International Journal of Statistics and Applied Mathematics 2025; 10(1): 59-61

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2025; 10(1): 59-61

https://www.mathsjournal.com
Received: 05-12-2024
Accepted: 08-01-2025

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W-distance fixed point theorems in fuzzy probabilistic metric space

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Abstract
In this paper, two common fixed point theorems are presented for non-commuting JSR* mappings with w-distance in complete fuzzy probabilistic metric space, supported by a suitable example.

Keywords: Fuzzy probabilistic metric space, w-distance, JSR* mapping, common fixed point

1. Introduction
The probabilistic notation was introduced by [2] in his basic paper and then [3] introduce fuzzy concept which provided important contribution in the field of pure and applied mathematics. A bulk of literature exists with commuting and non-commuting mappings. Fuzzy probabilistic metric space is used by R. Shrivastav, V. Patel and V.B. Dhagat [3] and Kada-Suziki-Takahashi [1] introduced the concept of w-distance on a metric space. In contribution we are defining non-commuting pair of maps JSR* maps which is more improved than the known mappings. Here we defined fuzzy probabilistic metric space with w-distance.

2. Preliminaries
Definition 2.1: Let (X, Fα, t) be a fuzzy Probabilistic metric space. Then the function pα: $X \times X \to [0,\infty)$ for $\alpha \in [0, 1]$ is called w-distance on X if

- $(x, z, t) \le p_\alpha(x, y, t) + p_\alpha(y, z, t)$ for any $x, y, z \in X$,
- For any $x, y, z \in X$, $p_\alpha(x, y, z) \to [0, \infty)$ is lower semi continuous and
- For any $\epsilon > 0$, there exists $\delta > 0$ such that $p_\alpha(x, z, t) \le \delta$ and $p_\alpha(z, y, t) \le \delta$ then $p_\alpha(x, y, t) \le \epsilon$.

Definition 2.2: Let S and T be two self-maps of a metric space (X, d). The pair {S, T} is said to be S-JSR* mappings iff for every sequence x_n in X such that $\lim n\to\infty Sx_n = \lim n\to\infty T xn = t$ for some t in X implies

$\alpha d(T Sx_n, T x_n) \le \alpha d(SSx_n, Sx_n)$, where α = lim Sup.

Example 2.1: Let X = [0, 1] with $d(x, y) = |x - y|$ and S,T are two self-mapping on X defined by $S(x) = 1 - x$,
$T(x) = \frac{1}{2x+1}$ for $x \in X$. Now we have the sequence $\{x_n\}$ in X is defined as $x_n = \frac{1}{n}$, $n \in N$. Then we have $\lim n\to\infty Sx_n = \lim n\to\infty T x_n = 1$. $|T Sx_n - T x_n| \to \frac{2}{3}$, $|SSxn - Sxn| \to 1$ as $n \to \infty$. Thus pair {S, T} is S − JSR mapping.

Example 2.2: Let X = [0, 1] with $p_\alpha(x, y, t) = \alpha.t.\max\{|\frac{x}{2} - y|, \frac{1}{2}|x - y|\}$ and S, T are two self mapping on X defined by $S(x) = 1 - x$, $T(x) = \frac{1}{2x+1}$. Now we have the sequence $\{x_n\}$ in X is defined as $x_n = \frac{1}{n}$, $n \in N$. Then we have $\lim n\to\infty Sx_n = \lim n\to\infty T x_n = 1$. Now

$p_\alpha(T Sx_n, T x_n, t) = \alpha.t.m\{|(T Sx_n)/2 - T x_n|, |T Sx_n - T x_n|/2\} = \alpha.t.m.\{5/6, 1/3\} = \alpha.5/6$

$p_\alpha(T x_n, T Sx_n, t) = \alpha.t.m\{|(T x_n)/2-T Sx_n|, |T x_n - T Sx_n|/2\} = \alpha.t.m.\{5/6, 1/3\} = \alpha.1/3$
$p_\alpha(SSx_n, Sx_n, t) = \alpha.t.m\{|(SSx_n)/2 - Sx_n|, |SSx_n - Sx_n|/2\} = \alpha.t.m.\{1, 1/2\} = \alpha.1$
$p_\alpha(Sx_n, SSx_n, t) = \alpha.t.m\{|(Sx_n)/2-SSx_n|, |Sx_n - SSx_n|/2\} = \alpha.t.m.\{1/2, 1/2\} = \alpha.1/2.$
Clearly pair {S,T} is $S - JSR * (p)$ mappings. Also $(x, y) \neq p_\alpha(y, x)$.
Before going to main results, we require to establish the following lemmas:

Lemma 2.1: Let (X, Fα, t) be a fuzzy probabilistic metric space and Pα be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequence in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequence in $(0,\infty)$ converging to 0 and for x, y, z ∈ X. Then the following conditions hold:
1. If $(x_n, y_n, t) \leq \alpha_n$ and $P\alpha(x_n, z, t) \leq \beta_n$ for any n ∈ N then y = z.
In particular, if (x, y, t) = 0 and (x, z, t) = 0 then y = z.
II. If $(x_n, y_n, t) \leq \alpha_n$ and $P\alpha(x_n, z, t) \leq \beta n$ for any n ∈ N then $\{y_n\}$ converges to z.
III. If $P\alpha\ (x_n, x_m, t) \leq \alpha_n$ for any n, m ∈ N with m > n, then $\{x_n\}$ is Cauchy sequence and
IV. If $P\alpha(y, x_n, t) \leq \alpha_n$ for any n ∈ N then $\{x_n\}$ is Cauchy sequence.

Lemma 2.2: Let (X, F_α, t) be a fuzzy probabilistic metric space with a w-distance p_α and let S and T be self-mappings on X, satisfying $T x_n = Sx_{n+1}$ for n = 0, 1, 2, ... Assume that there exist a continuous self-mapping Φ of $[0, \infty)$ such that

$$p_\alpha (T x, T y, t) \leq \Phi(p_\alpha(Sx, Sy, t)) \text{ for all } x, y \in X \quad (2.1)$$

$$\text{and for each } r > 0, \Phi(r) < r \quad (2.2)$$

Then
A. For an arbitrary $\epsilon > 0$, there exist positive integer m, s such that $m \leq n < s$ implies $p\alpha(T x_n, T x_s, t) < \epsilon$.
B. The sequence $\{Tx_n\}$ is a Cauchy sequence.

Proof: We have $p_\alpha(T, T x_{n+1}, t) \leq \Phi(p_\alpha(Sx_n, Sx_{n+1}, t)) = \Phi(p_\alpha(T x_{n-1}, T x_n, t)) < (p\alpha(T x_n, T x_{n+1}, t))$ for n = 1, 2, 3, ... Thus $\{(T x_n, T x_{n+1}, t)\}$ is a decreasing sequence of non-negative real number and there exists non-negative real number λ such that $\lim n\to\infty\ p_\alpha(T x_n, T x_{n+1}, t) = \lambda$. Let λ > 0, then the inequality $p_\alpha(T, T x_{n+1}, t) \leq \Phi(p_\alpha(T x_{n-1}, T x_n, t))$
Now by the continuity of Φ we have $\lambda \leq \Phi(\lambda) < \lambda$, which is a contradiction. Therefore λ = 0 so $p_\alpha(T, T x_{n+1}, t) \to 0$ as $n \to \infty$. Now suppose that (A) does not hold. Then, there exists an $\epsilon > 0$ such that for all sufficiently large positive integer k, there exist positive integer sk, nk with $k \leq nk < sk$ such that

$$\epsilon \leq p_\alpha(T x, T x_{sk}, t), p_\alpha(T x_{nk}, T x_{nk-1}, t) < \epsilon \quad (2.3)$$

From the above result, we have $p_\alpha(T x, T x_{sk}, t) \to \epsilon$ and $p_\alpha(T x_{nk}, T x_{nk-1}, t) \to 0$ as $k \to \infty$ and
$(T x_{nk}, T x_{sk}, t) \leq p_\alpha(T x_{nk}, T x_{nk+1}, t) + p_\alpha(T x_{nk+1}, t, T x_{sk}, t) \leq (T x_{nk}, T x_{nk+1}, t) + \Phi(p_\alpha(Sx_{nk+1}, t, Sx_{sk}, t))$

$$= (T x_{nk}, T x_{nk+1}, t) + \Phi(p_\alpha(T x_{nk}, T x_{nk-1}, t)) \quad (2.4)$$

By the hypothesis and (2.4), we obtain $\epsilon \leq \Phi(\epsilon) < \epsilon$. This is contradiction therefore (A) hold.
(B) By the third condition of the definition of a w-distance p_α and (A), we have that $\{Tx_n\}$ is a Cauchy sequence.

Lemma 2.3: Let (X, F_α, t) be a fuzzy probabilistic metric space with a w-distance p and let S and T be self mappings on X, satisfying $T x_n = Sx_{n+1}$ for n = 0, 1, 2, ... and the following conditions: for given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq p_\alpha(Sx, Sy, t) < \epsilon + \delta \Rightarrow p_\alpha(T x, T y, t) < \epsilon \quad (2.5)$$

and

$$p\alpha(Sx, Sy, t) < \epsilon \Rightarrow p\alpha(T x, T y, t) \leq \tfrac{1}{2}\, p_\alpha(Sx, Sy, t) \quad (2.6)$$

Then
A. For an arbitrary $\epsilon > 0$, there exists positive integer M such that $M \leq n < s$ implies $p\alpha(T, T x_s, t) < \epsilon$.
B. The sequence $\{Tx_n\}$ is a Cauchy sequence.

Proof: The proof is same as the proof of lemma 2.2

3. Main Results

Theorem 3.1: Let (X, F_α, t) be a fuzzy probabilistic metric space with a w-distance p and let T and S be $S - JSR(p)$ self mappings on X, satisfying $T(X) \subset S(X)$, (2.1), (2.2) and for each z ∈ X with $z \neq T z$ or $z \neq Sz$

$$\inf\{p_\alpha(T x, z, t) + p_\alpha(Sx, z, t) + p_\alpha(ST x, T x, t) + p_\alpha(SSx, Sx, t), x \in X\} \quad (3.1)$$

Then T and S have a unique common fixed point.

Proof: By the assumption, we have all the conditions of Lemma 2.2. Thus by (B) $\{T x_n\}$ is a Cauchy's sequence. Since X is a complete metric space and $T x_n = Sx_{n+1}$, $\{T x_n\}$ and $\{Sx_n\}$ have a limit point z in X. Suppose that $z \neq T z$ or $z \neq Sz$. Now, since $\lim_{n\to\infty} T x_n = \lim_{n\to\infty} Sx_n = z$, therefore by (A) and the lower semi continuity, we have
$\lim_{n\to\infty} p_\alpha(T x_n, z, t) = \lim_{n\to\infty} p_\alpha(Sx_n, z, t)$. Now,
$0 < \inf\{p_\alpha(T x, z, t) + p_\alpha(Sx, z, t) + p_\alpha(ST x, T x, t) + p_\alpha(SSx, Sx, t), x \in X\}$
$\leq \inf\{p_\alpha(T x_n, z, t) + p_\alpha(Sx_n, z, t) + p\alpha(STx_n, T x_n, t) + p\alpha(SSx_n, Sx_n, t)\}$
$\leq \inf\{p_\alpha(T x_n, z, t) + p_\alpha(Sx_n, z, t) + \max[\alpha p_\alpha(ST x_n, T x_n), \alpha p_\alpha(SSx_n, Sx_n)] + p_\alpha(SSx_n, Sx_n)\} < 0$ which is a contradiction. Thus z is a common fixed point of T and S. The uniqueness can be proved by the use of (2.1), (2.2) and (I) of lemma 2.1.

Theorem 3.2: Let (X, Fα, t) be a fuzzy probabilistic metric space with a w-distance p and let T and S be $S - JSR * (p)$ self-mappings on X, satisfying $T(X) \subset S(X)$, (2.1), (2.2) and for each z ∈ X with $z \neq T z$ or $z \neq Sz$

$$\inf\{p(T x, z, t) + p(Sx, z, t) + p(TSx, ST x, t) + p(SSx, TT x, t), x \in X\} \quad (3.2)$$

Then T and S have a unique common fixed point.

Proof: Since $\{T(X) \subset X\}$, we obtain a sequence in X such that $Tx_n = Sx_{n+1}$. Since X is complete and $Tx_n = Sx_{n+1}$ there exists z in X such that $Tx_n \to z$ and $Sx_n \to z$. Suppose that $z \neq Tz$ or $z \neq Sz$, Since $\lim_{n\to\infty} T x_n = \lim_{n\to\infty} Sx_n = z$, therefore by (A) and the lower semi continuity, we have $\lim_{n\to\infty} p_\alpha (Tx_n, z) = \lim_{n\to\infty} p_\alpha (Sx_n, z)$. Now,
$0 < \inf\{p_\alpha(Tx, z, t) + p_\alpha(Sx, z, t) + p_\alpha (TSx, ST x, t) + p_\alpha(SSx, T Tx, t), x \in X\}$
$\leq \inf\{p_\alpha(Tx_n, z, t) + p_\alpha(Sx_n, z, t) + p_\alpha(TSx_n, STx_n, t) + p_\alpha(SSx_n, TT x_n, t)\}$

$\leq$ inf{$p_\alpha(Tx_n, z, t)$+$p_\alpha(Sx_n, z, t)$+max[$\alpha p_\alpha(TSx_n, STx_n, t)$, $\alpha p_\alpha(SSx_n, TTx_n, t)$]+$p_\alpha(SSx_n, TTx_n, t)$}
< 0. which is a contradiction. Thus z is a common fixed point of T and S. The uniqueness of the common fixed point is clear by (I) of lemma 2.1 and (3.1), (3.2).

Example 3.1: Let X = [0, 1] with $p_\alpha(x, y, t) = \alpha.t.\max\{|\frac{x}{2} - y|, \frac{1}{2}|x - y|\}$and S,T are two self mapping on X defined by $S(x) = 1 - x$, $T(x) = \frac{1}{2x+1}$. Now we have the sequence $\{x_n\}$ in X is defined as $x_n = \frac{1}{n}$, $n \in N$. Then we have $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 1$. The pair (S, T) satisfy all the conditions of the theorem and $\frac{1}{2}$ is the fixed point.

4. Author's Contributions

Promila prepared initial draft, Vinod Bhatia and Vishvajit Singh demonstrated the key aspects and suggested remarkable modifications, Authors approved the final draft of the article.

5. Conflicts of Interest

All the authors declare that they have no conflict of interest.

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