \in A_2$ for all $y_1 \in U_1 \wedge\ y_2 \in U_2, a_1 \in A_1\ \wedge\ a_2 \in\ A_2, t \in I$.

2) A continuous function $\varphi_i: Y_i \to I$ such that $A_1 = \varphi_1^{-1}(0)$, $\varphi_1(y_1) = 1$ and $A_2 = \varphi_2^{-1}(0)$, $\varphi_2(y_2) = 1$ for all $y_1 \in Y_1 - U_1$, $y_2 \in Y_2 - U_2$.

Proof: Suppose that (Y_i, A_i) is a M-Serre cofibered pair. Then there exists a retraction

$r_i: Y_i \times I \to (Y_i \times 0) \cup (A_i \times I)$

Where $i = 1,2$, and U_i, H_i and φ_i may be chosen as follows:

$U_i = \{pr_i r_i(y_i, 1) \in A_i\}$

$H_i = pr_i r_i | U_i \times I$

$\varphi_i(y_i) = sup\ t \in I\ |t - pr_2 r_i(y_i, t)|$

pr_1 and pr_2 denoting projections on Y_i and I, respectively.
Conversely, suppose that U_i, H_i and φ_i are given and satisfy the conditions of the theorem. Since A_i is closed it suffices to prove the existence of a retraction.

$r_i: Y_i \times I \to (Y_i \times 0) \cup (A_i \times I)$

The required retraction may be constructed as follows:
a) If $\varphi_i(y_i) = 1$, $let\ r_i(y_i, t) = (y_i, 0)$.
b) If $\frac{1}{2} \leq \varphi_i(y_i) < 1$, let $r_i(y_i, t) = \{(H_1(y_1), 2(1 - \varphi_1(y_1))t), 0).(H_2(y_2), 2(1 - \varphi_2(y_2))t), 0)$.
c) If $0 < \varphi_i(y_i) \leq \frac{1}{2}$ and $0 \leq t \leq 2\varphi_i(y_i)$, let $r_i(y_i, t) = \{(H_1(y_1, \frac{t}{(2\varphi_1(y_1))})),0).(H_2(y_2, \frac{t}{(2\varphi_2(y_2))})),0)$.
d) If $0 < \varphi_i(y_i) \leq \frac{1}{2}$ and $2\varphi_i(y_i) \leq t \leq 1$, let $r_i(y_i, t) = \{(H_1(y_1, 1), t - 2\varphi_1(y_1)).(H_2(y_2, 1), t - 2\varphi_2(y_2))$.
e) If $\varphi_i(y_i) = 0$, let $r_i(y_i, t) = (y_i, t)$.

Lemma 3.5. If (Y_i, A_i) is mixed approximate Serre cofibered pair, where $i = 1,2$, then $(Y_i \times 0) \cup (A_i \times I)$ is a strong deformation retract of $Y_i \times I$.

Proof: Let $\eta_i: (Y_i \times 0) \cup (A_i \times I) \subset\ Y_i \times I$ be the inclusion map, and let

$r_i : Y_i \times I \to (Y_i \times 0) \cup (A_i \times I)$

be a retraction. A homotopy

$D_1: \eta_1 r_1 \simeq\ 1_{Y_1 \times I}\ rel(Y_1 \times 0) \cup (A_1 \times I)$

And

$D_2: \eta_2 r_2 \simeq\ 1_{Y_2 \times I}\ rel(Y_2 \times 0) \cup (A_2 \times I)$

Is given by

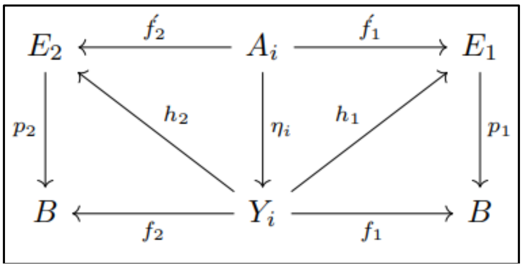
$D_i(y_i, t, t') =\ (pr_j r_i(y_i, (1 - t')t), (1 - t')pr_l r_i(y_2, t) + t't)$

Where $j = 1,2, l = 3$.

$D_1(y_1, t, \acute{t}) = (pr_1 r_1(y_1, (1-\acute{t})t), (1-\acute{t})pr_3 r_1(y_1, t) + \acute{t}t)$.

$D_2(y_2, t, \acute{t}) = (pr_2 r_2(y_2, (1-\acute{t})t), (1-\acute{t})pr_3 r_2(y_2, t) + \acute{t}t)$.

Theorem 3.6. Suppose that $p_i: E_i \to B$ is M-fibration, that A_i is a strong deformation retract of Y_i, and that there exists a map $\varphi_i: Y_i \to I$ such that $A_1 = \varphi_1^{-1}(0)$and $A_2 = \varphi_2^{-1}(0)$.Then any commutative diagram



may be filled in with a map $h_i: Y_i \to E_i$ such that $p_1 h_1 = f_1$, $p_2 h_2 = f_2$ and $h_1\eta_1 = \acute{f_1}$, $h_2\eta_2 = \acute{f_2}$. h_i is unique up to homotopy $relA_i$.

Proof: By hypothesis there exists a retraction $r_1: Y_1 \to A_1$,$r_2: Y_2 \to A_2$and a homotopy

$D_1: \eta_1 r_1 \simeq 1_{Y_1}\ relA_1$,

$D_2: \eta_2 r_2 \simeq 1_{Y_2}\ relA_2$.

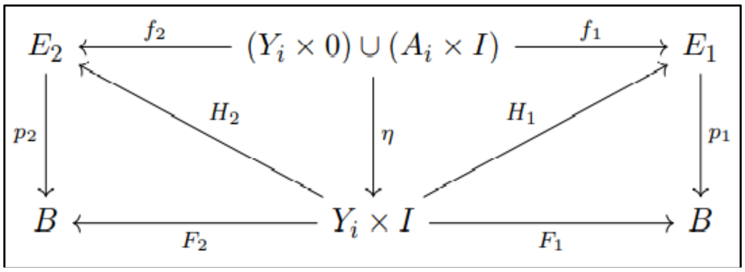
If $h_1: Y_1 \to E_1$ and $h_2: Y_2 \to E_2$ such that $h_i\eta_i = \acute{f_i}$, then $h_i \simeq h_i\eta_i r_i = \acute{f_i}\ relA$ which proves the last assertion of the theorem. Define $\underline{D_i}: Y_i \times I \to Y_i$ by

$\underline{D_1}(y_1, t) = \{D_1(y_1, \frac{t}{(\varphi_1(y_1))) \ t < \varphi_1(y_1)}\ D_1(y_1, 1)\ t \geq \ \varphi_1(y_1)$

$\underline{D_2}(y_2, t) = \{D_2(y_2, \frac{t}{(\varphi_2(y_2))) \ t < \varphi_2(y_2)}\ D_2(y_2, 1)\ t \geq \ \varphi_2(y_2)$

D_i is easily shown to be continuous. Because p_i is a fibration there exists a homotopy $\underline{F_1}: Y_1 \times I \to E_1$, $\underline{F_2}: Y_2 \times I \to E_2$ such that $p_1\underline{F_1} = f_1\underline{D_1}$, $p_1\underline{F_1} = f_1\underline{D_1}$ and $\underline{F_1}(y_1, 0) = \acute{f_1} r_1(y_1)$, $\underline{F_2}(y_2, 0) = \acute{f_2} r_2(y_2)$ for each $y_1 \in Y_1, y_2 \in Y_2$. h_i is given by $h_i(y_i) = \underline{F_i}(y_i, \varphi_i(y_i))$, where $i = 1,2$.

Theorem 3.7. Suppose that $p_i: E_i \to B$ is a mixed approximate fibration, that (Y_i, A_i) is a mixed approximate Serre cofibered pair, and that A_i is closed. Then any commutative diagram



may be filled in with a homotopy $\underline{F}: Y_i \times I \to E_i$ such that $p_1\underline{F_1} = F_1$, $p_2\underline{F_2} = F_2$and $\underline{F_i}|(Y_i \times 0) \cup (A_i \times I) = f_i$.

Proof: According to the lemma(4.5), and by theorem(4.4) there exists a Function $\psi_1: Y_1 \to I, \psi_2: Y_2 \to I$ such that $A_1 = \psi_1^{-1}(0), A_2 = \psi_2^{-1}(0)$.

Define $\varphi_1: Y_1 \times I \to I$ and $\varphi_2: Y_2 \times I \to I$ by $\varphi_1(y_1, t) = t\psi_1(y_1)$and $\varphi_2(y_2, t) = t\psi_2(y_2)$. Then $(Y_1 \times 0) \cup (A_1 \times I) = \varphi_1^{-1}(0)$and $(Y_2 \times 0) \cup (A_2 \times I) = \varphi_2^{-1}(0)$ and the theorem follows from theorem(4.6) The condition that A_i be closed is not very restrictive. For instance, A will always be closed if Y_i is Hausdorff. Not all M-Serre cofibration are closed. The most trivial example of a non-closed M-Serre cofibration is the pair $(Y_1, a_1), (Y_2, a_2)$ where Y_i is the two-point space a_i, b_i with the trivial topology

4. A mixed criterion for a map to be a mixed APPROXIMATE Serre Co-fibration

The M-criterion, which enables us to identify M-Serre Cofibration when we observe it, is discussed in this section. We frequently examine pairs (Y_i, A_i) that comprise a space A_i. and a subspace A_i. Pairs that "behave homologically" in the same way as the corresponding quotient spaces $\frac{Y_i}{A_i}$. are known as M-Serre Cofibration pairs.

Definition 4.1. A pair (Y_i, A_i) is an Mixed Neighborhood Deformation Retract pair (MNDR-pair) if there is a map $u: Y_1 \to I, v: Y_2 \to I$ such that $u^{-1}(0) = A_1, v^{-1}(0) = A_2$ and a homotopy $h: Y_1 \times I \to Y_1, k: Y_2 \times I \to Y_2$ such that $h_0 = id$, $h(a_1, t) = a_1$ and $k_0 = id, k(a_2, t) = a_2$ for $a_1 \in A_1, a_2 \in A_2$ and $t \in I$, and $h(y_1, 1) \in A_1$ if $u(y_1) < 1$, and $k(y_2, 1) \in A_2$ if $v(y_2) < 1$. (Y_i, A_i) is a MDR-pair if $u(y_1) < 1, v(y_2) < 1$ for all $y_1 \in Y_1, y_2 \in Y_2$, that means A_i is a deformation retract of Y_i where $i = 1,2$.

Lemma 4.2. If (h_i, u_i) and (k_i, v_i) represent (Y_i, A_i) and (Z_i, B_i) as (MNDR-pairs), then (l_i, w_i) represents the (product pair) $(Y_i \times Z_i, Y_i \times B_i \cup A_i \times Z_i)$ as an MNDR-pair, where $w_i(y_i, z_i) = min(u_i(y_i), v_i(z_i))$ and

$$l_i(y_i, z_i, t) = \{\left(h_i(y_i, t), k_i\left(z_i, \frac{tu_i(y_i)}{v_i(z_i)}\right)\right) \; if \; v_i(z_i) \geq u_i(y_i) \; (h_i\left(y_i, \frac{tv_i(z_i)}{u_i(y_i))}, k_i(z_i, t)\right) \; if \; u_i(y_i) \geq v_i(z_i)$$

If (Y_i, A_i) or (Z_i, B_i) is a DR-pair, then so is $(Y_i \times Z_i, Y_i \times B_i \cup A_i \times Z_i)$.

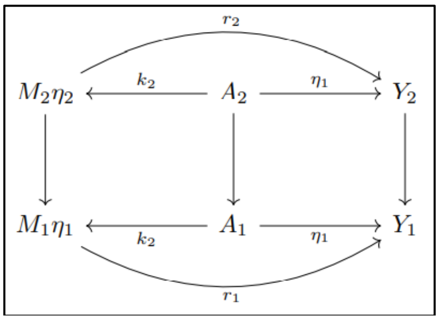
Proof: If $v_i(z_i) = 0$ and $v_i(z_i) \geq u_i(y_i)$, then $u_i(y_i) = 0$ and both (Z_i, B_i) and (Y_i, A_i), therefore we can and must understand $l_i(y_i, z_i, t)$ to be (y_i, z_i). This and the symmetric observation make it simple to verify that l_i is, as intended, a well-defined continuous homotopy.

Theorem 4.3. Let A_i be a closed subspace of Y_i, where $i = 1,2$. Then the following are equivalent:

- (Y_i, A_i) is an MNDR-pair.
- $(Y_i \times I, Y_i \times \{0\} \cup A_i \times I)$is a MDR-pair.
- $Y_i \times \{0\} \cup A_i \times I$ is a M-retract of $Y_i \times I$.
- The inclusion $\eta_i: A_i \to Y_i$ is a M-Serre cofibration.

Proof: According to the lemma, (a) entails (b), (b) simply implies (c), and (c) and (d) are equal, as we have previously determined. Assume given a retraction $r_i : Y_i \times I \to Y_i \times \{0\} \cup A_i \times I$.
Let $pr_i: Y_i \times I \to Y_i$ and $pr_2: Y_i \times I \to I$ be the projections and define $u: Y_1 \to I$ by
$u(y_1) = sup\, sup\, \{t - pr_2\, r_1(y_1, t) | t \in I\}$,
and $v: Y_2 \to I$ by
$v(y_2) = sup\, sup\, \{t - pr_2\, r_2(y_2, t) | t \in I\}$
$h: Y_1 \times I \to Y_1$ by
$h(y_1, t) = pr_1 r_1(y_1, t)$
and $k: Y_2 \times I \to Y_2$ by
$k(y_2, t) = pr_2 r_2(y_2, t)$
Then $(h, u), (k, v)$ represents (Y_i, A_i) as an MNDR-pair. Here $u^{-1}(0) = A_1$ since $u(y_1) = 0$ and $v^{-1}(0) = A_2$ since $v(y_2) = 0$ implies that $r_i(y_i, t) \in A_i \times I$ for $t > 0$ and thus also for $t = 0$ since $A_i \times I$ is closed in $Y_i \times I$,where $i = 1,2$.

Example 4.4. Let $\eta_1: A_1 \to Y_1$and $\eta_2: A_2 \to Y_2$ be a M-Serre cofibration, where $i = 1,2$. We then have the commutative diagram



Where $k_1(a_1) = (a_1, 1)$ and $k_2(a_2) = (a_2, 1)$ where $M_i\eta_i \equiv Y_i \cup_\eta (A_i \times I)$. The obviou homotopy inverse $l_i: Y_i \to M_i\eta_i$ has $l_i(y_i) = (y_i, 0)$ and is thus very far from being a map A_i. The proposition guarantees that l_i is homotopic to a map under A_i that is homotopy inverse to r_i under A_i

5. Conclusion

We mention the most important result that reached through our study mixed structure Serre cofibration as following:

- Every approximate Serre cofibration is Mixed approximate Serre cofibration.
- product Two Mixed approximate Serre cofibration is also Mixed approximate Serre cofibration.
- M-pullback of Mixed approximate Serre cofibration is also Mixed approximate Serre cofibration.

Let A_i be a closed subspace of Y_i, Then equivalent: The inclusion $\eta_i: A_i \to Y_i$ is a mixed approximate Serre cofibration.

6. References

1. Rainer A. Orbit projections as fibrations. Czechoslovak Mathematical Journal. 2009;59(134):538-592.
2. Mustafa HJ. Some theorems on fibration and cofibration. PhD Thesis, California University, Los Angeles; 1972.
3. Rohil P. An introduction to the Serre spectral sequence. Princeton University; 2018.
4. Lin Ch. Introductory differential topology and an application to the Hopf fibration. University of Chicago; June 2020.
5. May JP. A concise course in algebraic topology. 1st ed. Chicago: University of Chicago Press; 1999 Sep 1.
6. Al Baydli DW, Hussien AA. Mixed cofibration and mixed Hurewicz cofibration. University of Anbar - College of Education for Women, Wassit University - College of Science; 2009.
7. Strom A. Note on cofibrations. Mathematica Scandinavica. 1966;19:135-149.