International Journal of Statistics and Applied Mathematics 2025; 10(10): 17-24

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
NAAS Rating (2025): 4.49
Maths 2025; 10(10): 17-24

https://www.mathsjournal.com
Received: 7-08-2025
Accepted: 10-09-2025

Konan Charles Etienne Goli
Ecole Supérieure Africaine de Technologies de l'Information et de la Communication (ESATIC), 18 BP 1501 Abidjan 18, Cote D'Ivoire

Corresponding Author:
Konan Charles Etienne Goli
Ecole Supérieure Africaine de Technologies de l'Information et de la Communication (ESATIC), 18 BP 1501 Abidjan 18, Cote D'Ivoire

Existence of solutions for some second order equations with nonhomogeneous boundary conditions and a function ϕ continuous on $DOM(\phi) \subset \mathbb{R}$

Konan Charles Etienne Goli

DOI: https://www.doi.org/10.22271/maths.2025.v10.i10a.2173

Abstract
We study the second order differential equation

$$(\phi(v'(s)))' = k(s, v(s), v'(s)), a.e.s \in [0,\xi]$$

Submitted to nonlinear Neumann-Steklov boundary conditions on $[0,\xi]$ where $k: [0,\xi] \times \mathbb{R}^2 \to \mathbb{R}$ a L^1 −Carathéodory function. $\phi: \mathbb{R} \to \mathbb{R}$, is initially considered as an increasing homeomorphism such that $\phi(0) = 0$. In a second step ϕ is considered as a continuous function on $Dom(\phi) \subset \mathbb{R}$ and strictly increasing on $[a,b] \subset Dom(\phi)$. We show the existence of at least one solution using some sign conditions and lower and upper solution method. No Nagumo-like growth condition for the dependence of $f(s,w,z)$ with respect to v is required.

Keywords: $\phi-$ Laplacian; L^1 −Carathéodory function, nonlinear Neumann-Steklov problem, Leray-Schauder degree, Brouwer degree, lower and upper solutions

Introduction

This paper aims to study the existence of solutions for the differential equation

$$\left(\phi(v'(s))\right)' = k\left(t, v(s), v'(s)\right), a.e.s \in [0,\xi] \tag{1}$$

Subject to Neumann-Steklov type conditions

$$\phi(v'(0)) = l_0(v(0)), \phi(v'(\xi)) = l_\xi(v(\xi)), \tag{2}$$

Where:

- $l_0, l_\xi: \mathbb{R} \to \mathbb{R}$ are continuous functions,
- $k: [0,\xi] \times \mathbb{R}^2 \to \mathbb{R}$ a L^1 −Carathéodory function,
- $\phi: \mathbb{R} \to \mathbb{R}$ is either:
  - An increasing homeomorphism with $\phi(0) = 0$, or
  - A continuous function strictly increasing on $[a,b] \subset Dom(\phi)$.

The study of equation (1) is a classical topic with significant applications, attracting extensive research. A $\phi-$ Laplacian operator is classified as:

- Singular if ϕ has a finite domain (i.e., $\phi:]-d,d[\to \mathbb{R}$, with $0 < d < +\infty$),
- Regular otherwise.

Recent work has explored both singular and regular operators. Notably, Cristian B. and Jean M. [1, 2] established existence and multiplicity results for (1) under various boundary conditions, where ϕ is an increasing homeomorphism on $]-d,d[$ with $(d > 0)$.
In 2008, Cristian B. and Jean M. [3] studied (1) −(2) with:

- $\phi:]-d,d[\to \mathbb{R}$ ($d \in]0,+\infty]$) an increasing homeomorphism satisfying $\phi(0)=0$,
- $k: [0,\xi]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ continuous.

Their key findings include:
1) For $d < +\infty$, problem (1) $-$(2) admits at least one solution under Villari-type sign conditions on k, l_0, l_ξ [4].
2) For $d < +\infty$, existence is guaranteed if lower and upper solutions exist (ordered or not).
3) For $d = +\infty$, a solution exists when $|l_0|$ is bounded, l_ξ is bounded above and $k(s,u,v) \le \tau(s)$ for some $\tau \in C([0,\xi])$.

In 2015 and 2017, Eienne G. and Assohoun A. [5, 6] extended somes results of Cristian B. and Jean M. to L^1 $-$Carathéodory function k.
The paper is structured as follows:
In section 2, we some preliminary. In section 3, we prove that the problem (1) $-$(2) admits at least one solution when l_0 is bounded from below, l_ξ is bounded from above and there exists $\tau \in L^1(0,\xi)$ such that $k(s,u,v) \ge \tau(s)$ for a.e. $s\in[0,\xi]$ and $\forall(u,v)\in\mathbb{R}^2$ or when l_0 is bounded from above, l_ξ is bounded from below and there exists $\tau\in L^1(0,\xi)$ satisfying $f(s,u,v)\le\tau(s)$ for a.e. $s\in[0,\xi]$ and $\forall(u,v)\in\mathbb{R}^2$. After that, in section 4 we apply section 3's results to nonlinear beam equations. Finally, in section 5, we extend the study to the generalized problem

$$\left(\Phi\left(v'(s)\right)\right)' = k\left(s,v(s),v'(s)\right), a.e.\, s\in[0,\xi] \tag{3}$$

$$\Phi\left(v'(0)\right) = l_0\left(v(0)\right), \Phi\left(v'(\xi)\right) = l_\xi\left(v(\xi)\right) \tag{4}$$

where $\Phi: Dom(\Phi)\subset\mathbb{R}\to\mathbb{R}$ is continuous and strictly increasing on $[a,b]\subset Dom(\Phi)$,
$l_0, l_\xi: \mathbb{R}\to\mathbb{R}$ are two continuous functions and $k: [0,\xi]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ a L^1 $-$Carathéodory function.

Preliminary
Definition 2.1. $k: [0,\xi]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a L^1 $-$Carathéodory function if:
1. $k(.,w,z): [0,\xi] \to \mathbb{R}$ is measurable for all $(w,z)\in\mathbb{R}^2$;
2. $k(s,.,.): \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous for $a.e.\, s\in[0,\xi]$;
3. For each compact set $A\subset\mathbb{R}^2$ there is a function $\eta_A\in L^1(0,\xi)$ such that $|f(s,w,z)|\le\eta_A$ for $a.e.\, s\in[0,\xi]$ and all $(w,z)\in A$.

Let us consider the problem

$$\begin{aligned}&\left(\phi\left(v'(s)\right)\right)' = k\left(s,v(s),v'(s)\right), a.e. t\in[0,\xi]\\ &\phi\left(v'(0)\right)=l_0\left(v(0)\right), \phi\left(v'(\xi)\right)=l_\xi\left(v(\xi)\right),\end{aligned} \tag{3}$$

with $k: [0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a L^1 $-$Carathéodory function, $l_0, l_\xi: \mathbb{R}\to\mathbb{R}$ are two continuous functions and $\phi:]-d,d[\to\mathbb{R}$ ($d \in]0,+\infty[$), an increasing homeomorphism such that $\phi(0)=0$.

Definition 2.2. $v\in C^1([0,\xi])$ is a solution of problem (5) if $\phi(v')\in AC([0,\xi])$, $||u'||_\infty < d$ and v satisfies (5). $AC([0,\xi])$ is the set of absolutely continuous functions on $[0,\xi]$.

Definition 2.3. A lower-solution of the problem (5) is a function $\alpha\in C^1([0,\xi])$ such that

$||\alpha'||_\infty < d$, $\phi(\alpha')\in AC([0,\xi])$ and

$$\left(\phi\left(\alpha'(s)\right)\right)' \ge k\left(s,\alpha(s),\alpha'(s)\right), a.e.\, s\in[0,\xi],$$

$$\phi\left(\alpha'(0)\right)\ge l_0\left(\alpha(0)\right)\ and\ \phi\left(\alpha'(\xi)\right)\le l_\xi\left(\alpha(\xi)\right).$$

Definition 2.4. A lower-solution of the problem (5) is a function $\beta\in C^1([0,\xi])$ such that

$||\beta'||_\infty < d$, $\phi(\beta')\in AC([0,\xi])$ and

$$\left(\phi\left(\beta'(s)\right)\right)' \le k\left(s,\beta(s),\beta'(s)\right), a.e.\, s\in[0,\xi],$$

$$\phi\left(\beta'(0)\right)\le l_0\left(\beta(0)\right)\ and\ \phi\left(\beta'(\xi)\right)\ge l_\xi\left(\beta(\xi)\right).$$

Theorem 2.1. The existence of a lower solution α and an upper solution β for (5) implies that the problem (5) has at least one solution.

Proof. See [5] and [6].

Theorem 2.2. The existence of a lower-solution α and an upper-solution β of (5) such that $\forall\, s \in [0,\xi]$, $\alpha(s) \leq \beta(s)$, implies that the problem (5) admits at least one solution u such that $\forall\, s \in [0,\xi], \alpha(s) \leq u(s) \leq \beta(s)$.

Proof. See [5] and [6].

Existence result

To prove an analogous result for d = +∞, we will apply Theorem 2.1. With this goal in mind, let us consider the problem

$$\begin{aligned}&\left(\phi\left(v'(s)\right)\right)' = k\left(t,v(s),v'(s)\right), a.e. s\in[0,\xi]\\ &\phi(v'(0))=l_0(v(0)),\phi(v'(\xi))=l_\xi(v(\xi)),\end{aligned} \tag{10}$$

Where $k: [0,\xi]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a L^1 –Carathéodory function, $l_0, l_\xi: \mathbb{R}\to\mathbb{R}$ are two continuous functions and $\phi: \mathbb{R}\to\mathbb{R}$, an increasing homeomorphism such that $\phi(0) = 0$.

Definition 3.1. $u \in C^1([0,\xi])$ is a solution of problem (10) if $\phi(u') \in AC([0,\xi])$ and satisfies (10).

We need the following results.

Lemma 3.1. Suppose that:

a) There exists $g \in L^1(0,\xi)$ such that $k(s,w,z) \leq g(s)$ for a.e. $s \in [0,\xi]$ and $\forall (w,z) \in \mathbb{R}^2$.

b) There exists $(\vartheta_0,\vartheta_\xi) \in \mathbb{R}^2$ such that $\forall s \in [0,T]$, $l_0(s) \leq \vartheta_0$ and $l_\xi(s) \geq \vartheta_\xi$.

If v is a solution of (10), then $||v'||_\infty \leq e$ where

$$e = \max\{|\phi^{-1}[-(\max\{|\vartheta_0|,|\vartheta_\xi|\} + ||g||_{L^1(0,\xi)})]|, |\phi^{-1}[\max\{|\vartheta_0|,|\vartheta_\xi|\} + ||g||_{L^1(0,\xi)}]\,|\}.$$

Proof. Let u be a solution of (10). Then we have: $\forall s \in [0,\xi]$,

$$\begin{aligned}\phi\left(v'(s)\right) &= \phi\left(v'(0)\right) + \int_0^s k(z,v(z),v'(z))dz\\ &= l_0(\mathrm{v}(0)) + \int_0^s k(z,v(z),v'(z))dz\\ &\leq \vartheta_0 + \int_0^s g(z)dz\end{aligned}$$

and

$$\begin{aligned}\phi\left(v'(t)\right) &= \phi\left(v'(\xi)\right) - \int_s^\xi k(z,v(z),v'(z))dz\\ &= l_\xi(\mathrm{v}(\xi)) + \int_s^\xi k(z,v(z),v'(z))dz\\ &\geq \vartheta_\xi - \int_s^\xi g(z)dz\end{aligned}$$

Hence, $\forall s \in [0,\xi], \vartheta_\xi - \int_s^\xi g(z)dz \leq \phi\left(v'(s)\right) \leq \vartheta_0 + \int_0^s g(z)dz$.

Moreover $\forall s \in [0,\xi]$,

$$\left|\phi\left(v'(s)\right)\right| \leq \max\left\{|\vartheta_0| + ||g||_{L^1(0,\xi)}, |\vartheta_\xi| + ||g||_{L^1(0,\xi)}\right\}.$$

$$\max\left\{|\vartheta_0| + ||g||_{L^1(0,\xi)}, |\vartheta_\xi| + ||g||_{L^1(0,\xi)}\right\} = \max\{|\vartheta_0|,|\vartheta_\xi|\} + ||g||_{L^1(0,\xi)}$$

It follows that $\forall t \in [0,\xi]$,

$$|v'(t)| \leq \max\{|\phi^{-1}[-(\max\{|\vartheta_0|,|\vartheta_\xi|\} + ||g||_{L^1(0,\xi)})]|, |\phi^{-1}[\max\{|\vartheta_0|,|\vartheta_\xi|\} + ||g||_{L^1(0,\xi)}]|\} = e.$$

Lemma 3.2. Suppose that:

a) There exists $g \in L^1(0,\xi)$ such that $k(s,w,z) \geq g(s)$ for a.e. $s \in [0,\xi]$ and

$\forall (w,z) \in \mathbb{R}^2$.

b) There exists $(\vartheta_0, \vartheta_\xi) \in \mathbb{R}^2$ such that $\forall s \in [0,\xi]$, $l_0(s) \geq \vartheta_0$ and $l_\xi(s) \leq \vartheta_\xi$.

If v is a solution of (10), then $||v'||_\infty \leq e$, where

$e = \max\{|\phi^{-1}[-(\max\{|\vartheta_0|, |\vartheta_\xi|\} + ||g||_{L^1(0,\xi)})]|, |\phi^{-1}[\max\{|\vartheta_0|, |\vartheta_\xi|\} + ||g||_{L^1(0,\xi)}]\,|\}$.

Proof: One can prove this by adapting the proof of Lemma 3.1

Definition 3.2. $\alpha \in C^1([0,\xi])$ is a lower-solution of the problem (5) if $\phi(\alpha') \in AC([0,\xi])$ and

$\left(\phi(\alpha'(s))\right)' \geq k(s, \alpha(s), \alpha'(s)), a.e. s \in [0,\xi],$

$\phi(\alpha'(0)) \geq l_0(\alpha(0))$ *and* $\phi(\alpha'(\xi)) \leq l_\xi(\alpha(\xi)).$

Definition 3.3. $\beta \in C^1([0,\xi])$ is an upper-solution of the problem (5) if $\phi(\beta') \in AC([0,\xi])$ and

$\left(\phi(\beta'(s))\right)' \leq k(s, \beta(s), \beta'(s)), a.e. s \in [0,\xi],$

$\phi(\beta'(0)) \leq l_0(\beta(0))$ *and* $\phi(\beta'(\xi)) \geq l_\xi(\beta(\xi)).$

Theorem 3.1. Suppose that:
a) There exists $g \in L^1(0,\xi)$ such that $k(s,w,z) \leq g(s)$ for a.e. $s \in [0,\xi]$ and $\forall (w,z) \in \mathbb{R}^2$.
b) There exists $(\vartheta_0, \vartheta_\xi) \in \mathbb{R}^2$ such that $\forall s \in [0,\xi]$, $l_0(s) \leq \vartheta_0$ and $l_\xi(s) \geq \vartheta_\xi$.
c) The problem (10) admits a lower-solution α and an upper-solution β.

It follows that problem (10) has at least one solution.

Proof: Let
$e = \max\{|\phi^{-1}[-(\max\{|\vartheta_0|, |\vartheta_\xi|\} + \|g\|_{L^1(0,T)})]|, |\phi^{-1}[\max\{|\vartheta_0|, |\vartheta_\xi|\} + ||g||_{L^1(0,T)}]|\}$, $d' = \max\{\|\alpha'\|_\infty, \|\beta'\|_\infty, e\} + 1$ and $d = d' + 1$.
Let $\Upsilon:]-d, d[\to \mathbb{R}$ be an increasing homeomorphism such that $\phi = \Upsilon$ on $[-d', d']$. It is clear that α and β are respectively lower-solution and upper-solution of problem

$$\begin{cases} \left(\Upsilon(v'(s))\right)' = k(s, v(s), v'(s)), a.e. s \in [0,\xi] \\ \Upsilon(v'(0)) = l_0(v(0)), \Upsilon(v'(\xi)) = l_\xi(v(\xi)), \end{cases} \tag{11}$$

Then, using Theorem 2.2. we deduce that the problem (11) has a solution u which is also a solution of problem (10) by Lemma 3.1.

Theorem 3.2. Suppose that:
a) There exists $g \in L^1(0,\xi)$ such that $k(s,w,z) \geq g(s)$ for a.e. $s \in [0,\xi]$ and $\forall (w,z) \in \mathbb{R}^2$.
b) There exists $(\vartheta_0, \vartheta_\xi) \in \mathbb{R}^2$ such that $\forall s \in [0,\xi]$, $l_0(s) \geq \vartheta_0$ and $l_\xi(s) \leq \vartheta_\xi$.
c) The problem (10) admits a lower-solution α and an upper-solution β.

It follows that problem (10) has at least one solution.

Proof: The proof is similar to the proof Theorem 3.1.

Remark 3.1. In contrast to Theorem 5 in [3], $|g_0|$ bounded is not necessary; g_0 bounded above or g_0 bounded below is sufficient when $d = +\infty$.

Example 3.1. Consider the problem

$$\left(|v'(s)|^{p-2}v'(s)\right)' = -\gamma|v'(s)|^q - \frac{\delta \max\{0, v(s)\}}{\sqrt{t}} + s, a.e. s \in [0,\xi]$$
$$|v'(0)|^{p-2}v'(0) = -\left(v(0)\right)^2 \textit{and } |v'(\xi)|^{p-2}v'(\xi) = (v(\xi))^2$$

where $p \geq 2$, $\gamma > 0$, $\delta > 0$ and $q > 0$. $\alpha(s) = \frac{\xi\sqrt{\xi}}{\delta}$ and $\beta(s) = 0$ are lower and upper solutions. Taking $h(s) = s$ and $\vartheta_0 = \vartheta_\xi = 0$, by Theorem 3.1, we deduce that the problem has at least one solution.

Application to some nonlinear beam equations
In 2006, P. Pablo A. and Pedro Pablo Cardenas A. prove in [7] that the problem

$$\begin{aligned} &u''(s) + h(s, u(s), u'(s)) = 0, 0 < s < \xi, \\ &\quad u'(0) = -l(u(0)), u'(T) = l(u(\xi)) \end{aligned} \tag{12}$$

with $h: [0, \xi] \times \mathbb{R}^2 \to \mathbb{R}$, and $l: \mathbb{R} \to \mathbb{R}$ continuous, admits at least one solution, if h satisfies a Nagumo type condition and there exists an ordered couple of a lower and an upper solution of (12). The following result gives us some existence results without Nagumo type condition, when h is a L^1 −Carathéodory function and no ordering is assumed between the lower and upper solutions**.**

Theorem 4.1. Suppose that:
a) There exists $g \in L^1(0, \xi)$ such that for a.e. $s \in [0, \xi]$ and $\forall (w, z) \in \mathbb{R}^2$,
$h(s, w, z) \geq g(s)$ $(resp\ h(s, w, z) \leq g(s))$.
b) There exists $\vartheta, \in \mathbb{R}$ such that $\forall s \in [0, \xi]$, $l(s) \geq \vartheta$ and (resp $l(s) \leq \vartheta$).
c) The problem (12) admits a lower-solution α and an upper-solution β.

Then the problem (12) admits at least one solution.

Proof: Application of Theorem 3.1. (resp Theorem 3.2.) with $k = -h$, $l_0 = -l$, $l_\xi = l$ and $\phi(x) = x$.

5. Existence result for problem (3) −(4).
In this section we study the problem (3) −(4), where $\Phi: Dom(\Phi) \subset \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing on $[a, b] \subset Dom(\Phi)$, $l_0, l_\xi: \mathbb{R} \to \mathbb{R}$ are two continuous functions, and $k: [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ a L^1 −Carathéodory function.

Let $\theta: \mathbb{R} \to \mathbb{R}$ given by $\theta(s) = \begin{cases} a \ if\ s < a \\ s \ if\ a \leq s \leq b \\ b \ if\ s > b \end{cases}$

Definition 5.1. $\alpha \in C^1([0, \xi])$ is a lower-solution of the problem (3) −(4) if: $\alpha'([0, \xi]) \subset Dom(\Phi)$, $\Phi(\alpha') \in AC([0, \xi])$ and

$$\left(\Phi\left(\alpha'(s)\right)\right)' \geq f\left(s, \alpha(s), \alpha'(s)\right), a.e. s \in [0, \xi],$$

$$\Phi\left(\alpha'(0)\right) \geq l_0\left(\alpha(0)\right) \ and\ \Phi\left(\alpha'(\xi)\right) \leq l_\xi\left(\alpha(\xi)\right).$$

Definition 5.2. $\beta \in C^1([0, \xi])$ is an upper-solution of the problem (3) −(4) if: $\beta'([0, \xi]) \subset Dom(\Phi)$, $\Phi(\beta') \in AC([0, \xi])$ and

$$\left(\Phi\left(\beta'(s)\right)\right)' \leq k\left(s, \beta(s), \beta'(s)\right), a.e. s \in [0, \xi],$$

$$\Phi\left(\beta'(0)\right) \leq l_0\left(\beta(0)\right) \ and\ \Phi\left(\beta'(\xi)\right) \geq l_\xi\left(\beta(\xi)\right).$$

Theorem 3.1. Assume that:
1. There exist a lower-solution α and an upper-solution β of (3) −(4) such that $\forall\ s \in [0, T], a \leq \alpha'(s) \leq b, a \leq \beta'(s) \leq b\ and\ \alpha(s) \leq \beta(s)$;
2. There exists $g \in L^1(0, \xi)$ such that, for a.e. $s \in [0, \xi]$, and all (w, z) with $(s, w, z) \in \{(s, w, z) \in [0, \xi] \times \mathbb{R}^2, \alpha(s) \leq w \leq \beta(s), a \leq z \leq b\}$, $k(s, w, z) \leq g(s)$;
3. $\max\limits_{[\alpha(0), \beta(0)]} l_0 + \|g\|_{L^1(0,\xi)} \leq \Phi(\mathrm{b})$ and $\min\limits_{[\alpha(\xi), \beta(\xi)]} l_\xi - \|g\|_{L^1(0,\xi)} \geq \Phi(\mathrm{a})$.

Then, the problem (3) −(4) admits at least one solution U, with

$$\alpha(s) \leq U(s) \leq \beta(s)\ and\ a \leq U'(s) \leq b, \forall\ s \in [0, \xi].$$

Proof: We have three possible cases: $0 \in [a, b]$, $a > 0$ and $b < 0$.

Let $\vartheta_0 = \max\limits_{[\alpha(0), \beta(0)]} l_0$ and $\vartheta_T = \min\limits_{[\alpha(\xi), \beta(\xi)]} l_\xi$.

Case 1: $0 \in [a, b]$.

Let $p \in \mathbb{R}$ be such that $\Phi(0) + p = 0$. Let $a' = \max\{|a|, |b|\} + 1$.

Let $\Lambda:]-\mathrm{a}', \mathrm{a}'[\rightarrow \mathbb{R}$ given by $\Lambda(s) = \begin{cases} \Phi(\mathrm{a}) - \frac{1}{\sqrt{\mathrm{a}'+\mathrm{s}}} + \frac{1}{\sqrt{\mathrm{a}'+\mathrm{a}}} + \mathrm{p}\ if\ -\mathrm{a}' < \mathrm{s} < \mathrm{a} \\ \Phi(\mathrm{s}) + p\ if\ a \leq s \leq b \\ \Phi(\mathrm{b}) + \frac{1}{\sqrt{\mathrm{a}'-\mathrm{s}}} - \frac{1}{\sqrt{\mathrm{a}'-\mathrm{b}}} + \mathrm{p}\ if\ \mathrm{b} < \mathrm{s} < \mathrm{a}' \end{cases}$

Λ is an increasing homeomorphism such that $\Lambda\,(0) = 0$. Consider the functions

$G_0: \mathbb{R} \rightarrow \mathbb{R}$ *and* $G_\xi: \mathbb{R} \rightarrow \mathbb{R}$ given by $G_0(s) = l_0(s) + p$ *and* $G_\xi(s) = l_\xi(s) + p$.

We introduce the problem

$$\begin{aligned} &\left(\Lambda\left(v'(s)\right)\right)' = k\left(s, v(s), \theta\left(v'(s)\right)\right), a.e. s \in [0,\xi] \\ &\Lambda(v'(0)) = G_0(v(0)), \Lambda(v'(\xi)) = G_\xi(v(\xi)), \end{aligned} \tag{13}$$

We have:

$$\begin{aligned} &\left(\Lambda(\alpha'(\mathrm{s}))\right)' = \left(\Phi(\alpha'(\mathrm{s}))\right)' \geq \mathrm{k}\big(\mathrm{s}, \alpha(\mathrm{s}), \alpha'(\mathrm{s})\big), \mathrm{a.e.s} \in [0,\xi]; \\ &\quad \Lambda(\alpha'(0)) = \Phi(\alpha'(0)) + \mathrm{p} \geq l_0(\alpha(0)) + \mathrm{p} = G_0(\alpha(0)) \\ &\quad \Lambda(\alpha'(\xi)) = \Phi(\alpha'(\xi)) + \mathrm{p} \leq l_\xi(\alpha(\xi)) + \mathrm{p} = G_\xi(\alpha(\xi)) \end{aligned}$$

and

$$\begin{aligned} &\left(\Lambda(\beta'(\mathrm{s}))\right)' = \left(\Phi(\beta'(\mathrm{s}))\right)' \leq \mathrm{f}\big(\mathrm{s}, \beta(\mathrm{s}), \beta'(\mathrm{s})\big), \mathrm{a.e.s} \in [0,\xi]; \\ &\quad \Lambda(\beta'(0)) = \Phi(\beta'(0)) + \mathrm{p} \leq l_0(\beta(0)) + \mathrm{p} = G_0(\beta(0)) \quad . \\ &\quad \Lambda(\beta'(\xi)) = \Phi(\beta'(\xi)) + \mathrm{p} \geq l_\xi(\beta(\mathrm{T})) + \mathrm{p} = G_\xi(\beta(\xi)) \end{aligned}$$

Hence α is a lower-solution and β an upper-solution of problem (13) such that $\forall\, s \in [0,\xi], \alpha(s) \leq \beta(s)$. Using Theorem 2.1., there is at least one solution U, with $\alpha(s) \leq U(s) \leq \beta(s), \forall\, s \in [0,\xi]$. Therefore we have,

$$\forall\, s \in [0,\xi], \Lambda\big(U'(s)\big) = G_0\big(U(0)\big) + \int_0^s f\left(y, U(y), \theta\big(U'(y)\big)\right) dy$$

$$\leq \vartheta_0 + p + \int_0^s \mathrm{h(y)dy} \leq \vartheta_0 + p + \|g\|_{L^1(0,\xi)} \leq \Phi(\mathrm{b}) + \mathrm{p},$$

and $\Lambda\big(U'(s)\big) = G_T\big(U(\xi)\big) - \int_s^\xi k\left(y, U(y), \theta\big(U'(y)\big)\right) ds$

$$\geq \vartheta_\xi + p - \int_s^\xi \mathrm{h(y)dy} \geq \vartheta_T + p - \|g\|_{L^1(0,\xi)} \geq \Phi(\mathrm{a}) + \mathrm{p},$$

Hence, $\forall s \in [0,\xi], \Lambda(\mathrm{a}) \leq \Lambda(\mathrm{U}'(\mathrm{s})) \leq \Lambda(\mathrm{b})$. Moreover $\forall\, s \in [0,\xi], \mathrm{a} \leq \mathrm{U}'(\mathrm{s}) \leq \mathrm{b}$.

It follows that $\forall\, s \in [0,\xi], \Lambda(\mathrm{U}'(\mathrm{s})) = \Phi(\mathrm{U}'(\mathrm{s})) + \mathrm{p}$ and $\theta(U'(s)) = U'(s)$, hence U is also a solution of problem (3) $-$(4).

***Case 2*:** $a > 0$.

Let $q \in \mathbb{R}$ be such that $\Phi(a) + q > 0$. Let $a' = b + 1$.

Let $\Gamma:]-\mathrm{a}', \mathrm{a}'[\rightarrow \mathbb{R}$ given by $\Gamma(s) = \begin{cases} -\frac{1}{\sqrt{\mathrm{a}'+\mathrm{s}}} + 1 - \frac{\mathrm{b}(\Phi(\mathrm{a})+\mathrm{q})}{a}\ \text{if}\ -\mathrm{a}' < \mathrm{s} < -\mathrm{b} \\ \frac{(\Phi(\mathrm{a})+\mathrm{q})s}{a}\ \text{if}\ -\mathrm{b} \leq \mathrm{s} < \mathrm{a} \\ \Phi(\mathrm{s}) + q\ if\ a \leq s \leq b \\ \Phi(\mathrm{b}) + \frac{1}{\sqrt{\mathrm{a}'-\mathrm{s}}} - 1 + \mathrm{q}\ if\ \mathrm{b} < \mathrm{s} < \mathrm{a}' \end{cases}.$

Γ is an increasing homeomorphism such that $\Gamma(0) = 0$. Consider the functions

$G_0: \mathbb{R} \rightarrow \mathbb{R}$ *and* $G_\xi: \mathbb{R} \rightarrow \mathbb{R}$ given by $G_0(s) = l_0(s) + q$ *and* $G_\xi(s) = l_\xi(s) + q$.

Consider the problem

$$\begin{aligned}&\left(\Gamma\left(v'(s)\right)\right)' = k\left(s,v(s),\theta\left(v'(s)\right)\right), a.e. s\in[0,\xi]\\ &\Gamma(v'(0))=G_0\left(v(0)\right), \Gamma(v'(\xi))=G_\xi(v(\xi)),\end{aligned} \quad (14)$$

As in the proof of previous case, we can prove that α is a lower-solution and β an upper-solution of problem (14) such that $\forall\, s \in [0,\xi]$, $\alpha(s) \leq \beta(s)$. Using Theorem (2.1), there is at least one solution V, with $\alpha(s) \leq V(s) \leq \beta(s), \forall s \in [0,\xi]$.
We have,

$$\forall\, s \in [0,\xi], \Gamma\left(V'(s)\right) = G_0\left(V(0)\right) + \int_0^s k\left(y, V(y), \theta\left(V'(y)\right)\right) dy$$

$$\leq \vartheta_0 + q + \int_0^s \mathrm{h(y)dy} \leq \vartheta_0 + q + \|g\|_{L^1(0,T)} \leq \Phi(\mathrm{b}) + \mathrm{q},$$

and $\Gamma\left(V'(t)\right) = G_T\left(V(\xi)\right) - \int_s^\xi k\left(y, V(y), \theta\left(V'(y)\right)\right) dy$

$$\geq \vartheta_\xi + q - \int_s^\xi \mathrm{h(y)dy} \geq \vartheta_\xi + q - \|g\|_{L^1(0,\xi)} \geq \Phi(\mathrm{a}) + \mathrm{q},$$

Hence, $\forall\, s \in [0,\xi], \Gamma(\mathrm{a}) \leq \Gamma(\mathrm{V}'(\mathrm{s})) \leq \Gamma(\mathrm{b})$. Moreover $\forall\, s \in [0,\xi], \mathrm{a} \leq \mathrm{V}'(\mathrm{s}) \leq \mathrm{b}$.
It follows that $\forall\, s \in [0,\xi], \Gamma(\mathrm{V}'(\mathrm{s})) = \Phi(\mathrm{V}'(\mathrm{s})) + \mathrm{q}$ and $\theta(V'(s)) = V'(s)$, hence V is also a solution of problem (3) $-$(4).

***Case 3*:** $b < 0$.

Let $r \in \mathbb{R}$ be such that $\Phi(b) + r < 0$. Let $a' = -a + 1$.

Let $\Psi:]-\mathrm{a}', \mathrm{a}'[\to \mathbb{R}$ given by $\Psi(s) = \begin{cases} \Phi(\mathrm{a}) - \frac{1}{\sqrt{\mathrm{a}'+\mathrm{s}}} + 1 + r \text{ if } -\mathrm{a}' < \mathrm{s} < \mathrm{a} \\ \Phi(\mathrm{s}) + r \; if \; a \leq s \leq b \\ \frac{(\Phi(\mathrm{b})+\mathrm{r})s}{b} \; if \; b \leq s < -a \\ \frac{1}{\sqrt{\mathrm{a}'-\mathrm{s}}} - 1 - \frac{\mathrm{a}(\Phi(\mathrm{b})+\mathrm{s})}{b} \; if \; -\mathrm{a} < \mathrm{s} < \mathrm{a}' \end{cases}$.

Ψ is an increasing homeomorphism such that $\Psi(0) = 0$. Consider the functions

$G_0: \mathbb{R} \to \mathbb{R}$ *and* $G_\xi: \mathbb{R} \to \mathbb{R}$ given by $G_0(s) = l_0(s) + r$ *and* $G_\xi(s) = l_\xi(s) + r$.

Consider the problem

$$\begin{aligned}&\left(\Psi\left(v'(s)\right)\right)' = k\left(s, v(s), \theta\left(v'(s)\right)\right), a.e. s \in [0,\xi]\\ &\Psi\left(v'(0)\right) = G_0\left(v(0)\right), \Psi(v'(\xi)) = G_\xi(v(\xi)),\end{aligned} \quad (15)$$

As in the proof of the case 1, we can prove that α is a lower-solution and β an upper-solution of problem (15) such that $\forall s \in [0,\xi]$, $\alpha(s) \leq \beta(s)$. By Theorem (2.1), there is at least one solution W, with $\alpha(s) \leq \mathrm{W}(s) \leq \beta(s), \forall\, s \in [0,\xi]$. We have:

$$\forall\, s \in [0,\xi], \Psi\left(W'(s)\right) = G_0\left(W(0)\right) + \int_0^s k\left(y, W(y), \theta\left(W'(y)\right)\right) dy$$

$$\leq \vartheta_0 + r + \int_0^s \mathrm{h(y)dy} \leq \vartheta_0 + r + \|g\|_{L^1(0,\xi)} \leq \Phi(\mathrm{b}) + \mathrm{r},$$

and $\Psi\left(W'(s)\right) = G_\xi\left(W(T)\right) - \int_s^\xi k\left(y, W(y), \theta\left(W'(y)\right)\right) dy$

$$\geq \vartheta_T + r - \int_s^\xi \mathrm{g(y)dy} \geq \vartheta_T + r - \|g\|_{L^1(0,\xi)} \geq \Phi(\mathrm{a}) + \mathrm{r},$$

Hence, $\forall\, s \in [0,\xi], \Psi(\mathrm{a}) \leq \Psi(W'(\mathrm{s})) \leq \Psi(\mathrm{b})$. Moreover $\forall\, s \in [0,\xi], \mathrm{a} \leq \mathrm{V}'(\mathrm{s}) \leq \mathrm{b}$.

It follows that $\forall\, s \in [0,\xi], \Psi(W'(\mathrm{s})) = \Phi(W'(\mathrm{s})) + \mathrm{r}$ and $\theta(W'(s)) = W'(s)$, hence W is also a solution of problem (3) $-$(4).

Theorem 5.2. Assume that:
1) There exist a lower-solution α and an upper-solution β of (3) $-$(4) such that $\forall\, s \in [0,\xi], a \leq \alpha'(s) \leq b, a \leq \beta'(s) \leq b$ *and* $\alpha(s) \leq \beta(s)$;

2) There exists $g \in L^1(0,\xi)$ such that, for a.e. $s \in [0,\xi]$, and all (w,z) with $(s,w,z) \in \{(s,w,z) \in [0,\xi] \times \mathbb{R}^2, \alpha(s) \le u \le \beta(s), a \le v \le b\}$, $k(s,w,z) \ge g(s)$;
3) $\min_{[\alpha(0),\beta(0)]} l_0 - \|h\|_{L^1(0,\xi)} \ge \Phi(\mathrm{a})$ and $\max_{[\alpha(\xi),\beta(\xi)]} l_\xi + \|h\|_{L^1(0,\xi)} \le \Phi(\mathrm{b})$.

Then, the problem (3) $-$(4) admits at least one solution U, with

$\alpha(s) \le U(s) \le \beta(s)$ *and* $a \le U'(s) \le b, \forall s \in [0,\xi]$.

Proof: The proof is similar to the proof of Theorem 5.1.

Example 5 .1. Consider the problem
$$\begin{gathered}\left(\sin\left(\mathrm{v}'(\mathrm{s})\right)\right)' = \frac{|\mathrm{v}'(\mathrm{s})|}{8\sqrt{s}} + \mathrm{v}(\mathrm{s}) + \frac{s}{3} \text{ a.e. } \mathrm{s} \in [0,1] \\ \sin\left(\mathrm{v}'(0)\right) = \left(\mathrm{v}(0)\right)^3 \text{ and } \sin(\mathrm{v}'(1)) = -\mathrm{v}(1)\mathrm{e}^{\mathrm{v}(1)}\end{gathered}$$

It is easy to see that $\alpha(s) = -\frac{1}{2}$ *and* $\beta(s) = 0$ are respectively lower and upper solutions. Taking $g(s) = \frac{1}{3} + \frac{\pi}{16\sqrt{s}}$, $a = -\frac{\pi}{2}$ and $b = \frac{\pi}{2}$, from Theorem 5.1, we deduce the existence of a solution.

Example 5 .2. Consider the problem
$$\begin{gathered}\left(\left(\mathrm{v}'(\mathrm{s})\right)^2\right)' = \frac{|\mathrm{v}'(\mathrm{s})|}{8\sqrt{\mathrm{s}}} + \mathrm{v}(\mathrm{s}) - \frac{s}{3} \text{ a.e. } \mathrm{s} \in [0,1] \\ (\mathrm{v}'(0))^2 = 4(\mathrm{v}(0))^3 \text{ and } (\mathrm{v}'(1))^2 = |\mathrm{v}(1) - 3| + 5\end{gathered}$$

It is easy to see that $\alpha(s) = 0$ and $\beta(s) = 2s + 1$ are respectively lower and upper solutions. Taking $g(s) = 3 + \frac{1}{2\sqrt{s}}$, $a = 0$ and $b = 3$, from Theorem 5.1, we deduce the existence of a solution.

Example 5 .3. Consider the problem
$$\begin{gathered}-\left(\frac{1}{\mathrm{v}'(\mathrm{s})}\right)' = \frac{1}{24\left(\left(\mathrm{v}'(\mathrm{s})\right)^2+1\right)} + \frac{\mathrm{v}(\mathrm{s})}{36\sqrt{s}} - \frac{\sqrt{s}}{12} \text{ a.e. } \mathrm{s} \in [0,1] \\ -\left(\frac{1}{\mathrm{v}'(0)}\right)' = -\frac{1}{3}e^{-\mathrm{v}(0)} \text{ and } -\left(\frac{1}{\mathrm{v}'(1)}\right)' = -\mathrm{v}(1) + \frac{3}{2}\end{gathered}$$

It is easy to see that $\alpha(s) = 3s - \frac{3}{2}$ and $\beta(s) = 3s$ are respectively lower and upper solutions. Taking $(s) = \frac{1}{24} + \frac{1}{12\sqrt{s}}$, $a = \frac{1}{2}$ and $b = 9$, from Theorem 5.1, we deduce the existence of at least one solution.

Bibliography.

1. Cristian B, Jean M. Nonlinear Neumann boundary-value problems with ϕ-Laplacian operators. An. Stiint. Univ. Ovidius Constanta. 2004;12:73-92.
2. Cristian B, Jean M. Existence and multiplicity results for some nonlinear problems with singular ϕ-Laplacian. J Differential Equations. 2007;243:536-57.
3. Cristian B, Jean M. Nonhomogeneous boundary value problems for some nonlinear equations with singular ϕ-Laplacian. J Math Anal Appl. 2009;352:218-33.
4. Gaetano V. Soluzioni periodiche di una classe di equazioni differenziali del terz'ordine. Ann Mat Pura Appl. 1966;73:103-10.
5. Charles Etienne G, Assohoun A. Solvability for some boundary value problems with ϕ-Laplacian operators and general nonlinear boundary conditions. Far East J Math Sci. 2015;98(4):445-76.
6. Konan Charles Etienne G, Assohoun A. Existence of solutions of some nonlinear ϕ-Laplacian equations with Neumann-Steklov nonlinear boundary conditions. African Diaspora J Math. 2017;20(2):16-38.
7. Pablo A, Pablo Cardenas A. Existence of solutions for some nonlinear beam equations. Portugaliae Mathematica. 2006;63(1).
8. Jean M. Topological degree methods in nonlinear boundary value problems. CBMS Series, vol. 40. Providence, RI: American Mathematical Society; 1979.
9. De C, Patrick H. Two-point boundary value problems: lower and upper solutions. Amsterdam: Elsevier; 2006.