International Journal of Statistics and Applied Mathematics

**ISSN:** 2456-1452
Maths 2025; 10(1): 128-141

https://www.mathsjournal.com
Received: 25-10-2024
Accepted: 24-11-2024

**Boampong Asare**
Department of Engineering,
United Tribes Technical College,
Bismarck, North Dakota

**Corresponding Author:**
**Boampong Asare**
Department of Engineering,
United Tribes Technical College,
Bismarck, North Dakota

# Enhancing precision in nonlinear oscillator analysis: A comparative study of multiple-scale analysis and strained coordinates

**Boampong Asare**

**DOI:** https://dx.doi.org/10.22271/maths.2025.v10.i1b.1956

**Abstract**
This paper presents a comparative study of three prominent analytical techniques. These are multiple-scale analysis, the method of strained coordinates, and the Poincaré-Lindstedt method for analyzing nonlinear oscillators under singular perturbation. Focusing on the undamped Duffing equation as a representative nonlinear system, we investigate the efficacy, accuracy, and limitations of each method. Notably, all three methods were applied to a single Duffing oscillator problem to evaluate varying degrees of accuracy and computational efficiency. The study highlights the importance of selecting the most suitable analytical tool based on the specific characteristics of the nonlinear oscillator and the nature of the singular perturbation. Our findings contribute to the ongoing refinement of analytical approaches in nonlinear dynamics, offering insights to guide researchers and engineers in choosing the most appropriate method for their specific nonlinear oscillator problems.

**Keywords:** Multiple-scale analysis, Method of strained coordinates, Poincaré-Lindstedt method, Nonlinear oscillators, Singular perturbation, undamped duffing equation, analytical techniques, comparative study, efficacy, accuracy, computational efficiency, nonlinear dynamics

## 1. Introduction

Nonlinear oscillators are common in various fields of science and engineering, from mechanical systems to electromagnetic phenomena. Their complex behavior, characterized by amplitude-dependent frequencies and potential for chaotic motion, presents significant challenges in analysis and prediction. Traditional linear methods often fall short in capturing the rich dynamics of these systems, necessitating the development and application of specialized analytical techniques. This paper focuses on three prominent methods for analyzing nonlinear oscillators: multiple-scale analysis, the method of strained coordinates, and the Poincaré-Lindstedt method. Both multiple-scale analysis and the method of strained coordinates have emerged as powerful tools in the field of nonlinear dynamics, each offering unique approaches to overcoming the limitations of conventional perturbation methods. The Poincaré-Lindstedt method, which involves a perturbation expansion of the solution and a strained time variable, is particularly useful for analyzing nonlinear oscillators with small damping or nonlinear damping terms. Multiple-scale analysis, introduced by Nayfeh [1, 2, 3] in the 1960s, provides a systematic way to construct uniformly valid approximations by introducing multiple time scales. This method is particularly adept at capturing both fast oscillations and slow modulations in nonlinear systems, offering insights into phenomena such as amplitude-frequency interdependence and resonance behaviors. The method of strained coordinates, closely related to other perturbation techniques, involves a clever transformation of the time variable to maintain the uniformity of the solution. This approach has proven effective in eliminating secular terms that often plague regular perturbation methods, leading to more accurate periodic solutions. Our study aims to provide a comprehensive comparison of these three methods, using the undamped Duffing [4, 5, 6] equation as a prototypical nonlinear system. The Duffing equation, with its cubic nonlinearity, serves as an excellent benchmark for evaluating the strengths and limitations of each analytical technique. By conducting this comparative analysis, we seek to: Elucidate the fundamental principles and mathematical foundations of each method. Assess the accuracy and range of validity of the Elucidate the fundamental principles and mathematical foundations of each method. By conducting this comparative analysis, we seek to: assess the accuracy and range of validity of the solutions obtained through multiple-scale analysis, the method of strained coordinates, and the Poincaré-Lindstedt method, identify the specific strengths and potential limitations of each approach in the context of nonlinear oscillator analysis, including Multiple-scale analysis, the method of strained coordinates, and the Poincaré-Lindstedt method and guide researchers and practitioners in selecting the most appropriate method for their specific nonlinear problems, considering factors such as the type of nonlinearity, the presence of damping or external forcing, and the desired level of accuracy.

## 2. Comparative Analysis

Multiple-Scale Analysis (MSA) and Strained Coordinates Method (SCM) each have their strengths and limitations. MSA is particularly effective for systems with disparate time scales and addresses secular growth in perturbation solutions. Strained Coordinates Method (SCM), also known as the Lindstedt-Poincaré method, is useful for constructing periodic solutions and eliminating secular terms. The choice between these methods often depends on the specific nonlinear oscillator problem at hand.

### 2.1. Case Studies

Common case studies for applying these methods include:

- Duffing oscillator
- Van der Pol oscillator
- Mathieu equation

These oscillators are frequently used to demonstrate the effectiveness of various analytical techniques. For instance, the Duffing oscillator has been analyzed using both MSA and SCM to compare their accuracy in predicting nonlinear behavior

### 2.2. Advanced Techniques and Hybrid Approaches

Recent developments have led to more sophisticated methods for analyzing nonlinear oscillators:

- Least Square Homotopy Perturbation Method (LSHPM): This technique combines the homotopy perturbation method with least squares optimization, offering improved accuracy for certain nonlinear systems
- Hybrid Reservoir Computing: This approach combines reservoir computing with expert analytical models, showing promise in predicting dynamics of complex nonlinear oscillator networks
- Non-perturbative Methodology (NPM): This method transforms nonlinear ODEs into linear ones, simplifying the analysis of strong nonlinear oscillators
- He's Frequency Formula (HFF): This general formula has been applied to examine analytical representations of strong nonlinear oscillator systems. These advanced techniques and hybrid approaches, often outperform traditional methods in terms of accuracy and computational efficiency, especially for highly nonlinear or complex oscillator systems. They demonstrate the evolution of nonlinear oscillator analysis, aiming to bridge the gap between simplified analytical models and real-world complexity.

## 3. The method of multiple-scale analysis

The Method of Multiple Scales is a mathematical technique for solving singular perturbation problems by assuming the solution depends on multiple independent time or space scales. This method introduces two or more scales, typically represented as $\tau = \varepsilon t$ (slow scale) and $\theta = \mathrm{t}$ (fast scale), to separate the problem's slow and fast variations. The solution is then expanded in powers of ε, yielding a leading-order equation and higher-order corrections. The multiple scales method eliminates secular terms which cause non-uniformity, by requiring that each term in the expansion be bounded for all time. This is achieved by setting the coefficients of the secular terms to zero, resulting in a uniformly valid solution. The method is particularly effective for problems with oscillatory or periodic behavior, such as in quantum mechanics, nonlinear dynamics, and fluid mechanics, and is often used to study phenomena like resonance, instability, and pattern formation. To illustrate the method of multiple-scale analysis, let's consider the Van der Pol Oscillator equation to illustrate the method of multiple-scale analysis [7] [8] [9].

### 3.1. Use the method of multiple scales to obtain the first term in the approximation for $\ddot{\mathbf{x}} + \boldsymbol{\varepsilon}\mathbf{x}^3 + \mathbf{x} = \mathbf{0}$ with $\mathbf{x}(\mathbf{0}) = \mathbf{1}, \dot{\mathbf{x}}(\mathbf{0}) = \mathbf{0}$

Fast scale as $\tau = t$, slow scale $T = \varepsilon t$

Consider $x(t) \equiv x(\tau, T)$, so that:

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d\tau} \bullet \frac{d\tau}{dt} + \frac{dx}{dT} \bullet \frac{dT}{dt} = \frac{dx}{d\tau} + \varepsilon \frac{dx}{dT}$$

$$\ddot{x} = \frac{d}{dt}\left(\frac{dx}{d\tau} + \varepsilon \frac{dx}{dT}\right) = \frac{d}{dt} \bullet \frac{dx}{d\tau} + \varepsilon \frac{d}{dT} \bullet \frac{dx}{dt} = \frac{d}{dr} \bullet \frac{dx}{dt} + \varepsilon \frac{d}{dT} \bullet \frac{dx}{dt}$$

$$= \frac{d}{dr} \bullet \left(\frac{dx}{d\tau} + \varepsilon \frac{dx}{dT}\right) + \varepsilon \frac{d}{dT} \bullet \left(\frac{dx}{d\tau} + \varepsilon \frac{dx}{dT}\right) = \frac{d^2x}{d\tau^2} + \varepsilon \frac{d^2x}{d\tau dT} + \varepsilon \frac{d^2x}{dT d\tau} + \varepsilon^2 \frac{d^2x}{dT^2}$$

$$\ddot{x} = \frac{d^2x}{d\tau^2} + 2\varepsilon \frac{d^2x}{d\tau dT} + \varepsilon^2 \frac{d^2x}{dT^2}$$

Substituting into the DE, we obtain

$$\ddot{x} + \varepsilon x^3 + x = 0$$

$$\frac{d^2x}{d\tau^2} + 2\varepsilon\frac{d^2x}{d\tau dT} + \varepsilon^2\frac{d^2x}{dT^2} + \varepsilon(x(\tau,T))^3 + x(\tau,T) = 0$$

Using perturbation method, we let $x(\tau,T) = x_0(\tau,T) + \varepsilon x_1(\tau,T) + \cdots$

Substituting into the equation, we obtain

$$\frac{d^2(x_0(\tau,T) + \varepsilon x_1(\tau,T) + \cdots)}{d\tau^2} + 2\varepsilon\frac{d^2(x_0(\tau,T) + \varepsilon x_1(\tau,T) + \cdots)}{d\tau dT} + \varepsilon^2\frac{d^2(x_0(\tau,T) + \varepsilon x_1(\tau,T) + \cdots)}{dT^2} + \varepsilon(x_0(\tau,T) + \varepsilon x_1(\tau,T) + \cdots)^3 + x_0(\tau,T) + \varepsilon x_1(\tau,T) = 0$$

Equating coefficient of $\varepsilon^0$, we have

$$\varepsilon^0: \frac{d^2x_0}{d\tau^2} + x_0 = 0, x_0 = A\cos\tau + B\sin\tau$$

So, the solution of $\frac{d^2x_0}{d\tau^2} + x_0 = 0$ becomes $x_0(\tau,T) = A(T)\cos\tau + B(T)\sin\tau$

$$x_0(\tau,T) = A(T)\cos\tau + B(T)\sin\tau = R(T)\left[\frac{A(T)}{R(T)}\cos\tau + \frac{B(T)}{R(T)}\sin\tau\right]$$

Let $\cos\theta(T) = \frac{A(T)}{R(T)}, \sin\theta(T) = -\frac{B(T)}{R(T)}$ so that $\cos^2\theta(T) = \frac{A(T)^2}{R(T)^2}, \sin^2\theta(T) = \frac{B(T)^2}{R(T)^2}$

$$\cos^2\theta(T) + \sin^2\theta(T) = \frac{A(T)^2}{R(T)^2} + \frac{B(T)^2}{R(T)^2}, 1 = \frac{A(T)^2}{R(T)^2} + \frac{B(T)^2}{R(T)^2}, A(T)^2 + B(T)^2 = R(T)^2$$

$$R(T) = \sqrt{A(T)^2 + B(T)^2}$$

$$\cos\theta(T) = \frac{A(T)}{R(T)} = \frac{A(T)}{\sqrt{A(T)^2 + B(T)^2}}, \sin\theta(T) = -\frac{B(T)}{R(T)} = -\frac{B(T)}{\sqrt{A(T)^2 + B(T)^2}}$$

$$x_0(\tau,T) = R(T)\left[\frac{A(T)}{R(T)}\cos\tau + \frac{B(T)}{R(T)}\sin\tau\right] = R(T)[\cos\tau\cos\theta(T) - \sin\theta(T)\sin\tau]$$

$$x_0(\tau,T) = R(T)\cos[\tau + \theta(T)]$$

$$\frac{d(\tau,T)}{d\tau} = -R(T)\sin[\tau + \theta(T)]$$

**Boundary condition**

$t = 0, \tau = 0$ and $T = 0$

$$x_0(0,0) = 1, R(0)\cos\{\theta(0)\} = 1$$

$$\frac{dx_0(0,0)}{d\tau} = -R(0)\sin[\theta(0)] = 0$$

$$R^2(0)\cos^2\{\theta(0)\} = 1, R^2(0)\sin^2\{\theta(0)\} = 0$$

$$R^2(0)\cos^2\{\theta(0)\} + R^2(0)\sin^2\{\theta(0)\} = 1$$

$$R^2(0)[\cos^2\{\theta(0)\} + \sin^2\{\theta(0)\}] = 1$$

Thus, we take $R(0) = 1$, $\theta(0) = 0$ since $\sin[\theta(0)] = 0$

Therefore, $x_0(\tau,T) = R(T)\cos[\tau + \theta(T)]$

Also, equating coefficient of $\varepsilon^1$, we have

$$\varepsilon^1: \frac{d^2x_1}{d\tau^2} + 2\frac{d^2x_0}{d\tau dT} + {x_0}^3 + x_1 = 0$$

Note:
$\frac{dx_0(\tau,T)}{d\tau} = -R(T)\sin[\tau + \theta(T)]$ so that $\frac{d^2x_0}{d\tau T} = -R'(T)\sin[\tau + \theta(T)] - R(T)\,\theta'(T)\cos[\tau + \theta(T)]$

Therefore, $\frac{d^2x_1}{d\tau^2} + 2\frac{d^2x_0}{d\tau dT} + {x_0}^3 + x_1 = 0$ becomes

$$\frac{d^2x_1}{d\tau^2} - 2R'(T)\sin[\tau + \theta(T)] - 2R(T)\,\theta'(T)\cos[\tau + \theta(T)] + \{R(T)\cos[\tau + \theta(T)]\}^3 + x_1 = 0$$

$$\frac{d^2x_1}{d\tau^2} + x_1 = 2R'(T)\sin[\tau + \theta(T)] + 2R(T)\,\theta'(T)\cos[\tau + \theta(T)] - R^3(T)\cos^3[\tau + \theta(T)]$$

Note: $\cos^3[\tau + \theta(T)] = \frac{3}{4}\cos[\tau + \theta(T)] + \frac{1}{4}\cos 3[\tau + \theta(T)]$

$$\frac{d^2x_1}{d\tau^2} + x_1 = 2R'(T)\sin\big(\tau + \theta(T)\big) + 2R(T)\,\theta'(T)\cos[\tau + \theta(T)] - R(T)^3\left\{\frac{3}{4}\cos[\tau + \theta(T)] + \frac{1}{4}\cos 3[\tau + \theta(T)]\right\}$$

$$\frac{d^2x_1}{d\tau^2} + x_1 = 2R'(T)\sin\big(\tau + \theta(T)\big) + \left[2R(T)\theta'(T) - \frac{3}{4}R(T)^3\right]\cos[\tau + \theta(T)] - \frac{1}{4}R(T)^3\cos 3[\tau + \theta(T)]$$

To avoid resonant terms, we require in the DE,

$2R'(T) = 0$ and $2R(T)\theta'(T) - \frac{3}{4}R(T)^3 = 0$

Now, since $R(0) = 1$, we cannot have $R(T) \equiv 0$

$$R'(T) = 0, R(T) = c$$

Applying the boundary conditions $R(0) = 1$, we have , $R(T) = 1$

$$2R(T)\theta'(T) - \frac{3}{4}R(T)^3 = 0$$

$$2\theta'(T) - \frac{3}{4} = 0$$

$$2\theta'(T) = \frac{3}{4}, \theta'(T) = \frac{3}{8}, \theta(T) = \frac{3}{8}T + C$$

Applying the boundary condition $\theta(0) = 0$, we have

$$\theta(T) = \frac{3}{8}T + C, C = 0$$

$$\theta(T) = \frac{3}{8}T = \frac{3}{8}\tau\varepsilon$$

Now,

$$\frac{d^2x_1}{d\tau^2} + x_1 = -\frac{1}{4}R(T)^3\cos 3[\tau + \theta(T)]$$

$$\frac{d^2x_1}{d\tau^2} + x_1 = -\frac{1}{4}\cos 3[\tau + \theta(T)]$$

Thus, the zero order $x_0(\tau, T) = R(T)\cos[\tau + \theta(T)]$

$$x(t) = \cos\left[\tau + \frac{3}{8}\tau\varepsilon\right] = \cos\left[\left(1 + \frac{3}{8}\varepsilon\right)\tau\right]$$

$\mathrm{x(t)} = cos\left[\left(1+\frac{3}{8}\varepsilon\right)t\right]$

**4. The method of strained coordinates**

The Method of Strained Coordinates is a mathematical technique for solving singular perturbation problems by transforming the original coordinates to eliminate the small parameter, ε. This method involves introducing a stretching transformation, typically of the form $X = \frac{x}{\varepsilon^a}$, and $\tau = \frac{t}{\varepsilon^b}$, which compresses the boundary layer, and enables the derivation of a uniformly valid solution. The transformed equations are then expanded in powers of $\varepsilon$, yielding a leading-order equation and higher-order corrections. By choosing the stretching parameters, a and b, appropriately the method eliminates the singularity, allowing for a regular perturbation expansion. The solution is then transformed back to the original coordinates, providing an approximate solution that captures the essential features of the original problem. This method is particularly effective for problems with thin boundary layers, such as in fluid dynamics, heat transfer, and electromagnetic theory. It is often used in conjunction with other perturbation methods [1, 2, 3, 10, 11]. To explain the method, let's consider Duffing's Oscillator problem below.

**4.1. Use the method of strained coordinates to solve the problem $\ddot{\mathbf{u}} + \boldsymbol{\varepsilon}\mathbf{u}^3 + \mathbf{u} = \mathbf{0}, \mathbf{t} \geq \mathbf{0}$ and subject to initial conditions $\mathbf{u} = \mathbf{1}, \dot{\mathbf{u}} = \mathbf{0}$ at $\mathbf{t} = \mathbf{0}$**

Let $\mathrm{t = s + \varepsilon t_1(s) + \varepsilon^2 t_2(s) + \cdots}$

$\mathrm{u(t) = u_0(s) + \varepsilon u_1(s) + \varepsilon^2 u_2(s) + \cdots}$

Invoking the initial conditions to $\mathrm{t = s + \varepsilon t_1(s) + \varepsilon^2 t_2(s) +}$ and $\mathrm{u(t) = u_0(s) + \varepsilon u_1(s) + \varepsilon^2 u_2(s) +}$, we have

When $\mathrm{t = 0, s = 0, t_1 = 0, t_2 = 0}$

When $\mathrm{u(0) = 1}$

$\mathrm{1 = u_0(0) + \varepsilon u_1(0) + \varepsilon^2 u_2(0) + \cdots}$

$\mathrm{u_0(0) = 1, u_1(0) = 0, u_2(0) = 0}$

Differentiating (1) and (2), we have

$\mathrm{t = s + \varepsilon t_1(s) + \varepsilon^2 t_2(s) + \cdots}$

$$\frac{dt}{ds} = 1 + \varepsilon\frac{dt_1}{ds} + \cdots$$

$$\frac{ds}{dt} = \frac{1}{1+\varepsilon\frac{dt_1}{ds}} = 1 - \varepsilon\frac{dt_1}{ds}$$

$$\frac{du}{dt} = \frac{du_0}{ds}\bullet\frac{ds}{dt} + \varepsilon\frac{du_1}{ds}\bullet\frac{ds}{dt} + \varepsilon^2\frac{du_2}{ds}\bullet\frac{ds}{dt} + \cdots$$

Substituting $\frac{ds}{dt}$ into $\frac{du}{dt}$, we obtain

$$\frac{du}{dt} = \frac{du_0}{ds}\left[1-\varepsilon\frac{dt_1}{ds}\right] + \varepsilon\frac{du_1}{ds}\left[1-\varepsilon\frac{dt_1}{ds}\right] + \varepsilon^2\frac{du_2}{ds}\left[1-\varepsilon\frac{dt_1}{ds}\right] + \cdots$$

$$\frac{du}{dt} = \frac{du_0}{ds}\left[1-\varepsilon\frac{dt_1}{ds}\right] + \varepsilon\frac{du_1}{ds}\bullet 1 + \cdots \text{ up to } O(\varepsilon) \text{ terms}$$

$$\frac{du}{dt} = \frac{du_0}{ds} - \varepsilon\frac{dt_1}{ds}\bullet\frac{du_0}{ds} + \varepsilon\frac{du_1}{ds} + \cdots$$

$$\frac{du}{dt} = \frac{du_0}{ds} + \varepsilon\left\{\frac{du_1}{ds} - \frac{du_0}{ds}\bullet\frac{dt_1}{ds}\right\} + \cdots$$

When $\mathrm{t = 0, s = 0, \dot{u}(0) = 0}$

Therefore,

$$0 = \left\{\frac{du_0}{ds}|s = 0\right\} + \varepsilon\left\{\frac{du_1}{ds}|s = 0 - \frac{du_0}{ds}\bullet\frac{dt_1}{ds}|s = 0\right\}$$

$$u_0(0) = 1, \left\{\frac{du_0}{ds}|s = 0\right\} = 0, u_1(0) = 0, \left\{\frac{du_1}{ds}|s = 0\right\} = 0$$

Also, $\frac{d^2u}{dt^2} = \frac{d}{dt}\left(\frac{du}{dt}\right) = \frac{d}{ds}\left(\frac{du}{dt}\right)\bullet\frac{ds}{dt} = \frac{\frac{d}{ds}\left(\frac{du}{dt}\right)}{\frac{dt}{ds}}$

$$\frac{d^2u}{dt^2} = \frac{\frac{d}{ds}\left\{\frac{du_0}{ds} + \varepsilon\left\{\frac{du_1}{ds} - \frac{du_0}{ds}\bullet\frac{dt_1}{ds}\right\}\right\}}{1 + \varepsilon\frac{dt_1}{ds}} = \frac{\frac{d^2u_0}{ds^2} + \varepsilon\left\{\frac{d^2u_1}{ds^2} - \frac{du_0}{ds}\bullet\frac{d^2t_1}{ds^2} - \frac{d^2u_0}{ds^2}\bullet\frac{dt_1}{ds}\right\}}{1 + \varepsilon\frac{dt_1}{ds}}$$

$$\frac{d^2u}{dt^2} = \left[\frac{d^2u_0}{ds^2} + \varepsilon\left\{\frac{d^2u_1}{ds^2} - \frac{du_0}{ds}\bullet\frac{d^2t_1}{ds^2} - \frac{d^2u_0}{ds^2}\bullet\frac{dt_1}{ds}\right\}\right]\left\{1 - \varepsilon\frac{dt_1}{ds}\right\}$$

$$\frac{d^2u}{dt^2} = \frac{d^2u_0}{ds^2} - \varepsilon\frac{d^2u_0}{ds^2}\frac{dt_1}{ds} + \varepsilon\left\{\frac{d^2u_1}{ds^2} - \frac{du_0}{ds}\bullet\frac{d^2t_1}{ds^2} - \frac{d^2u_0}{ds^2}\bullet\frac{dt_1}{ds}\right\}$$

$$\frac{d^2u}{dt^2} = \frac{d^2u_0}{ds^2} + \varepsilon\left\{\frac{d^2u_1}{ds^2} - \frac{du_0}{ds}\bullet\frac{d^2t_1}{ds^2} - \frac{d^2u_0}{ds^2}\bullet\frac{dt_1}{ds} - \frac{d^2u_0}{ds^2}\frac{dt_1}{ds}\right\}$$

$$\frac{d^2u}{dt^2} = \frac{d^2u_0}{ds^2} + \varepsilon\left\{\frac{d^2u_1}{ds^2} - \frac{du_0}{ds}\bullet\frac{d^2t_1}{ds^2} - 2\frac{d^2u_0}{ds^2}\bullet\frac{dt_1}{ds}\right\}$$

Substituting $u(t)$, $\dot{u}(t)$ and $\frac{d^2u}{dt^2}$ into the DE, $\ddot{u} + \varepsilon u^3 + u = 0$, we obtain

$$\frac{d^2u_0}{ds^2} + \varepsilon\left\{\frac{d^2u_1}{ds^2} - \frac{du_0}{ds}\bullet\frac{d^2t_1}{ds^2} - 2\frac{d^2u_0}{ds^2}\bullet\frac{dt_1}{ds}\right\} + \varepsilon\{u_0(s) + \varepsilon u_1(s)\}^3 + u_0(s) + \varepsilon u_1(s) = 0$$

Equating coefficients of equal powers of $\varepsilon$

$\varepsilon^0$: $\frac{d^2u_0}{ds^2} + u_0 = 0$ solution is $u_0(s) = a\cos(s) + b\sin(s)$

Applying the boundary condition $u_0(0) = 1$

$u_0(0) = a\cos(0) + b\sin(0), 1 = a, b = 0$

Therefore $u_0(s) = \cos(s)$

$\varepsilon^1$: $\frac{d^2u_1}{ds^2} - \frac{du_0}{ds}\bullet\frac{d^2t_1}{ds^2} - 2\frac{d^2u_0}{ds^2}\bullet\frac{dt_1}{ds} + \left(u_0(s)\right)^3 + u_1(s) = 0$

$$\frac{d^2u_1}{ds^2} + \sin(s)\bullet\frac{d^2t_1}{ds^2} + 2\bullet\cos(s)\frac{dt_1}{ds} + \cos^3(s) + u_1(s) = 0$$

$$\frac{d^2u_1}{ds^2} + u_1(s) = -\sin(s)\frac{d^2t_1}{ds^2} - 2\cos(s)\frac{dt_1}{ds} - \cos^3(s)$$

$$\frac{d^2u_1}{ds^2} + u_1(s) = -\sin(s)\frac{d^2t_1}{ds^2} - 2\cos(s)\frac{dt_1}{ds} - \frac{1}{4}\{3\cos(s) + \cos(3s)\}$$

$$\frac{d^2u_1}{ds^2} + u_1(s) = -\sin(s)\frac{d^2t_1}{ds^2} + \left\{-\frac{3}{4} - 2\frac{dt_1}{ds}\right\}\cos(s) - \frac{1}{4}\cos(3s)$$

Applying the initial condition $u_1(0) = 0, \frac{du_1}{ds}(0) = 0$

$$\frac{d^2u_1}{ds^2} + u_1(s) = -\sin(s)\frac{d^2t_1}{ds^2} + \left\{-\frac{3}{4} - 2\frac{dt_1}{ds}\right\}\cos(s) - \frac{1}{4}\cos(3s)$$

Homogenous solution is $u_{1H} = A\cos(s) + B\sin(s)$

A particular solution is $u_{1p} = Ds\bullet\cos(s) + Es\bullet\sin(s) + F\cos(3s) + G\sin(3s) + H$

By Poincare, suppose we require $\frac{d^2t_1}{ds^2} = 0$, and $\frac{3}{4} - 2\frac{dt_1}{ds} = 0$

Therefore

$$\frac{3}{4} - 2\frac{dt_1}{ds} = 0, \frac{dt_1}{ds} = -\frac{3}{8}$$

$$\int dt_1 = -\frac{3}{8}\int ds$$

$t_1(s) = -\frac{3}{8}s + C$, but $t_1(0) = 0 \rightarrow C = 0$. Therefore $t_1(s) = -\frac{3}{8}s$

And

$\frac{d^2u_1}{ds^2} + u_1(s) = -\frac{1}{4}\cos(3s)$, and

$$u_{1p} = Ds\bullet\cos(s) + Es\bullet\sin(s) + F\cos(3s) + G\sin(3s) + H$$

$$\dot{u}_{1p} = -Ds\sin(s) + D\cos(s) + Es\cos(s) + E\sin(s) - 3F\sin(3s) + 3G\cos(3s)$$

$$\ddot{u}_{1p} = -D\sin(s) - Ds\cos(s) - D\sin(s) + E\cos(s) - Es\sin(s) + E\cos(s) - 9F\cos(3s) - 9G\sin(3s)$$

Substituting into $\frac{d^2u_1}{ds^2} + u_1(s) = \frac{1}{4}\cos(3s)$, we obtain

$$-2Ds\cos(s) + 2E\cos(s) - 8F\cos(3s) - 8G\sin(3s) + H = -\frac{1}{4}\cos(3s)$$

$$-8F = -\frac{1}{4}, F = \frac{1}{32}, D = 0, E = 0, H = 0$$

$$u_{1p} = \frac{1}{32}\cos(3s)$$

$$u_{1n} = A\cos(s) + B\sin(s) + \frac{1}{32}\cos(3s)$$

Using the boundary condition $u_1(0) = 0, \frac{du_1}{ds}(0) = 0$

$$0 = A\cos(0) + B\sin(0) + \frac{1}{32}\cos(3*0), A = -\frac{1}{32}$$

$$\dot{u}_{1n} = -A\sin(s) + B\cos(s) - \frac{3}{32}\sin(3s)$$

$$0 = -\frac{1}{32}\sin(0) + B\cos(0) - \frac{3}{32}\sin(3*0), B = 0$$

$$u_{1n} = -\frac{1}{32}\cos(s) + \frac{1}{32}\cos(3s) = \frac{1}{32}\{\cos(3s) - \cos(s)\}$$

Therefore

$$u(t) = u_0(s) + \varepsilon u_1(s)$$

$$u(t) = \cos(s) + \frac{1}{32}\varepsilon\{\cos(3s) - \cos(s)\}$$

$$t = s - \frac{3}{8}\varepsilon s = \left\{1 - \frac{3}{8}\varepsilon\right\} s$$

$$s = \frac{t}{1 - \frac{3}{8}\varepsilon} = t\left\{1 + \frac{3}{8}\varepsilon\right\}$$

Thus,

$$u(t) = \cos(s) + \tfrac{1}{32}\varepsilon\{\cos(s) - \cos(3s)\}$$

Substituting $s = t\left\{1 + \frac{3}{8}\varepsilon\right\}$, we have

$$u(t) = \cos\left[t\left\{1 + \frac{3}{8}\varepsilon\right\}\right] + \frac{1}{32}\varepsilon\left[\cos 3\left\{1 + \frac{3}{8}\varepsilon\right\}t - \cos\left\{1 + \frac{3}{8}\varepsilon\right\}t\right]$$

$$= \cos t\cos\left(\frac{3}{8}t\varepsilon\right) - \sin t\sin\left(\frac{3}{8}t\varepsilon\right) + \frac{1}{32}\varepsilon\left[\cos 3t\cos\left(\frac{9}{8}t\varepsilon\right) - \sin t\sin\left(\frac{9}{8}t\varepsilon\right) - \cos t\cos\left(\frac{3}{8}t\varepsilon\right) + \sin(t)t\sin\left(\frac{3}{8}t\varepsilon\right)\right]$$

Since we want up to $O(\varepsilon)$ terms, we need to find the series for sine and cosine.

$$\cos\left(\frac{9}{8}t\varepsilon\right) = 1 - \frac{1}{2}\left(\frac{9}{8}\varepsilon t\right)^2 + \cdots$$

up to $O(\varepsilon)$, we have $\cos\left(\frac{3}{8}t\varepsilon\right) = 1$

$$\sin\left(\frac{3}{8}t\varepsilon\right) = \frac{3}{8}\varepsilon t - \frac{1}{6}\left(\frac{3}{8}\varepsilon t\right)^3 + \cdots$$

up to $O(\varepsilon)$, we have $\sin\left(\frac{3}{8}t\varepsilon\right) = \frac{3}{8}\varepsilon t$

Thus, up to the first term, we obtain

$$u(t) = \cos t - \frac{3}{8}t\varepsilon\sin t + \frac{1}{32}\varepsilon\left[\cos 3t - \frac{9}{8}t\varepsilon\sin t - \cos t + \frac{3}{8}t\varepsilon\sin t\right]$$

$$u(t) = \cos t - \frac{3}{8}t\varepsilon\sin t + \frac{1}{32}\varepsilon[\cos 3t - \cos t]$$

$$u(t) = \cos t + \varepsilon\left[\frac{1}{32}\{\cos 3t - \cos t\} - \frac{3}{8}t\sin t\right]$$

**5. Poincaré-Lindstedt method**

The Poincaré-Lindstedt Method is a mathematical technique for solving nonlinear oscillation problems and singular perturbation problems involving periodic solutions. This method assumes a solution of the form $x(t;\varepsilon) = X(\tau;\varepsilon)$, where $\tau = \omega(\varepsilon)t$ is a strained time variable and expands both the solution X and the frequency ω in powers of ε. The method eliminates secular terms, which cause non-uniformity, by requiring that each term in the expansion be periodic. This is achieved by setting the coefficients of the secular terms to zero, resulting in a uniformly valid solution. The Poincaré-Lindstedt method is particularly effective for problems involving nonlinear resonances, frequency locking, and bifurcations, and is widely used in nonlinear dynamics, physics, and engineering. To explain the method, let's consider Duffing's Oscillator problem below.

**5.1. $\ddot{u} + u + \varepsilon u^3 = 0, t > 0$ and subject to initial conditions $u(0) = 1, \dot{u}(0) = 0$**

This is an oscillator with a slightly nonlinear restoring force, which means that the force opposing the oscillation is not entirely proportional to the displacement, leading to a more complex and interesting behavior. We shall solve for $u(t)$ satisfying.

$\tau = \omega t$ where $\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots$

By changing variable

$$u(t) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \cdots$$

$$v(\tau) = v_0(\tau) + \varepsilon v_1(\tau) + \varepsilon^2 v_2(\tau) + \cdots$$

$$v'(\tau) = v'_0(\tau) + \varepsilon v'_1(\tau) + \varepsilon^2 v'_2(\tau) + \cdots$$

$$\nu''(\tau) = \nu_0'(\tau) + \varepsilon\nu_1'(\tau) + \varepsilon^2\nu_2'(\tau) + \cdots$$

$$\nu''(\tau) = \nu_0''(\tau) + \varepsilon\nu_1''(\tau) + \varepsilon^2\nu_2''(\tau) + \cdots$$

$$\tau = \omega t, \frac{d\tau}{dt} = \omega, u(t) = \nu(\tau)$$

$$\dot{u} = \frac{du}{dt} = \frac{d\nu}{d\tau} \bullet \frac{d\tau}{dt} = \frac{d\nu}{d\tau} \bullet \omega = \omega\frac{d\nu}{d\tau} = \omega\nu',$$

$$\ddot{u} = \frac{d^2u}{dt^2} = \frac{d}{dt}\left(\frac{d\nu}{d\tau} \bullet \frac{d\tau}{dt}\right) = \frac{d}{dt}\left(\frac{d\nu}{d\tau}\right)\frac{d\tau}{dt} = \frac{d}{d\tau}\left(\frac{d\nu}{dt}\right)\frac{d\tau}{dt} = \frac{d}{d\tau}\left(\frac{d\nu}{d\tau} \bullet \frac{d\tau}{dt}\right)\frac{d\tau}{dt} = \frac{d}{d\tau}\left(\frac{d\nu}{d\tau}\omega\right)\omega = \omega^2\frac{d^2\nu}{d\tau^2} = \omega^2\nu''$$

Now, , $u(t) = \nu(\tau)$ so $u(0) = \nu(0) = 1, \dot{u}(0) = \nu'(0) = 0$

Substituting the differential equation, we obtain

$$\omega^2\nu'' + \varepsilon(\omega\nu')^3 + \nu = 0, \omega^2\nu'' + \varepsilon\omega^3(\nu)^3 + \nu = 0$$

$$\{1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots\}^2\{\nu_0''(\tau) + \varepsilon\nu_1''(\tau) + \varepsilon^2\nu_2''(\tau) + \cdots\} + \varepsilon\{1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots\}^3(\nu_0(\tau) + \varepsilon\nu_1(\tau) + \varepsilon^2\nu_2(\tau) + \cdots)^3 + \nu_0(\tau) + \varepsilon\nu_1(\tau) + \varepsilon^2\nu_2(\tau) + \cdots = 0$$

Equating coefficients of equal powers of $\varepsilon$, we have

$$\varepsilon^0: \nu_0''(\tau) + \nu_0(\tau) = 0, \nu_0(0) = 1, \nu'_1(0) = 0$$

$$\nu_0(\tau) = A\cos(\tau) + B\sin(\tau)$$

Applying the boundary condition $\nu_0(0) = 1$, and $\nu'_0(0) = 0$, we have $A = 1, B = 0$

$$\nu_0(\tau) = \cos(\tau)$$

$$\varepsilon^1: \nu_1''(\tau) + 2\omega_1\nu_0''(\tau) + \{\nu_0(\tau)\}^3 + \nu_1(\tau) = 0$$

$$\nu_1''(\tau) + \nu_1(\tau) = -\{\nu_0(\tau)\}^3 - 2\omega_1\nu_0''(\tau)$$

$$\nu_1''(\tau) + \nu_1(\tau) = -\cos^3(\tau) + 2\omega_1\cos(\tau)$$

$$\nu_1''(\tau) + \nu_1(\tau) = -\left\{\frac{3}{4}\cos(\tau) + \frac{1}{4}\cos(3\tau)\right\} + 2\omega_1\cos(\tau)$$

$$\nu_1''(\tau) + \nu_1(\tau) = \left\{2\omega_1 - \frac{3}{4}\right\}\cos(\tau) - \frac{1}{4}\cos(3\tau)$$

Homogenous solution is $\nu_{1H} = A\cos(\tau) + B\sin(\tau)$

A particular solution is $\nu_{1p} = D\tau\cos(\tau) + E\tau\sin(\tau) + F\cos(3\tau) + G\sin(3\tau) + H$

$$\dot{\nu}_{1p} = -D\tau\sin(\tau) + D\cos(\tau) + E\tau\cos(\tau) + E\sin(\tau) - 3F\sin(3\tau) + 3G\cos(3\tau)$$

$$\ddot{\nu}_{1p} = -D\sin(\tau) - D\tau\cos(\tau) - D\sin(\tau) + E\cos(\tau) - E\tau\sin(\tau) + E\cos(\tau) - 9F\cos(3\tau) - 9G\sin(3\tau)$$

Substituting into

$\nu_1''(\tau) + \nu_1(\tau) = \left\{2\omega_1 - \frac{3}{4}\right\}\cos(\tau) - \frac{1}{4}\cos(3\tau)$, we obtain

$$-D\sin(\tau) - D\tau\cos(\tau) - D\sin(\tau) + E\cos(\tau) - E\tau\sin(\tau) + E\cos(\tau) - 9F\cos(3\tau) - 9G\sin(3\tau) + D\tau\cos(\tau) + E\tau\sin(\tau) + F\cos(3\tau) + G\sin(3\tau) + H = \left\{2\omega_1 - \frac{3}{4}\right\}\cos(\tau) - \frac{1}{4}\cos(3\tau)$$

$$-D\sin(\tau) - D\sin(\tau) + E\cos(\tau) + E\cos(\tau) - 9F\cos(3\tau) - 9G\sin(3\tau) + F\cos(3\tau) + G\sin(3\tau) + H = \left\{2\omega_1 - \frac{3}{4}\right\}\cos(\tau) - \frac{1}{4}\cos(3\tau)$$

$$-2\mathrm{D}\sin(\tau)+2\mathrm{E}\cos(\tau)-8\mathrm{F}\cos(3\tau)-8\mathrm{G}\sin(3\tau)+\mathrm{H}=\left\{2\omega_1-\frac{3}{4}\right\}\cos(\tau)-\frac{1}{4}\cos(3\tau)$$

Equating coefficients, we have

$$\mathrm{D}=0,2\mathrm{E}=2\omega_1-\frac{3}{4},\mathrm{E}=\omega_1-\frac{3}{8},\mathrm{H}=0,\mathrm{G}=0,-8\mathrm{F}=-\frac{1}{4},\mathrm{F}=\frac{1}{32}$$

$$\nu_{1p}=\left(\omega_1-\frac{3}{8}\right)\tau\sin(\tau)+\frac{1}{32}\cos(3\tau)$$

Since $\cos(\tau)$ is a solution to the homogeneous equation, this term on the right-side leads to a particular solution, leading to a secular term of the form $\tau\cos(\tau)$. This term is eliminated by taking $\omega_1=0$

$$\nu_{1p}=-\frac{3}{8}\tau\sin(\tau)+\frac{1}{32}\cos(3\tau)$$

Therefore

$$\nu_{1H}=\mathrm{A}\cos(\tau)+\mathrm{B}\sin(\tau)-\frac{3}{8}\tau\sin(\tau)+\frac{1}{32}\cos(3\tau)$$

Using the boundary conditions, $\nu_{10}(0)=0$

$$0=\mathrm{A}\cos(0)+\mathrm{B}\sin(0)+\frac{1}{32}\cos(3\bullet 0),\mathrm{A}=-\frac{1}{32}$$

$$\nu_{1H}=-\frac{1}{32}\cos(\tau)+\mathrm{B}\sin(\tau)-\frac{3}{8}\tau\sin(\tau)+\frac{1}{32}\cos(3\tau)$$

$$\nu'_{1H}=\frac{1}{32}\sin(\tau)+\mathrm{B}\cos(\tau)-\frac{3}{32}\sin(\tau)-\frac{3}{32}\tau\cos(\tau)-\frac{3}{32}\sin(3\tau)$$

Using the boundary conditions, $\nu'_1(0)=0,B=0$

$$\nu_{1H}=-\frac{1}{32}\cos(\tau)-\frac{3}{8}\tau\sin(\tau)+\frac{1}{32}\cos(3\tau)$$

$$\nu(\tau)=\nu_0(\tau)+\varepsilon\nu_1(\tau)+\varepsilon^2\nu_2(\tau)+\cdots$$

$$\tau=\omega t=(1+\varepsilon\omega_1)t=\left(1+\frac{3}{8}\varepsilon\right)t,\nu(\tau)=\cos(\tau)+\frac{1}{32}\varepsilon\{\cos(3\tau)-\cos(\tau)\}$$

But $\tau=\omega t$ where $\omega=1+\varepsilon\omega_1+\varepsilon^2\omega_2+\cdots$

$$\tau=\omega t=(1+\varepsilon\omega_1+\varepsilon^2\omega_2+\cdots)t$$

$$\omega_1=\tfrac{3}{8},\tau=\omega t=\left(1+\tfrac{3}{8}\varepsilon+\cdots\right)t$$

$$\nu(\tau)=\cos\left(1+\frac{3}{8}\varepsilon\right)t+\frac{1}{32}\varepsilon\left\{-\cos\left(1+\frac{3}{8}\varepsilon\right)t+\cos 3\left(1+\frac{3}{8}\varepsilon\right)t\right\}$$

$$=\mathrm{cost}\cos\left(\frac{3}{8}t\varepsilon\right)-\mathrm{sint}\sin\left(\frac{3}{8}t\varepsilon\right)+\frac{1}{32}\varepsilon\left[-\mathrm{cott}\cos\left(\frac{3}{8}t\varepsilon\right)+\mathrm{sint}\sin\left(\frac{3}{8}t\varepsilon\right)+\cos 3t\cos\left(\frac{9}{8}t\varepsilon\right)-\sin 3(t)t\sin\left(\frac{9}{8}t\varepsilon\right)\right]$$

Since we want up to $O(\varepsilon)$ terms, we need to find the series for sine and cosine.

$$\cos\left(\frac{3}{8}t\varepsilon\right)=1-\frac{1}{2}\left(\frac{3}{8}\varepsilon t\right)^2=1$$

$$\cos\left(\frac{9}{8}t\varepsilon\right)=1-\frac{1}{2}\left(\frac{9}{8}\varepsilon t\right)^2=1$$

$$\sin\left(\frac{3}{8}t\varepsilon\right)=\frac{3}{8}\varepsilon t-\frac{1}{6}\left(\frac{3}{8}\varepsilon t\right)^3=\frac{3}{8}\varepsilon t$$

$$\sin\left(\frac{9}{8}t\varepsilon\right) = \frac{9}{8}\varepsilon t - \frac{1}{6}\left(\frac{9}{8}\varepsilon t\right)^3 = \frac{9}{8}\varepsilon t$$

up to $O(\varepsilon)$, we have $\sin\left(\frac{3}{8}t\varepsilon\right) = \frac{3}{8}\varepsilon t$

Thus, up to the first term, we obtain

$$u(t) = \cos t\cos\left(\frac{3}{8}t\varepsilon\right) - \sin t\sin\left(\frac{3}{8}t\varepsilon\right) + \frac{1}{32}\varepsilon\left[-\cos t\cos\left(\frac{3}{8}t\varepsilon\right) + \sin t\sin\left(\frac{3}{8}t\varepsilon\right) + \cos 3t\cos\left(\frac{9}{8}t\varepsilon\right) - \sin 3(t) t\sin\left(\frac{9}{8}t\varepsilon\right)\right]$$

$$u(t) = \cos t - \frac{3}{8}t\varepsilon\sin t + \frac{1}{32}\varepsilon\left[-\cos t + \frac{3}{8}t\varepsilon\sin t + \cos 3t - \frac{9}{8}t\varepsilon\sin(3t)\right] + \cdots$$

$$u(t) = \cos t - \frac{3}{8}t\varepsilon\sin t + \frac{1}{32}\varepsilon\left[\cos 3t - \frac{9}{8}t\varepsilon\sin 3t - \cos t + \frac{3}{8}\varepsilon t\sin t\right] + \cdots$$

Hence, for up to $O(\varepsilon)$, we obtain

$$u(t) = \cos t - \frac{3}{8}t\varepsilon\sin t + \frac{1}{32}\varepsilon[\cos 3t - \cos t] = \cos t + \varepsilon\left[\frac{1}{32}\{\cos 3t - \cos t\} - \frac{3}{8}t\sin t\right]$$

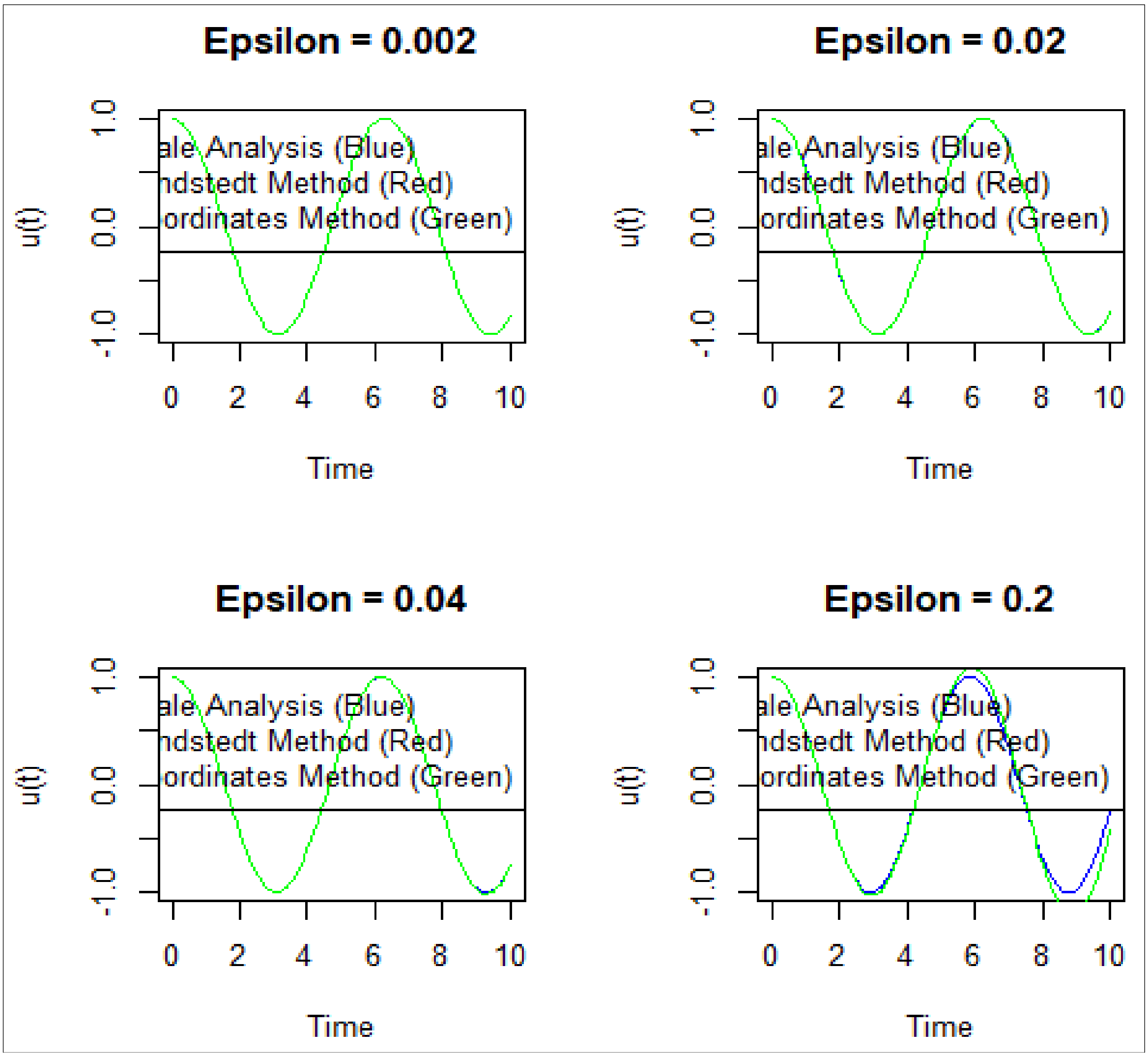


**Fig 1:** The analysis of graphs depicting

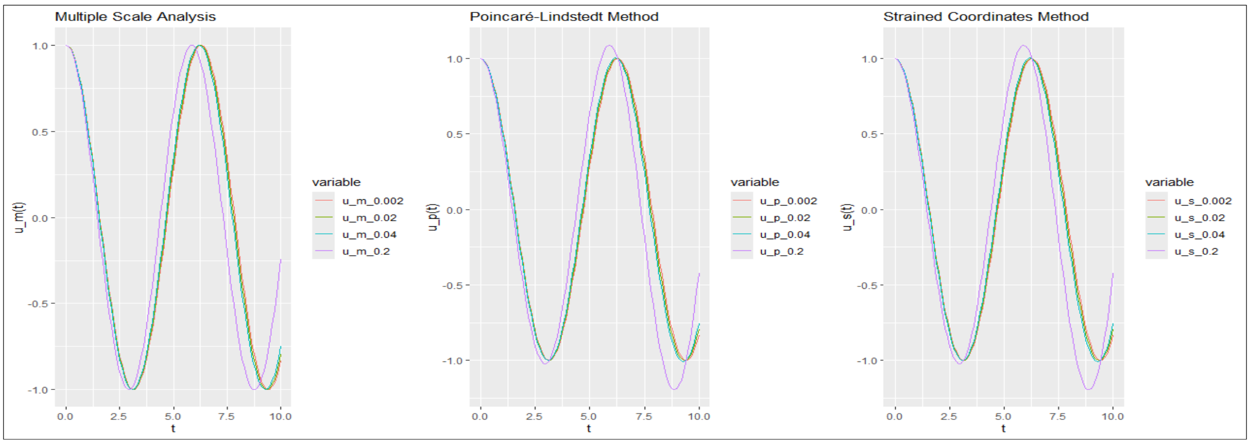


**Fig 2:** Method should consider the specific

The analysis of graphs depicting the behavior of Multiple Scale Analysis, Poincaré-Lindstedt Method, and Strained Coordinates Method across varying epsilon values (0.002 to 0.2) reveals significant insights into their performance. For small epsilon values (0.002 to 0.04), all methods exhibit sinusoidal waveforms, with the Strained Coordinates and Poincaré-Lindstedt methods showing identical results and more pronounced amplitude variations compared to the Multiple Scale Analysis. As epsilon increases, particularly at 0.2, the waveforms become highly complex for all methods, with significant amplitude variations and deviations from sinusoidal behavior. While the Multiple Scale Analysis method demonstrates more consistent sinusoidal waveforms and less sensitivity to changes in epsilon, it's important to note that this consistency might not accurately represent the true nonlinear behavior of the system for larger epsilon values. The Poincaré-Lindstedt and Strained Coordinates methods, despite showing more pronounced amplitude variations, may capture more of the nonlinear effects present in the system. Given these observations, rather than recommending a single method, researchers are advised to adopt a nuanced approach. For weakly nonlinear systems (small epsilon), the Multiple Scale Analysis method is suitable due to its consistency and ease of interpretation. For moderately nonlinear systems, the Strained Coordinates or Poincaré-Lindstedt methods may be more appropriate as they capture more nonlinear effects. In strongly nonlinear systems (large epsilon), a combination of methods should be employed and compared with numerical solutions to ensure accuracy. Ultimately, the choice of method should consider the specific characteristics of the system under study and the desired balance between simplicity and capturing nonlinear effects.

**Table 1:** $\varepsilon = 0.002$

| *t* | Multiple scales | Strained Coordinates | Poincare- Lindstedt |
|---|---|---|---|
| 0 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.1 | 0.9949967 | 0.9949942 | 0.9949942 |
| 0.2 | 0.9800368 | 0.9800271 | 0.9800271 |
| 0.3 | 0.9552700 | 0.9552491 | 0.9552491 |
| 0.4 | 0.9209441 | 0.9209092 | 0.9209092 |
| 0.5 | 0.8774027 | 0.8773523 | 0.8773523 |

The analysis of the results table for ε=0.002 reveals interesting insights into the performance of Multiple Scales, Strained Coordinates, and Poincaré-Lindstedt methods for nonlinear oscillator analysis. All three methods start with the same initial condition of 1.0000000 at t=0, providing a common baseline for comparison. As time progresses, the solutions from all methods show a decreasing trend in the oscillator's amplitude, which is consistent with typical nonlinear oscillator behavior. A striking observation is the exact agreement between the Strained Coordinates and Poincaré-Lindstedt methods to seven decimal places across all time steps. This suggests that for this problem and parameter value, these two methods are essentially equivalent. Their consistency implies that either method could be reliably used for this specific scenario. The Multiple Scales method, while closely aligned with the other two, shows slight deviations. These differences, though small, become more noticeable as time advances. At t=0.1, the deviation is about 2.5e-6, growing to approximately 5.04e-5 by t=0.5. This growing divergence hints at a cumulative effect of small discrepancies over time, which could become more significant in longer time-scale analyses. Given the small value of ε (0.002), it's expected that all these perturbation methods would provide reasonably accurate results, which is indeed reflected in the close agreement among them. The high precision of the results, presented to seven decimal places, allows for a detailed comparison of these subtle differences. However, it's important to note that for short time scales, all methods appear to be in close agreement, and the choice of method may only become critical for longer time scales or higher precision requirements. In conclusion, while all three methods provide very close results for this case, the slight deviations in the Multiple Scales method and the perfect agreement between Strained Coordinates and Poincaré-Lindstedt methods are noteworthy. For this scenario with a small ε, the latter two methods might be preferred for their consistency. However, a more comprehensive analysis over longer time scales and with different ε values would be necessary to draw definitive conclusions about the relative accuracies and applicability of these methods in various nonlinear oscillator scenarios.

**Table 2:** $\varepsilon = 0.02$

| $t$ | Multiple Scale Analysis | Strained Coordinates | Poincare- Lindstedt |
|---|---|---|---|
| 0 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.1 | 0.9797675 | 0.9949045 | 0.9949045 |
| 0.2 | 0.9546692 | 0.9796719 | 0.9796719 |
| 0.3 | 0.9198886 | 0.9544630 | 0.9544630 |
| 0.4 | 0.5352888 | 0.9195435 | 0.9195435 |
| 0.5 | 0.8757785 | 0.8752804 | 0.8752804 |

The analysis of Table 2, presenting results for ε=0.02, reveals significant insights into the performance of Multiple Scale Analysis, Strained Coordinates, and Poincaré-Lindstedt methods for nonlinear oscillator analysis. All methods start with the same initial condition of 1.0000000 at t=0, providing a common baseline. However, as time progresses, stark differences emerge. The Strained Coordinates and Poincaré-Lindstedt methods maintain perfect agreement to seven decimal places throughout all time steps, demonstrating remarkable consistency even with this larger ε value. Their results show a steady, decreasing trend in the oscillator's amplitude over time, which aligns with expected nonlinear oscillator behavior. In contrast, the Multiple Scale Analysis exhibits substantial deviations from the other two methods, with differences becoming increasingly pronounced over time. The deviation is already significant at t=0.1 (about 0.015137) and grows larger as time advances. A particularly anomalous result occurs at t=0.4, where the Multiple Scale Analysis yields 0.5352888, drastically deviating from both the other methods and the overall trend. Interestingly, at t=0.5, the Multiple Scale Analysis result seems to partially realign with the other methods, though still showing a notable difference. These results highlight the critical impact of the ε value on method performance. With ε=0.02, which is ten times larger than in the previous table, the Multiple Scale Analysis shows heightened sensitivity and potentially compromised accuracy. This suggests that for problems with larger nonlinearities or perturbations, the choice of analytical method becomes crucial. The Strained Coordinates and Poincaré-Lindstedt methods demonstrate greater robustness and consistency for this larger ε value, indicating they might be more suitable for analyzing oscillators with stronger nonlinearities. This comparison underscores the importance of carefully selecting the analytical method based on the specific characteristics of the nonlinear oscillator problem, particularly the magnitude of the nonlinearity or perturbation parameter.

**Table 3:** $\varepsilon = 0.04$

| $t$ | Multiple Scale Analysis | Strained Coordinates | Poincare- Lindstedt |
|---|---|---|---|
| 0 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.1 | 0.9948533 | 0.9948048 | 0.9948048 |
| 0.2 | 0.9794662 | 0.9792772 | 0.9792772 |
| 0.3 | 0.9539970 | 0.9535895 | 0.9535895 |
| 0.4 | 0.9187079 | 0.9180261 | 0.9180261 |
| 0.5 | 0.8739622 | 0.8729783 | 0.8729783 |

The analysis of Table 3, presenting results for ε=0.04, reveals important insights into the performance of Multiple Scale Analysis, Strained Coordinates, and Poincaré-Lindstedt methods for nonlinear oscillator analysis. All three methods begin with the same initial condition of 1.0000000 at t=0, providing a common starting point. As time progresses, we observe a consistent decreasing trend in the oscillator's amplitude across all methods, which aligns with typical nonlinear oscillator behavior.The Strained Coordinates and Poincaré-Lindstedt methods continue to show perfect agreement to seven decimal places throughout all time steps, maintaining their consistency even with this further increased ε value. This persistent agreement reinforces the equivalence of these two methods for this problem, regardless of the magnitude of ε within the range tested. Their results demonstrate a steady, predictable decrease in amplitude over time.The Multiple Scale Analysis, while showing closer alignment with the other methods compared to the case where ε=0.02, still exhibits noticeable deviations. These differences, though smaller than in the previous table, are consistent and grow slightly as time progresses. For instance, at t=0.1, the difference is about 0.0000485, and by t=0.5, it has grown to approximately 0.0009839. This pattern suggests that while the Multiple Scale Analysis performs better for this ε value than for ε=0.02, it still introduces some cumulative error over time. The improved performance of Multiple Scale Analysis in this case, compared to when ε=0.02, indicates that the relationship between the method's accuracy and the magnitude of ε is not simply linear, and there may be an optimal range for its application.

**Table 1:** $\boldsymbol{\varepsilon = 0.2}$

| $t$ | Multiple Scale Analysis | Strained Coordinates | Poincare- Lindstedt |
|---|---|---|---|
| 0 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.1 | 0.9942274 | 0.9940075 | 0.9940075 |
| 0.2 | 0.9769764 | 0.9761195 | 0.9761195 |
| 0.3 | 0.9484460 | 0.9466015 | 0.9466015 |
| 0.4 | 0.9089657 | 0.9058865 | 0.9058865 |
| 0.5 | 0.8589913 | 0.8545613 | 0.8545613 |

The analysis of Table 1 for ε=0.2 reveals significant insights into the performance of Multiple Scale Analysis, Strained Coordinates, and Poincaré-Lindstedt methods for nonlinear oscillator analysis under a much larger perturbation parameter. All three methods start with the same initial condition of 1.0000000 at t=0, providing a common baseline for comparison. As time progresses, all methods show a decreasing trend in the oscillator's amplitude, which is consistent with nonlinear oscillator behavior.The Strained Coordinates and Poincaré-Lindstedt methods continue to show perfect agreement to seven decimal places across all time steps, maintaining their consistency even with this substantially larger ε value. This persistent agreement further reinforces the equivalence of these two methods for this problem, regardless of the magnitude of ε within the range tested. Their results demonstrate a steady, predictable decrease in amplitude over time. The Multiple Scale Analysis, while following the same general trend, shows notable deviations from the other two methods. These differences are more pronounced than in the cases with smaller ε values and grow consistently as time progresses. For instance, at t=0.1, the difference is about 0.0002199, and by t=0.5, it has grown to approximately 0.0044300. This pattern indicates that the Multiple Scale Analysis introduces a cumulative error over time, which becomes more significant with this larger ε value. The increasing deviation suggests that for strongly nonlinear systems (represented by larger ε values), the Multiple Scale Analysis may be less accurate compared to the Strained Coordinates and Poincaré-Lindstedt methods, especially over longer time scales.

## 6. Conclusion

Based on the analysis of the four tables comparing Multiple Scale Analysis, Strained Coordinates, and Poincaré-Lindstedt methods for nonlinear oscillator analysis across various ε values (0.002, 0.02, 0.04, and 0.2), several key insights emerge. The Strained Coordinates and Poincaré-Lindstedt methods demonstrate remarkable consistency, producing identical results to seven decimal places across all ε values tested. This consistency suggests these methods are robust and reliable for a wide range of nonlinearity strengths. In contrast, the Multiple Scale Analysis shows increasing deviation from the other methods as ε increases, indicating a higher sensitivity to the magnitude of nonlinearity. For weakly nonlinear systems (small ε values), all three methods provide similar results, suggesting that any of these approaches could be appropriate. However, as ε increases, the choice of method becomes more critical. The Multiple Scale Analysis exhibits significant deviations and potential accuracy issues for larger ε values, especially over extended time scales. This observation underscores the importance of method selection based on the specific characteristics of the nonlinear oscillator problem, particularly the magnitude of the nonlinearity or perturbation parameter. Given these findings, researchers are recommended to carefully consider their method selection based on the nature of the system under study. For weakly nonlinear systems, any of the three methods can be used effectively. However, for strongly nonlinear systems, the Strained Coordinates or Poincaré-Lindstedt methods are preferable due to their consistency and apparent robustness. Regardless of the chosen method, it is crucial to validate results using multiple approaches, especially for systems with significant nonlinearities. This multi-method validation can provide valuable insights into the system's behavior and the limitations of each analytical approach. Researchers should also conduct thorough error analysis, particularly when using Multiple Scale Analysis for systems with larger ε values, as cumulative errors over time can become significant. For analyses over extended time scales, caution is advised with Multiple Scale Analysis, and the Strained Coordinates or Poincaré-Lindstedt methods may prove more reliable. Additionally, hybrid approaches that combine analytical methods with numerical techniques can be beneficial for complex nonlinear systems. Recent advancements like the least square homotopy perturbation method (LSHPM) or non-perturbative methodology (NPM) may offer improved accuracy and efficiency. To enhance the precision and reliability of nonlinear oscillator analyses, researchers should utilize advanced computational tools and software to implement these methods efficiently and verify results through numerical simulations. It's essential to tailor the choice of method to the specific characteristics of the nonlinear oscillator being studied, considering factors such as the type of nonlinearity, desired accuracy, and time scale of interest. By following these recommendations, researchers can achieve more accurate predictions and gain deeper insights into complex dynamical systems, advancing our understanding of nonlinear phenomena across various scientific and engineering disciplines.

## 7. References

1. Nayfeh AH. Perturbation methods. 1st ed. New York: John Wiley & Sons; c2000. ISBN: 9783527617609. Available from: https://doi.org/10.1002/9783527617609.fmatter.
2. Dalla Riva M, Lanza de Cristoforis M, Musolino P. Singularly perturbed boundary value problems: A functional analytic approach. Cham: Springer; c2021. ISBN: 978-3-030-76258-2. DOI: https://doi.org/10.1007/978-3-030-76259-9.
3. Nayfeh AH. Introduction to perturbation techniques. New York: Wiley; c2011. ISBN: 978-3-527-61845-3.
4. Kevorkian J, Cole JD. Perturbation methods in applied mathematics. 1st ed. New York: Springer-Verlag; 1981. ISBN: 978-1-4757-4213-8. DOI: https://doi.org/10.1007/978-1-4757-4213-8.
5. Hinch EJ. Perturbation methods. Cambridge: Cambridge University Press; 1991. ISBN: 9781139172189. DOI: https://doi.org/10.1017/CBO9781139172189.
6. Holmes M. Introduction to perturbation methods. New York: Springer; 1995. ISBN: 978-1-4612-5347-1. DOI: https://doi.org/10.1007/978-1-4612-5347-1.
7. Simmonds JG, Mann Jr JE. A first look at perturbation theory. Mineola (NY): Dover Publications Inc.; 1997. ISBN: 978-0486675510.
8. O'Malley RE. Singular perturbation methods for ordinary differential equations. New York: Springer-Verlag; 1991. ISBN: 978-0-387-97556-6. DOI: https://doi.org/10.1007/978-1-4612-0977-5.
9. JH. Asymptotic analysis and singular perturbation theory. University of California at Davis; c2004.
10. Asare B, Sah M, Hona RK. Alternative methods of regular and singular perturbation problems. Appl Math. 2024;15:687-708. DOI: 10.4236/am.2024.1510040.
11. Asare B, Sah M, Hona RK. Epsilon variations and their impact on solution accuracy: A matched asymptotic approach. Int J Stat Appl Math. 2024;9(6):34-44. DOI: 10.22271/maths.2024.v9.i6a.1883.