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Stanzin Kunga
University of Jammu, Jammu and Kashmir, India

Corresponding Author:
Stanzin Kunga
University of Jammu, Jammu and Kashmir, India

Weighted composition operators on Bicomplex bloch space

Stanzin Kunga

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Abstract
Let $\mathbb{U}$ be the bidisk in $\mathbb{BC}$ and $\phi: \mathbb{U} \to \mathbb{U}$ as well as $f: \mathbb{U} \to \mathbb{U}$ be a bicomplex holomorphic self map of $\mathbb{U}$. For a bicomplex holomorphic map f on $\mathbb{U}$ the weighted composition operator $u.C_\phi$ is defined by $u.(u.C_\phi f)(z) = u(z).f \circ \phi(z)$ for every $z \in \mathbb{U}$, where ube a fixed bicomplex holomorphic function on $\mathbb{U}$. The present study deals the condition for boundedness and compactness of Weighted Composition Operators on bicomplex Bloch space which define over the unit ball $\mathbb{U}$, an extension of the classical Bloch space. The bicomplex Bloch is a higher-dimensional generalization of the Bloch space, structured to accommodate bicomplex analysis. The bicomplex Bloch space over the unit ball $\mathbb{U}$ is useful for studying more advanced problems in analysis.

Keywords: Bicomplex, studying, problems

Introduction
The theory of bicomplex numbers is an area of active research for quite a long time since the innovative work of Segre in search of special algebras. In 1892, Segre proposed the concept of bicomplex numbers which can be thought of as a generalization of complex numbers. Because not every non zero bicomplex number has a multiplicative inverse, the set of bicomplex numbers is a commutative ring with unity that contains the field of complex numbers but does not form a field, the study of zero divisors in bicomplex analysis is introduced. For a long time, bicomplex numbers have been explored. In recent years, a large amount of work has been published. G. B. Price's work [8] is a thorough examination of bicomplex numbers. One can refer to [1, 5, 8] for a systematic study of basic of bicomplex functional analysis.
Composition operators are extensively studied on various holomorphic function spaces of different domains in $\mathbb{C}$ and $\mathbb{C}^n$. There are many papers on the study of composition operators on Bloch space and Bloch type spaces of holomorphic functions, see [2, 6] and references therein. The boundedness and the compactness of weighted composition operators on Bloch space was characterised by Ohno, see [7]. My results can be viewed as a generalisations of their results.

Preliminaries
In this section, we summarize some common facts about bicomplex numbers and bicomplex holomorphic functions. The set of bicomplex numbers $\mathbb{BC}$ is defined as.

$$\mathbb{BC} = \{Z = w_1 + jw_2 | w_1, w_2 \in \mathbb{C}(i)\},$$

where i and j are imaginary units such that $ij = ji, i^2 = j^2 = -1$ and $\mathbb{C}(i)$ is the set of complex numbers with the imaginary unit i. The set $\mathbb{BC}$ of bicomplex numbers form a ring under the usual addition and multiplication of bicomplex numbers. " "Moreover, $\mathbb{BC}$ is a module over itself. The set of positive hyperbolic number is denoted by $\mathbb{D}^+$ which is a subset of $set\ of\ hyperbolic\ numbers\ \mathbb{D}$ is given by

$$\mathbb{D}^+ = \{\beta_1 + k\beta_2 : \beta_1^2 - \beta_2^2 \geq 0, \beta_1 \geq 0\}.$$

We can discuss three conjugations for bicomplex numbers in the same way we can do for usual complex numbers, because $\mathbb{BC}$ comprises two imaginary units with squares equal to -1 and a hyperbolic units with square equal to 1.

1. $\bar{Z} = \bar{w}_1 + j\bar{w}_2 (the\ bar - conjugation)$;
2. $Z^{\dagger} = w_1 - jw_2, (the\ \dagger - conjugation)$;
3. $Z^{*} = \bar{w}_1 - j\bar{w}_2, (the * - conjugation)$,

where $\bar{w}_1, \bar{w}_2$ denote the usual complex conjugates to $w_1, w_2 \in \mathbb{C}(i)$.
If $Z = w_1 + jw_2 \neq 0$ is such that $Z\,.\,Z^{\dagger} = {w_1}^2 + {w_2}^2 = 0$, then Z is a zero divisor. The set of zero divisors $\mathbb{NC}$ of $\mathbb{BC}$ is, thus, given by

$$\mathbb{NC} = \{Z \mid Z \neq 0, {w_1}^2 + {w_2}^2 = 0\},$$

and is called the null cone and $\mathbb{NC}_0 = \mathbb{NC} \cup \{0\}$.
The hyperbolic numbers e and $e^{\dagger}$ defined as

$$e = \frac{1+k}{2}\ and\ e^{\dagger} = \frac{1-k}{2},$$

are zero divisors, which are linearly independent in the $\mathbb{C}(i)$- vector space and satisfy the following properties:

$$e^2 = e, (e^{\dagger})^2 = e^{\dagger}, e^{*} = e, (e^{\dagger})^{*} = e^{\dagger}, e + e^{\dagger} = 1, e \cdot e^{\dagger} = 0.$$

Any bicomplex number $Z = w_1 + jw_2$ can be uniquely written as " "

$$Z = z_1 e + z_2 e^{\dagger} \tag{2.1}$$

where $z_1 = w_1 - iw_2$ and $z_2 = w_1 + iw_2$ are elements of $\mathbb{C}(i)$. Formula (2.1) is called the idempotent representation of a bicomplex number Z. A hyperbolic number $\alpha = \beta_1 + k\beta_2$ in idempotent representation can be written as

$$\alpha = \alpha_1 e + \alpha_2 e^{\dagger},$$

where $\alpha_1 = \beta_1 + \beta_2$ and $\alpha_2 = \beta_1 - \beta_2$ are real numbers. We say that α is a positive hyperbolic number if $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$. Writing these hyperbolic numbers in their idempotent form $\alpha = \alpha_1 e + \alpha_2 e^{\dagger}$ and $\gamma = \gamma_1 e + \gamma_2 e^{\dagger}$, with real numbers $\alpha_1, \alpha_2, \gamma_1$ and γ_2, we have that

$\alpha \preceq \gamma$ iff $\alpha_1 \leq \gamma_1$ and $\alpha_2 \leq \gamma_2$.

If $\gamma - \alpha \in \mathbb{D}^{+}\backslash\{0\}$, we write $\gamma \succ \alpha$. This imples that $z \in \mathbb{D}^{+}$ is equivalent to $z \succeq 0$ and that $z \in \mathbb{D}^{+}\backslash\{0\}$ is equivalent to $z \succ 0$.
A module defined over the ring of bicomplex numbers $\mathbb{BC}$ (or ring of hyperbolic numbers $\mathbb{D}$) is called a $\mathbb{BC}$-module (or $\mathbb{D}$-module). Consider the set $\mathfrak{X}_1 = \mathfrak{X}e$ and $\mathfrak{X}_2 = \mathfrak{X}e^{\dagger}$.Then $\mathfrak{X}_1 \cap \mathfrak{X}_2 = \{0\}$. Thus, we can write

$$\mathfrak{X} = \mathfrak{X}_1 e + \mathfrak{X}_2 e^{\dagger}, \tag{2.2}$$

where $\mathfrak{X}_1 = \mathfrak{X}e$ and $\mathfrak{X}_2 = \mathfrak{X}e^{\dagger}$ are $\mathbb{C}(i)$-vector (or $\mathbb{R}$-vector) spaces. Equation (2.2) is called the idempotent decomposition of $\mathfrak{X}$.

Definition 2.1 Let $\mathfrak{X}$ be a $\mathbb{BC}$ module. A function $\|\cdot\|_{\mathbb{D}}: \mathfrak{X} \to \mathbb{D}^{+}$ is said to be a hyperbolic-valued norm (or $\mathbb{D}$-valued norm) on $\mathfrak{X}$ if it satisfies the following properties:

1. $\|x\|_{\mathbb{D}} = 0$ if and only if $x = 0$.
2. $|\mu x|_{\mathbb{D}} = |\mu|_k \|x\|_{\mathbb{D}}, \forall\, x \in \mathfrak{X}, \forall\, \mu \in \mathbb{BC}$.
3. $\|x + y\|_{\mathbb{D}} \prec \|x\|_{\mathbb{D}} + \|y\|_{\mathbb{D}}, \forall x, y \in \mathfrak{X}$.

The $\mathbb{BC}$ module can be endowed canonically with the hyperbolic, or $\mathbb{D}$-valued norm denoted by $\|\cdot\|_{\mathbb{D}}$ as follows:

$$\|x\|_{\mathbb{D}} = \left\|x_1 e + x_2 e^{\dagger}\right\|_{\mathbb{D}} = \|x_1\|_1 e + \|x_2\|_2 e^{\dagger}. \tag{2.3}$$

Definition 2.2 Let $\Omega \subset \mathbb{BC} \to \mathbb{BC}$ be a function defined on the domain Ω. The derivative $f'(z_0)$of the function f at a point $z_0 \in \Omega$ is the limit, if it exists,

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

such that $z - z_0$ is an invertible bicomplex number. If f is derivable for all $z \in \Omega$, we say that it is a bicomplex holomorphic function in Ω.

Theorem 2.3 A bicomplex valued function f: $\Omega \to \mathbb{BC}$ defined as

$$f(z) = f_1(z_1)e + f_2(z_2)e^{\dagger}$$

is a bicomplex holomorphic function on Ω if and only if f_1 and f_2 are complex holomorphic functions on Ω_1 and Ω_2 respectively. It derivatives is given by

$$f'(z) = f_1'(z_1)e + f_2'(z_2)e^{\dagger}.$$

The unit disk in $\mathbb{C}(i)$ will be denoted by U. The bidisk $\mathbb{U}$ in $\mathbb{BC}$ is the product-type domain:

$$\mathbb{U} = U(e + e^{\dagger}) = U_1 e + U_1 e^{\dagger} = \{z_1 e + z_2 e^{\dagger} : z_1, z_2 \in U\}.$$

Note that such bidisk should be seen as the cartesian product $U \times U$ in the idempotent $\mathbb{C}^2(i)$.

3 Bio-complex Bloch spaces

Definition 3.1 [9] A bicomplex holomorphic function f in $\mathbb{U}$ is said to be bicomplex Bloch space $\mathcal{B}$ if

$$\| f \|_{\mathcal{B}} = \sup\{(1 - \| z \|_{\mathbb{D}}^2) \| f'(z) \|_{\mathbb{D}} : z \in \mathbb{U}\} \prec \infty.$$

The little bicomplex Bloch space $\mathcal{B}_0$ is the closed subspace of $\mathcal{B}$ consisting of functions f with :

$$\lim_{\| z \|_{\mathbb{D}} \to 1^-} (1 - \| z \|_{\mathbb{D}}^2) \| f'(z) \|_{\mathbb{D}} = 0.$$

The hyperbolic norm in the bicomplex Bloch space is defined as

$$\| f \|_{\mathbb{D}} = \| f(0) \|_{\mathbb{D}} + \| f \|_{\mathcal{B}}.$$

With this hyperbolic norm, the bicomplex Bloch space is complete.
By idempotent decomposition it is immediate that:

$\mathcal{B} = \mathcal{B}_1 e + \mathcal{B}_2 e^{\dagger}$ and $\mathcal{B}_0 = \mathcal{B}_{0,1} e + \mathcal{B}_{0,2} e^{\dagger}$, where $\mathcal{B}_l$ and $\mathcal{B}_{0,l}$ are the complex Bloch spaces, where $l = 1,2$.

Lemma 3.2 [9] Let a $\in \mathbb{U}$ and define the bicomplex Mobius transformation T: $\mathbb{U} \to \mathbb{U}$ as:

$$T(z) = \Delta \frac{a - z}{1 - a^* z}, with \ \| \Delta \|_{\mathbb{D}} = 1.$$

Then

$$(1 - \| z \|_{\mathbb{D}}^2) \| T'(z) \|_{\mathbb{D}} = 1 - \| T(z) \|_{\mathbb{D}}^2.$$

In particular $\| T(z) \|_{\mathbb{D}} = 1$ if and only if $\| z \|_{\mathbb{D}} = 1$, that is, if z belongs to the distinguished boundary of $\mathbb{U}$.

Lemma 3.3 [9] Let f: $\mathbb{U} \to \mathbb{U}$ be a bicomplex holomorphic function. Then

$$\| f'(z) \|_{\mathbb{D}} \leq \frac{1 - \| f(z) \|_{\mathbb{D}}^2}{1 - \| z \|_{\mathbb{D}}^2}, for\ all\ z \in \mathbb{U}.$$

The equality occurs if and only if f is a Mobius transformation of the bidisk.
Let $f: \mathbb{U} \to \mathbb{BC}$ be a bicomplex holomorphic function and $\phi: \mathbb{U} \to \mathbb{U}$ be a bicomplex holomorphic self map of $\mathbb{U}$. Thus, with $z = z_1 e + z_2 e^{\dagger} \in \mathbb{U}$, we can write $f(z) = f_1(z_1)e + f_2(z_2)e^{\dagger}$, where f_1 and f_2 are complex valued holomorphic functions on the unit disk U and $\phi(z) = \phi_1(z_1)e + \phi_2(z_2)e^{\dagger}$, where ϕ_1 and ϕ_2 are holomorphic self maps of the unit disk U.

Definition 3.4 [4] Let $\phi: \mathbb{U} \to \mathbb{U}$ be a bicomplex holomorphic self map of $\mathbb{U}$. We define the composition operator C_ϕ induced from ϕ on $\mathcal{B}$ by

$$C_\phi f = f o \phi, \forall\ f \in \mathcal{B}.$$

Further, from lemma [see [4]], we can write

$C_\phi f(z) = C_{\phi_1} f_1(z_1)e + C_{\phi_2} f_2(z_2)e^\dagger,$

where C_{ϕ_l}, for $l = 1,2$ is a composition operator on complex Bloch spaces.

Definition 3.5 Let u be a fixed bicomplex holomorphic function on $\mathbb{U}$ and $\phi: \mathbb{U} \to \mathbb{U}$ be a bicomplex holomorphic self map of $\mathbb{U}$. Then we can define weighted composition operator $u.C_\phi$ on $\mathcal{B}$ by

$u.C_\phi f = u.f\ o\ \phi, f \in \mathcal{B}.$

We can write $u.C_\phi f$ as

$u.C_\phi f(z) = u_1 C_{\phi_1} f_1(z_1)e + u_2 C_{\phi_2} f_2(z_2)e^\dagger,$

and its derivatives is given by

$[u(z).fo\phi(z)]' = [u_1(z_1).f_1 o\phi_1(z_1)]'e + [u_2(z_2).f_2 o\phi_2(z_2)]'e^\dagger$

$= (u_1'(z_1).f_1 o\phi_1(z_1) + u_1(z_1).f_1'(\phi_1(z_1)).\phi_1'(z_1))e + (u_2'(z_2).f_2 o\phi_2(z_2) + u_2(z_2).f_2'(\phi_2(z_2)).\phi_2'(z_2))e^\dagger$

$= [(u_1'(z_1).f_1 o\phi_1(z_1))e + (u_2'(z_2).f_2 o\phi_2(z_2))e^\dagger] + [(u_1(z_1).f_1'(\phi_1(z_1)).\phi_1'(z_1))e + (u_2(z_2).f_2'(\phi_2(z_2)).\phi_2'(z_2)e^\dagger)]$

$= (u'(z).fo\phi(z)) + (u(z).f'(\phi(z)).\phi'(z)).$

Proposition 3.6 Let f: $\mathbb{U} \to \mathbb{BC}$ be a bicomplex holomorphic function. If f $\in \mathcal{B}$, then

$\| f(z) - f(w) \|_{\mathbb{D}} \preceq \| f \|_{\mathcal{B}}\ \beta(z,w).$

Proof. For any $f \in \mathcal{B}$, there exist two holomorphic functions $f_1, f_2: U \to \mathbb{C}(i)$ such that $f(z) = f_1(z_1)e + f_2(z_2)e^\dagger$. Since $f \in \mathcal{B}$, it follows that f_1 and f_2 lie in the complex Bloch spaces. So by using [7], it follows that

$\| f(z) - f(w) \|_{\mathbb{D}} = \| (f_1(z_1) - f_1(w_1))e + (f_2(z_2) - f_2(w_2))e^\dagger \|_{\mathbb{D}}$

$= \| f_1(z_1) - f_1(w_1) \|_1\ e + \| f_2(z_2) - f_2(w_2) \|_2\ e^\dagger \preceq \| f_1 \|_{\mathcal{B}_1}\ \beta_1(z_1,w_1)e + \| f_2 \|_{\mathcal{B}_2}\ \beta_2(z_2,w_2)e^\dagger$

$= (\| f_1 \|_{\mathcal{B}_1}\ e + \| f_2 \|_{\mathcal{B}_2}\ e^\dagger)(\beta_1(z_1,w_1)e + \beta_2(z_2,w_2)e^\dagger)$

$= \| f \|_{\mathcal{B}}\ \beta(z,w).$

Thus, $\| f(z) - f(w) \|_{\mathbb{D}} \preceq \| f \|_{\mathcal{B}}\ \beta(z,w).$

Proposition 3.7 *Let $f: \mathbb{U} \to \mathbb{BC}$ be a bicomplex holomorphic function. If $f \in \mathcal{B}$, then*

$\| f(z) \|_{\mathbb{D}} \preceq \frac{1}{\log 2} \| f \|_{\mathcal{B}} \log \frac{2}{1 - \| z \|_{\mathbb{D}}^2}.$ (3.1)

Proof. For each $f \in \mathcal{B}$ and $z \in \mathbb{U}$, we can write $f(z)$ as

$f(z) = f_1(z_1)e + f_2(z_2)e^\dagger,$

so by using [7], it follows that

$$\| f(z) \|_{\mathbb{D}} = \| f_1(z_1) \|_1\ e + \| f_2(z_2) \|_2\ e^\dagger \preceq \frac{1}{\log 2} \| f_1 \|_{\mathcal{B}_1} \log \frac{2}{1 - \| z_1 \|_1^2}\ e + \frac{1}{\log 2} \| f_2 \|_{\mathcal{B}_2} \log \frac{2}{1 - \| z_2 \|_2^2} e^\dagger$$

$$= \frac{1}{\log 2} (\| f_1 \|_{\mathcal{B}_1}\ e + \| f_2 \|_{\mathcal{B}_2}\ e^\dagger)(\log \frac{2}{1 - \| z_1 \|_1^2} e + \log \frac{2}{1 - \| z_2 \|_2^2} e^\dagger)$$

$$= \frac{1}{\log 2} \| f \|_{\mathcal{B}} \log \frac{2}{1 - \| z \|_{\mathbb{D}}^2}.$$

Proposition 3.8 Let $\mathcal{B}$ be a bicomplex Bloch space and let $f: \mathbb{U} \to \mathbb{BC}$ be a bicomplex holomorphic function. *If* $f \in \mathcal{B}$, then $fo\,\phi \in \mathcal{B}$.

Proof. If $f \in \mathcal{B}$, then $(1-\| z \|_{\mathbb{D}}^2) \| f'(z) \|_{\mathbb{D}} \prec \infty$. Now.

$\| (fo\,\phi)'(z) \|_{\mathbb{D}} = \| f'(\phi(z)).\phi'(z) \|_{\mathbb{D}} = \| f'(\phi(z)) \|_{\mathbb{D}}.\| \phi'(z) \|_{\mathbb{D}}$

$(1-\| z \|_{\mathbb{D}}^2) \| (fo\phi)'(z) \|_{\mathbb{D}} = (1-\| z \|_{\mathbb{D}}^2) \| f'(\phi(z)) \|_{\mathbb{D}} \| \phi'(z) \|_{\mathbb{D}}$.

Since, $\phi(z)$ is bicomplex mobius transformation on $\mathbb{U}$, so

$(1-\| z \|^2) \| \phi'(z) \|_{\mathbb{D}} = (1-\| \phi(z) \|_{\mathbb{D}}^2)$.

$(1-\| z \|_{\mathbb{D}}^2) \| (fo\,\phi)'(z) \|_{\mathbb{D}} = (1-\| \phi(z) \|_{\mathbb{D}}^2) \| f'(\phi(z)) \|_{\mathbb{D}}$

$= \| f \|_{\mathcal{B}} \prec \infty$.

Thus, $fo\,\phi \in \mathcal{B}$.

Theorem 3.9 Let $\mathcal{B}$ be a bicomplex Bloch space and let $u.C_\phi: \mathcal{B} \to \mathcal{B}$ be a weighted composition operator on $\mathcal{B}$. Then $u.C_\phi$ *is* bounded on $\mathcal{B}$ if and only *if* $u_1.C_{\phi_1}$ and $u_2.C_{\phi_2}$ are bounded on $\mathcal{B}_1$ and $\mathcal{B}_2$.

Proof. Suppose that $u_1.C_{\phi_1}$ and $u_2.C_{\phi_2}$ are bounded on $\mathcal{B}_1$ and $\mathcal{B}_2$. We can write $u.C_\phi$ as

$u.C_\phi = u_1.C_{\phi_1}e + u_2.C_{\phi_2}e^\dagger$.

Since, both $u_1.C_{\phi_1}$ and $u_2.C_{\phi_2}$ are bounded on $\mathcal{B}_1$ and $\mathcal{B}_2$. This implies that $\| u_1.C_{\phi_1} \|_{\mathcal{B}_1} \prec \infty$ and $\| u_2.C_{\phi_2} \|_{\mathcal{B}_2} \prec \infty$. Then we have

$\| u.C_\phi \|_{\mathcal{B}} = \| u_1.C_{\phi_1} \|_{\mathcal{B}_1} e + \| u_2.C_{\phi_2} \|_{\mathcal{B}_2} e^\dagger$

$\prec \infty$.

Thus, $u.C_\phi$ is bounded on bicomplex Bloch space $\mathcal{B}$.
Conversely, suppose that $u.C_\phi$ is bounded on bicomplex Bloch space $\mathcal{B}$. Now, we have to show that $u_1.C_{\phi_1}$ and $u_2.C_{\phi_2}$ are bounded on $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively.There exist some constants M_1 and M_2 such that $M = M_1e + M_2e^\dagger$. Then we have

$\| u.C_\phi \|_{\mathcal{B}} \prec M = M_1e + M_2e^\dagger$.

Further, $\| u.C_\phi \|_{\mathcal{B}} = \| u_1.C_{\phi_1} \|_{\mathcal{B}_1} e + \| u_2.C_{\phi_2} \|_{\mathcal{B}_2} e^\dagger$. Then

$\| u_1.C_{\phi_1} \|_{\mathcal{B}_1} e + \| u_2.C_{\phi_2} \|_{\mathcal{B}_2} e^\dagger \prec M_1e + M_2e^\dagger$.

Therefore, $\| u_1.C_{\phi_1} \|_{\mathcal{B}_1} \prec M_1$ and $\| u_2.C_{\phi_2} \|_{\mathcal{B}_2} \prec M_2$. Hence" " $u_1.C_{\phi_1}$ and $u_2.C_{\phi_2}$ are bounded on $\mathcal{B}_1$ and $\mathcal{B}_2$.

4. Conditions for boundedness and compactness

Theorem 4.1 Let u be a bicomplex holomorphic function on $\mathbb{U}$ and $\phi: \mathbb{U} \to \mathbb{U}$ be a bicomplex holomorphic self map of $\mathbb{U}$. Then $u.C_\phi$ is bounded on a bicomplex Bloch space $\mathcal{B}$ if and only if the following are satisfied:

1. $\sup_{z\in\mathbb{U}}(1-\| z \|_{\mathbb{D}}^2) \| u'(z) \|_{\mathbb{D}} \log\frac{2}{1-\|\phi(z)\|_{\mathbb{D}}^2} \prec \infty$.
2. $\sup_{z\in\mathbb{U}} \frac{1-\|z\|_{\mathbb{D}}^2}{1-\|\phi(z)\|_{\mathbb{D}}^2} \| u(z)\phi'(z) \|_{\mathbb{D}} \prec \infty$.

Proof. Suppose $u.C_\phi$ is bounded on a bicomplex Bloch space $\mathcal{B}$. By above theorem, $u.C_\phi$ is bounded on bicomplex Bloch space if and only if $u_1.C_{\phi_1}$ and $u_2.C_{\phi_2}$ are bounded on complex Bloch spaces. We can decompose the L.H.S of (i) and (ii) part as:

$$\sup_{z\in\mathbb{U}}(1-\| z \|_{\mathbb{D}}^2) \| u'(z) \|_{\mathbb{D}} \log\frac{2}{1-\| \phi(z) \|_{\mathbb{D}}^2} = \sup_{z_1\in U}(1-\| z_1 \|_1^2) \| u_1'(z_1) \|_1 \log\frac{2}{1-\| \phi_1(z_1) \|_1^2} e + \sup_{z_2\in U}(1-\| z_2 \|_2^2) \| u_2'(z_2) \|_2 \log\frac{2}{1-\| \phi_2(z_2) \|_2^2} e^\dagger$$

$$\sup_{z\in\mathbb{U}}\frac{1-\| z \|_{\mathbb{D}}^2}{1-\| \phi(z) \|_{\mathbb{D}}^2}\| u(z)\phi'(z) \|_{\mathbb{D}}=\sup_{z_1\in U}\frac{1-\| z_1 \|_1^2}{1-\| \phi_1(z_1) \|_1^2}\| u_1(z_1)\phi_1{}'(z_1) \|_1\, e+\sup_{z_2\in U}\frac{1-\| z_2 \|_2^2}{1-\| \phi_2(z_2) \|_2^2}\| u_2(z_2)\phi_2{}'(z_2) \|_2\, e^{\dagger}.$$

By [[7], theorem 1], $\sup_{z_l\in U}(1-\| z_l \|_l^2)\| u_l{}'(z_l) \|_l \log\frac{2}{1-\|\phi_l(z_l)\|_l^2}<\infty$ and

$\sup_{z_l\in U}\frac{1-\|z_l\|_l^2}{1-\|\phi_l(z_l)\|_l^2}\| u_l(z_l)\phi_l{}'(z_l) \|_l<\infty$ if and only if $u_lC_{\phi_l}$ is bounded on complex Bloch spaces, where" "$l=1,2$.

Thus, it follows that

$$\sup_{z\in\mathbb{U}}\frac{1-\| z \|_{\mathbb{D}}^2}{1-\| \phi(z) \|_{\mathbb{D}}^2}\| u(z)\phi'(z) \|_{\mathbb{D}}<\infty\ and$$

$$\sup_{z\in\mathbb{U}}(1-\| z \|_{\mathbb{D}}^2)\| u'(z) \|_{\mathbb{D}} \log\frac{2}{1-\| \phi(z) \|_{\mathbb{D}}^2}<\infty.$$

Conversely, suppose that (i) and (ii) conditions are hold. We have to show that $u.C_\phi$ is bounded on bicomplex Bloch space $\mathcal{B}$. Further,

$$\| u.C_\phi f \|_{\mathcal{B}}=(1-\| z \|_{\mathbb{D}}^2)\| (u.C_\phi f)'(z) \|_{\mathbb{D}}$$

$$=(1-\| z \|_{\mathbb{D}}^2)\| (u.fo\phi)'(z) \|_{\mathbb{D}}$$

$$=(1-\| z_1 \|_1^2)\| (u_1.f_1o\phi_1)'(z_1) \|_1\, e+(1-\| z_2 \|_2^2)\| (u_2.f_2o\phi_2)'(z_2) \|_2\, e^{\dagger}$$

$$=(1-\| z_1 \|_1^2)\| u_1{}'(z_1).f_1o\phi_1(z_1)+u_1(z_1).f_1{}'(\phi_1(z_1))(\phi_1{}'(z_1)) \|_1\, e+(1-\| z_2 \|_2^2)$$
$$\| u_2{}'(z_2).f_2o\phi_2(z_2)+u_2(z_2).f_2{}'(\phi_2(z_2))(\phi_2{}'(z_2)) \|_2\, e^{\dagger}$$

$$\preceq(1-\| z_1 \|_1^2)[\| u_1{}'(z_1) \|_1.\| f_1o\phi_1(z_1) \|_1+\| u_1(z_1)(\phi_1{}'(z_1)) \|_1.\| f_1{}'(\phi_1(z_1)) \|_1]e+(1-\| z_2 \|_2^2)[\| u_2{}'(z_2) \|_2.$$
$$\| f_2o\phi_2(z_2) \|_2+\| u_2(z_2)(\phi_2{}'(z_2)) \|_2.\| f_2{}'(\phi_2(z_2)) \|_2\, e^{\dagger}]$$

$$=(1-\| z_1 \|_1^2)\| u_1{}'(z_1) \|_1.\| f_1o\phi_1(z_1) \|_1\, e+(1-\| z_2 \|_2^2)\| u_2{}'(z_2) \|_2.\| f_2o\phi_2(z_2) \|_2\, e^{\dagger}$$

$$+(1-\| z_1 \|_1^2)\| u_1(z_1)(\phi_1{}'(z_1)) \|_1.\| f_1{}'(\phi_1(z_1)) \|_1\, e+(1-\| z_2 \|_2^2)\| u_2(z_2)(\phi_2{}'(z_2)) \|_2.$$

$$\| f_2{}'(\phi_2(z_2)) \|_2\, e^{\dagger}$$

$$=(1-\| z \|_{\mathbb{D}}^2)\| u'(z) \|_{\mathbb{D}}.\| fo\phi(z) \|_{\mathbb{D}}+(1-\| z \|_{\mathbb{D}}^2)\| u(z)(\phi'(z)) \|_{\mathbb{D}}.\| f'(\phi(z)) \|_{\mathbb{D}}$$

$$=(1-\| z \|_{\mathbb{D}}^2)\| u'(z) \|_{\mathbb{D}}.\| fo\phi(z) \|_{\mathbb{D}}+\frac{(1-\| z \|_{\mathbb{D}}^2)}{(1-\| \phi(z) \|_{\mathbb{D}}^2)}(1-\| \phi(z) \|_{\mathbb{D}}^2)\| f'(\phi(z)) \|_{\mathbb{D}}.$$

$$\| u(z)(\phi'(z)) \|_{\mathbb{D}}$$

$$\preceq(1-\| z \|_{\mathbb{D}}^2)\| u'(z) \|_{\mathbb{D}}\frac{1}{\log 2}\| f \|_{\mathcal{B}}\log\frac{2}{1-\| z \|_{\mathbb{D}}^2}+\frac{(1-\| z \|_{\mathbb{D}}^2)}{(1-\| \phi(z) \|_{\mathbb{D}}^2)}\| f \|_{\mathcal{B}}$$

$$\| u(z)(\phi'(z)) \|_{\mathbb{D}},$$

where we use (3.1) in last inequality. So, (i) and (ii) condition implies that R.H.S is bounded by some constant times $\| f \|_{\mathcal{B}}$. Thus, $u.C_\phi$ is bounded on a bicomplex Bloch space $\mathcal{B}$.

Theorem 4.2 Let u be a bicomplex holomorphic function on $\mathbb{U}$ and $\phi:\mathbb{U}\to\mathbb{U}$ be a bicomplex holomorphic self map of $\mathbb{U}$. Then $u.C_\phi$ is compact on a bicomplex Bloch space $\mathcal{B}$ if and only if $u_1.C_{\phi_1}$ and $u_2.C_{\phi_2}$ are compact on a bicomplex Bloch space $\mathcal{B}_1$ and $\mathcal{B}_2$.

Proof. Suppose that $u.C_\phi$ is compact on a bicomplex Bloch space $\mathcal{B}$. Let $\{f_n\}$ be a bounded sequence in $\mathcal{B}$, each $\{f_n\}$ has complex idempotent decomposition $f_n=f_{n,1}e+f_{n,2}e^{\dagger}$. Then $f_{n,1}$ and $f_{n,2}$ are bounded on $\mathcal{B}_1$ and $\mathcal{B}_2$. Since, $u.C_\phi$ is compact on a bicomplex Bloch space $\mathcal{B}$, then $\| u.C_\phi f_n \|_{\mathcal{B}}\to 0$ as $n\to 0$.

We have $\| u.C_\phi f_n \|_{\mathcal{B}}=\| u_1.C_{\phi_1}f_{n,1} \|_{\mathcal{B}_1}\, e+\| u_2.C_{\phi_2}f_{n,2} \|_{\mathcal{B}_2}\, e^{\dagger}$ which means as $n\to\infty$, $\| u_1.C_{\phi_1}f_{n,1} \|_{\mathcal{B}_1}\to 0$ and $\| u_2.C_{\phi_2}f_{n,2} \|_{\mathcal{B}_2}\to 0$.Thus, both $u_1.C_{\phi_1}$ and $u_2.C_{\phi_2}$ are compact on $\mathcal{B}_1$ and $\mathcal{B}_2$.

Conversely, suppose that $u_1.C_{\phi_1}$ and $u_2.C_{\phi_2}$ are compact on $\mathcal{B}_1$ and $\mathcal{B}_2$. So, $\| u_1.C_{\phi_1}f_{n,1} \|_{\mathcal{B}_1} \to 0$ and $\| u_2.C_{\Phi_2}f_{n,2} \|_{\mathcal{B}_2} \to 0$, as $n \to \infty$.
Now, we have $\| u.C_\phi f_n \|_{\mathcal{B}} = \| u_1.C_{\phi_1}f_{n,1} \|_{\mathcal{B}_1} e + \| u_2.C_{\phi_2}f_{n,2} \|_{\mathcal{B}_2} e^\dagger$. Thus, as $n \to \infty$, $\| u.C_\phi f_n \|_{\mathcal{B}} \to 0$.

Theorem 4.3 Let u be a bicomplex holomorphic function on $\mathbb{U}$ and $\phi: \mathbb{U} \to \mathbb{U}$ be a bicomplex holomorphic self map of $\mathbb{U}$. Suppose that $u.C_\phi$ is bounded on a bicomplex Bloch space $\mathcal{B}$. Then $u.C_\phi$ is compact on a bicomplex Bloch space $\mathcal{B}$ if and only if the following are satisfied.

1. $\lim_{\|\phi(z)\|_{\mathbb{D}} \to 1^-} (1-\| z \|_{\mathbb{D}}^2) \| u'(z) \|_{\mathbb{D}} \log \frac{2}{1-\|\phi(z)\|_{\mathbb{D}}^2} = 0$

2. $\lim_{\|\phi(z)\|_{\mathbb{D}} \to 1^-} \frac{1-\|z\|_{\mathbb{D}}^2}{1-\|\phi(z)\|_{\mathbb{D}}^2} \| u(z)\phi'(z) \|_{\mathbb{D}} = 0$

Proof. Suppose that $u.C_\phi$ is compact on a bicomplex Bloch space $\mathcal{B}$. Let $\{f_n\}$ be a bounded sequence in $\mathcal{B}$, each $\{f_n\}$ has complex idempotent decomposition $f_n = f_{n,1}e + f_{n,2}e^\dagger$. Then $f_{n,1}$ and $f_{n,2}$are bounded on $\mathcal{B}_1$ and $\mathcal{B}_2$. Since, $u.C_\phi$ is compact on a bicomplex Bloch space $\mathcal{B}$, we have

$$\| u.C_\phi f_n \|_{\mathcal{B}} \to 0 \ as \ n \to \infty.$$

Further, $\| u.C_\phi f_n \|_{\mathcal{B}} = \| u_1.C_{\phi_1}f_{n,1} \|_{\mathcal{B}_1} e + \| u_2.C_{\phi_2}f_{n,2} \|_{\mathcal{B}_2} e^\dagger$. Then, by above theorem, $\| u_1.C_{\phi_1}f_{n,1} \|_{\mathcal{B}_1} \to 0$ and $\| u_2.C_{\phi_2}f_{n,2} \|_{\mathcal{B}_2} e^\dagger \to 0, as \ n \to \infty$. This implies that $u_1.C_{\phi_1}$ and $u_2.C_{\phi_2}$ are compact on complex Bloch spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Now,

$$\| u.C_\phi f_n \|_{\mathcal{B}} "" = \| u_1.C_{\phi_1}f_{n,1} \|_{\mathcal{B}_1} e + \| u_2.C_{\phi_2}f_{n,2} \|_{\mathcal{B}_2} e^\dagger$$

$$= (1-\| z_1 \|_1^2) \| (u_1.f_{n,1} o \phi_1)'(z_1) \|_1 e + (1-\| z_2 \|_2^2) \| (u_2.f_{n,2} o \phi_2)'(z_2) \|_2 e^\dagger$$

$$= [(1-\| z_1 \|_1^2) \| u_1'(z_1).f_1 o \phi_1(z_1) + u_1(z_1).f_1'(\phi_1(z_1)).\phi_1'(z_1) \|_1]e$$

$$+[(1-\| z_2 \|_2^2) \| u_2'(z_2).f_2 o \phi_2(z_2) + u_2(z_2).f_2'(\phi_2(z_2)).\phi_2'(z_2) \|_2]e^\dagger$$

$$\preceq [(1-\| z_1 \|_1^2) \| u_1'(z_1).f_1 o \phi_1(z_1) \|_1 e + (1-\| z_2 \|_2^2) \| u_2'(z_2).f_2 o \phi_2(z_2) \|_2 e^\dagger]$$

$$+[(1-\| z_1 \|_1^2) \| u_1(z_1).f_1'(\phi_1(z)).\phi_1'(z_1) \|_1 e + (1-\| z_2 \|_2^2) \| u_2(z_2).f_2'(\phi_2(z_2)).\phi_2'(z_2) \|_2 e^\dagger]$$

$$\preceq (1-\| z_1 \|_1^2) \| u_1'(z_1 \|_1.\frac{1}{\log 2} \| f_1 \|_{\mathcal{B}_1} \log \frac{2}{(1-\| z_1 \|_1^2)} e$$

$$+(1-\| z_2 \|_2^2)"" \| u_2'(z_2 \|_2."" \frac{1}{\log 2} \| f_2 \|_{\mathcal{B}_2} \log \frac{2}{(1-\| z_2 \|_2^2)} e^\dagger$$

$$+\frac{(1-\| z_1 \|_1^2)}{(1-\| \phi(z_1) \|_1^2)} (1-\| \phi(z_1) \|_1^2) \| f_1'(\phi_1(z_1)) \|_1 "" \| u_1(z_1)\phi_1'(z_1) \|_1 e$$

$$+\frac{(1-\| z_2 \|_2^2)}{(1-\| \phi(z_2) \|_2^2)} (1-\| \phi(z_2) \|_2^2) \| f_2'(\phi_2(z_2)) \| \| (u_2(z_2).\phi_2'(z_2)) \|_2 e^\dagger$$

$$= (1-\| z_1 \|_1^2) \| u_1'(z_1) \|_1.\frac{1}{\log 2} \| f_1 \|_{\mathcal{B}_1} \log \frac{2}{(1-\| z_1 \|_1^2)} e$$

$$+(1-\| z_2 \|_2^2) \| u_2'(z_2) \|.\frac{1}{\log 2} \| f_2 \|_{\mathcal{B}_2} \log \frac{2}{(1-\| z_2 \|_2^2)} e^\dagger$$

$$+\frac{(1-\| z_1 \|_1^2)}{(1-\| \phi(z_1) \|_1^2)} \| f_1 \|_1 \| u_1(z_1)\phi_1'(z_1) \|_1 e$$

$$+\frac{(1-\| z_2 \|_2^2)}{(1-\| \phi(z_2) \|_2^2)} \| f_2 \|_2 \| u_2(z_2).\phi_2'(z_2) \|_2 e^\dagger.$$

Therefore, from [7], we have $\lim_{|\phi_l(z)|_l \to 1^-} (1-\| z_l \|_l^2) u_l'(z_l) \frac{1}{\log 2} \| f_l \|_l \log \frac{2}{(1-\|z_l\|_l^2)} = 0 \ and \lim_{|\phi_l(z)|_l \to 1-} \frac{(1-\|z_l\|_l^2)}{(1-\|\phi_l(z_l)\|_l^2)}$

$\| f_l \|_l (u_l(z_l)\phi_l'(z_l)) = 0, where\ l = 1,2.$

Thus, it follows that

$$\lim_{|\phi(z)|\to 1^-}(1-\| z \|_{\mathbb{D}}^2) \| u'(z) \|_{\mathbb{D}} \log\frac{2}{1-\| \phi(z) \|_{\mathbb{D}}^2} = 0\ and$$

$$\lim_{|\phi(z)|\to 1^-}\frac{1-\| z \|_{\mathbb{D}}^2}{1-\| \phi(z) \|_{\mathbb{D}}^2} \| u(z)\phi'(z) \|_{\mathbb{D}} = 0.$$

Conversely suppose that (i) and (ii) conditions are holds. Now, we have

$\| u.C_\phi f_n \|_{\mathcal{B}} = \| u_1.C_{\phi_1} f_{n,1} \|_{\mathcal{B}_1} e + \| u_2.C_{\phi_2} f_{n,2} \|_{\mathcal{B}_2} e^\dagger$

$= \| u_1.f_{n,1} o\phi_1 \|_{\mathcal{B}_1} e + \| u_2.f_{n,2} o\phi_2 \|_{\mathcal{B}_2} e^\dagger$

$= (1-\| z_1 \|_1^2) \| [u_1(z_1).f_{n,1} o\phi_1(z_1)]' \|_1 e + (1-\| z_2 \|_2^2) \| [u_2(z_2).f_{n,2} o\phi_2(z_2)]' \|_2 e^\dagger$

$= [(1-\| z_1 \|_1^2) \| u_1'(z_1).f_{n,1} o\phi_1(z_1) + u_1(z_1).f_{n,1}'(\phi_1(z_1)).\phi_1'(z_1) \|_1]e$

$+[(1-\| z_2 \|_2^2) \| u_2'(z_2).f_{n,2} o\phi_2(z_2) + u_2(z_2).f_{n,2}'(\phi_2(z_2)).\phi_2'(z_2) \|_2]e^\dagger$

$\preceq [(1-\| z_1 \|_1^2) \| u_1'(z_1).f_{n,1} o\phi_1(z_1) \|_1 e + (1-\| z_2 \|_2^2) \| u_2'(z_2).f_{n,2} o\phi_2(z_2) \|_2 e^\dagger]$

$+[(1-\| z_1 \|_1^2) \| u_1(z_1).f_{n,1}'(\phi_1(z_1)).\phi_1'(z_1) \|_1 e$

$+(1-\| z_2 \|_2^2) \| u_2(z_2).f_{n,2}'(\phi_2(z_2)).\phi_2'(z_2) \|_2 e^\dagger]$

$= [(1-\| z \|_{\mathbb{D}}^2) \| u'(z).f_n o\phi(z) \|_{\mathbb{D}}] + [(1-\| z \|_{\mathbb{D}}^2) \| u(z).f_n'(\phi(z)).\phi'(z) \|_{\mathbb{D}}]$

$\preceq (1-\| z \|_{\mathbb{D}}^2) \| u'(z) \|_{\mathbb{D}}.\frac{1}{\log 2} \| f_n \|_{\mathcal{B}} \log\frac{2}{(1-\| z \|_{\mathbb{D}}^2)}$

$+\frac{(1-\| z \|^2)}{(1-\| \phi(z) \|^2)} \| f_n \|_{\mathcal{B}} \| u(z).\phi'(z) \|_{\mathbb{D}}.$

Since, conditions (i) and (ii) holds. So, $\| u.C_\phi f_n \|_{\mathcal{B}} \to 0\ as\ n \to \infty$. Thus, $u.C_\phi$ is a compact on bicomplex Bloch space $\mathcal{B}$.

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