International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
NAAS Rating (2025): 4.49
Maths 2025; 10(11): 07-11

www.mathsjournal.com
Received: 12-09-2025
Accepted: 15-11-2025

Pramod Kumar Rajak
Department of Mathematics, Rabindranath Tagore University, Bhopal, Madhya Pradesh, India

Bhawna Agrawal
Department of Mathematics, Rabindranath Tagore University, Bhopal, Madhya Pradesh, India

Sanjit Kumar
Department of Mathematics, Lakshmi Narain College of Technology & Science, Bhopal, Madhya Pradesh, India

Corresponding Author:
Pramod Kumar Rajak
Department of Mathematics, Rabindranath Tagore University, Bhopal, Madhya Pradesh, India

Coupled fixed point theorems for mappings satisfies contractive condition in cone metric space

Pramod Kumar Rajak, Bhawna Agrawal and Sanjit Kumar

DOI: https://www.doi.org/10.22271/maths.2025.v10.i11a.2191

Abstract
In the past few years, there are several authors have come up with coupled fixed point theorems in cone metric space. The purpose of this paper to prove the existence of a coupled fixed point of some type of contraction mappings defined on a complete cone metric space. It extends and generalizes many previous coupled fixed point theorems.

Keywords: Metric space, cone metric space, coupled fixed point, mixed monotone property

1. Introduction
Fixed point theory is a well-known and significant area in mathematics, with a wide range of applications. In 2007, Huang and Zhang [9] introduced the concept of cone metric spaces as a generalization of traditional metric spaces. They proved the existence of a unique fixed point for contractive mappings in complete cone metric spaces. Dajun Guo and V. Lakshmikantham [2] established existence theorems for coupled fixed points for both continuous and discontinuous operators, with applications to initial value problems of ordinary differential equations with discontinuous right-hand sides. Bhaskar and Lakshmikantham [19] further developed the theory by proving the existence of coupled fixed points for mixed monotone mappings in partially ordered metric spaces.
In 2008, C. Di Bari [1] presented a common fixed point theorem in cone metric spaces. Later, in 2009 and 2010, I. Altun [6, 7] established several common fixed point theorems in cone metric spaces and ordered cone metric spaces. Additionally, M. Arshad [13] and S. Radenović [16], both in 2009, contributed further by proving common fixed point theorems in cone metric spaces.

Preliminaries
Definition 2.1: [8] Let E be a real Banach Spaces. A subset P of E is called a cone if and only if
A). P is closed, non-empty and $p \neq 0$
$B)$ $a, b \in R, a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$
$C)$ $P \cap (-P) = \{0\}$

Given a cone $P \subset E$ we define the partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to denote that $x \leq y$ but $\neq y$, while $x << y$ will stand for $y - x \in int. P$.

Definition 2.2: [9] Let X be a nonempty set. Suppose the mapping $d: X \times X \to E$ satiafies the following condition:
- $0 < \mathrm{d(x, y)}\ \forall\ \mathrm{x, y} \in \mathrm{X}$ and $\mathrm{d(x, y)} = 0 \Leftrightarrow \mathrm{x} = \mathrm{y}$
- $\mathrm{d(x, y)} = \mathrm{d(y, x)}, \forall\ \mathrm{x, y} \in \mathrm{X}$
- $\mathrm{d(x, y)} \leq \mathrm{d(x, y)} + \mathrm{d(x, y)}, \forall\ \mathrm{x, y} \in \mathrm{X}$

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.3:[9] Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X, $\{x_n\}$ is a convergent sequence if there is some $k \in \mathbb{N}$ such that, for all $n \geq k$,

$d(x_n, x) \ll c$;

Then x is called limit of the sequence $\{x_n\}$

Definition 2.4:[9] Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X, $\{x_n\}$ is a Cauchy sequence if there is some $k \in \mathbb{N}$ such that, for all $n, m \geq k$,

$d(x_n, x_m) \ll c$;

Note that:
(i) Every convergent sequence in a cone metric space X is a Cauchy sequence.
(ii) A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X

Bhashkar and Lakshmikantham in [19] introduced the concept of coupled fixed point of a mapping $F: X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered sets. They also discussed an application of their result by investigating the existence and uniqueness of solution for a periodic boundary value problem. Sabetghadam *et al.* in [5] introduced this concept in cone metric spaces.

Definition 2.5: [5] Let $F: X \times X \to X$ be mapping, an element $(x, y) \in X \times X$ is called a coupled fixed point of mapping F if $x = F(x, y)$ and $y = F(y, x)$

3. Main Results
In this theorem, we extend and unify several well-known comparable results in the literature and results of M. Abbas *et al.* [12].

Theorem 3.1: Let X a non-empty set, (X, d) be a Cone metric space with cone P having non empty interior, $F, G: X \times X \to X$ be mapping satisfying the following conditions:

$$d\big(G(x, y), F(u, v)\big) \leq \alpha\{d(x, u) + d(y, v)\} + \beta\,\{d(u, F(u, v)) + d(u, G(x, y))\} + \gamma\{d(x, G(x, y))\}$$

$\forall\, x, y, u, v \in X$ where $\alpha, \beta, \gamma\, \epsilon (0, \frac{1}{2}]$such that $0 < h = \frac{\alpha}{(1-\beta-\gamma)} < 1$ and Then F and G has a coupled fixed point in X.

Proof: Let x_0 and y_0 be arbitrary points in X. Let
$x_{k+1} = F(x_k, y_k)\; y_{k+1} = F(y_k, x_k)$
$x_{k+2} = G(x_{k+1}, y_{k+1})\; y_{k+2} = G(y_{k+1}, x_{k+1})$

Now,
$d(x_{k+1}, x_{k+2}) = d(F(x_k, y_k), G(x_{k+1}, y_{k+1}))$

$\leq \alpha\{d(x_{k-1}, x_k) + d(y_{k-1}, y_k)\} + \beta\,\{d(x_k, F(x_k, y_k) + d(x_k, G(x_{k+1}, y_{k+1})\} + \gamma\{d(x_{k-1}, G(x_{k+1}, y_{k+1})\}$

$\leq \alpha\{d(x_{k+1}, x_k) + d(y_{k+1}, y_k)\} + \beta\,\{d(x_k, x_{k+1}) + d(x_k, x_{k+2})\} + \gamma\{d(x_{k+1}, x_{k+2}\}$

$\leq \alpha\{d(x_{k+1}, x_k) + d(y_{k+1}, y_k)\} + \beta\,\{d(x_{k+1}, x_k) + d(x_k, x_{k+2})\} + \gamma\{d(x_{k+1}, x_{k+2}\}$

$\leq \alpha\{d(x_{k+1}, x_k) + d(y_{k+1}, y_k)\} + \beta\,\{d(x_{k+1}, x_{k+2})\} + \gamma\{d(x_{k+1}, x_{k+2}\}$
$(1 - \beta - \gamma)\, d(x_{k+1}, x_{k+2}) \leq \alpha\{d(x_{k+1}, x_k) + d(y_{k+1}, y_k)\}$

$$d(x_{k+1}, x_{k+2}) \leq \frac{\alpha}{(1-\beta-\gamma)}\{d(x_k, x_{k+1}) + d(y_k, y_{k+1})\} \ldots (1)$$

$d(y_{k+1}, y_{k+2}) = d(g(y_k), g(y_{k+1})) = d(F(y_k, x_k), G(y_{k+1}, x_{k+1}))$

$\leq \alpha\{d(y_{k+1}, y_k) + d(x_{k+1}, x_k)\} + \beta\,\{d\big(y_k, F(y_k, x_k)\big) + d\big(y_k, G(y_{k+1}, x_{k+1})\big)\} + \gamma\{d(y_{k+1}, G(y_{k+1}, x_{k+1}))\}$

$\leq \alpha\{d(y_{k+1}, y_k) + d(x_{k+1}, x_k)\} + \beta\,\{d(y_k, y_{k+1}) + d(y_k, y_{k+2})\} + \gamma\{d(y_{k+1}, y_{k+1})\}$

$\leq \alpha\{d(y_{k+1}, y_k) + d(x_{k+1}, x_k)\} + \beta\,\{d(y_{k+1}, y_k) + d(y_k, y_{k+2})\} + \gamma\{d(y_{k+1}, y_{k+1})\}$

$\leq \alpha\{d(y_{k+1}, y_k) + d(x_{k+1}, x_k)\} + \beta\,\{d(y_{k+1}, y_{k+2})\} + \gamma\{d(y_{k+1}, y_{k+1})\}$

$d(y_{k+1}, y_{k+2}) - \beta\{d(y_{k+1}, y_{k+2})\} - \gamma\{d(y_{k+1}, y_{k+2})\} \leq \alpha\{d(y_k, y_{k+1}) + d(x_k, x_{k+1})\}$

$(1-\beta-\gamma)\, d(y_{k+1}, y_{k+2}) \leq \alpha\{d(y_k, y_{k+1}) + d(x_k, x_{k+1})\}$

$d(y_{k+1}, y_{k+2}) \leq \frac{\alpha}{(1-\beta-\gamma)}\{d(y_k, y_{k+1}) + d(x_k, x_{k+1})\} \ldots (2)$

Now adding (1) and (2) we have

$d(x_{k+1}, x_{k+2}) + d(y_{k+1}, y_{k+2}) \leq \frac{2\alpha}{(1-\beta-\gamma)}\{d(x_k, x_{k+1}) + d(y_k, y_{k+1})\}$

Similarly,

$d(x_{k+2}, x_{k+3}) + d(y_{k+2}, y_{k+3}) \leq \frac{2\alpha}{(1-\beta-\gamma)}\{d(x_{k+1}, x_{k+2}) + d(y_{k+1}, y_{k+2})\}$

And so on. Therefore

$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq \frac{2\alpha}{(1-\beta-\gamma)}\{d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\}$

$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq h\{d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\}$

Where $0 < h = \frac{2\alpha}{(1-\beta-\gamma)} < 1$

$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq h^2\{d(x_{n-2}, x_{n-1}) + d(y_{n-2}, y_{n-1})\}$

$\leq \ldots\ldots\ \ldots\ldots\ldots\ldots$

$\leq h^n\{d(x_0, x_1) + d(y_0, y_1)\}$

Now if $d(x_n, x_{n+1}) + d(y_n, y_{n+1}) = \delta_n$

Then,

$\delta_n \leq h\delta_{n-1} \leq h^2\delta_{n-2} \leq \cdots \ldots\ldots\ldots\ldots \leq h^n\delta_0$ for $m > n$

$d(x_n, x_m) + d(y_n, y_m) \leq \delta_{m-1} + \delta_{m-2} + \ldots\ldots\ldots\ldots\ldots + \delta_n$

$\leq \delta_0(h^{m-1} + h^{m-2} + \ldots\ldots\ldots + h^n)$

$\leq \delta_0(1 + h + h^2 + h^3 + \ldots\ldots\ldots + h^{m-n-1})h^n$

$\leq \frac{h^n\{1 - h^{m-n}\}\delta_0}{1-h}$

$d(x_n, x_m) + d(y_n, y_m) \leq \frac{\{h^n - h^m\}\delta_0}{1-h}$

$d(x_n, x_m) + d(y_n, y_m) \leq \frac{h^n}{1-h}\,\delta_0 \to 0\ as\ n \to \infty$

It follows that for $0 << c$ and for large n, we have

$\frac{h^n}{1-h}\delta_0 << c$

$d(x_n, x_m) + d(y_n, y_m) << c$

Hence by definition $\{\, d(x_n, x_m) + d(y_n, y_m)\}$ is Cauchy sequence.

Since,

$d(x_n, x_m) \leq d(x_n, x_m) + d(y_n, y_m)$

And

$d(y_n, y_m) \leq d(x_n, x_m) + d(y_n, y_m)$

Again $\{x_n\}$ and $\{y_n\}$ are cauchy sequence in x so $\exists\ x, y\ \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.
Now we have to show that $x = F(x, y)$ and $y = F(y, x)$, on the contrary let we assume that $x \neq F(x, y)$ and $y \neq F(y, x)$
So that,

$d(F(x, y), x) = k > 0$ and $d\big(y, F(y, x)\big) = l > 0$

Consider,

$\alpha_1 = d(F(x, y), x)$

$\leq d(F(x, y), x_{k+2}) + d(x_{k+2}, x)$

$\leq d(x_{k+2}, x) + d(F(x, y), x_{k+2})$

$\leq d(x_{k+2}, x) + d(G(x_{k+1}, y_{k+1}), F(x, y))$

$\leq d(x_{k+2}, x) + \alpha\{d(x_{k+1}, x) + d(y_{k+1}, y)\} + \beta\left\{d\big(x, F(x, y)\big) + d\big(x, G(x_{k+1}, y_{k+1})\big)\right\} + \gamma\{d(x_{k+1}, G(x_{k+1}, y_{k+1}))\}$

$\leq d(x_{k+2}, x) + \alpha\{d(x_{k+1}, x) + d(y_{k+1}, y)\} + \beta\left\{d\big(x, F(x, y)\big) + d(x, x_{k+2})\right\} + \gamma\{d(x_{k+1}, x_{k+2})\}$

As $k \to \infty$

$\leq 0 + \alpha.0 + \beta d\big(x, F(x, y)\big) + \gamma\{0\}$

$\alpha_1 = (1 - \beta)d(F(x, y), x) \leq 0$

Which is a contradiction.

Therefore $d(F(x, y), x) = 0$

$x = F(x, y)$

Similarly we can prove that $y = F(y, x)$
It follows that $x = G(x, y)$ and $y = G(y, x)$
So we have proved that (x, y) is common coupled fixed point of F and G.

Corollary: Let X a non-empty set, (X, d) be a Cone metric space with cone P having non empty interior, $F, G: X \times X \to X$ be mapping satisfying the following conditions:

$d\big(G(x, y), G(u, v)\big) \leq \alpha\{d(x, u) + d(y, v)\} + \beta\{d(u, G(u, v)) + d(u, G(x, y))\} + \gamma\{d(x, G(x, y))\}$

$\forall\ x, y, u, v \in X$ where $\alpha, \beta, \gamma\ \epsilon (0, \frac{1}{2}]$such that $0 < h = \frac{\alpha}{(1-\beta-\gamma)} < 1$ and Then F and G has a coupled fixed point in X.

References

1. Di Bari C, Vetro P. U-pairs and common fixed points in cone metric spaces. Rend Circ Mat Palermo. 2008;57:279-85.
2. Guo D, Lakshmikantham V. Coupled fixed points of nonlinear operation with applications. Nonlinear Anal Theory Methods Appl. 1987;11:623-32.
3. Ilic D, Rakocevic V. Common fixed points for maps on cone metric space. J Math Anal Appl. 2008;341:876-82.
4. Ilic D, Rakocevic V. Quasi-contraction on cone metric space. Appl Math Lett. 2009;22:728-31.
5. Sabetghadam F, Masiha HP, Sanatpour AH. Some coupled fixed point theorems in cone metric space. Fixed Point Theory Appl. 2009;2009:1254260.
6. Altun I, Damjanovic B, Djoric D. Fixed point and common fixed point theorems on ordered cone metric spaces. Appl Math Lett. 2010;23:310-6.
7. Altun I, Durmaz G. Some fixed point results in cone metric spaces. Rend Circ Mat Palermo. 2009;58:319-25.
8. Deimling K. Nonlinear Functional Analysis. Berlin: Springer-Verlag; 1985.

9. Huang LG, Zhang X. Cone metric spaces and fixed point theorems of contractive mappings. J Math Anal Appl. 2007;332:1467-75.
10. Abbas M, Rhoades BE. Fixed and periodic point results in cone metric spaces. Appl Math Lett. 2009;22:511-5.
11. Abbas M, Rhoades BE, Nazir T. Common fixed points for four maps in cone metric spaces. Appl Math Comput. 2010;216:80-6.
12. Abbas M, Khan MA, Radenovic S. Common coupled fixed point theorems in cone metric spaces for w-compatible mappings. Appl Math Comput. 2010;217:195-202.
13. Arshad M, Azam A, Vetro P. Some common fixed point results in cone metric spaces. Fixed Point Theory Appl. 2009;2009:493965.
14. Jankovic S, Kadelburg Z, Radenovic S, Rhoades BE. Assad-Kirk-type fixed point theorems for a pair of nonself mappings on cone metric spaces. Fixed Point Theory Appl. 2009;2009:761086.
15. Radenovic S, Rhoades BE. Fixed point theorem for two non-self mappings in cone metric spaces. Comput Math Appl. 2009;57:1701-7.
16. Radenovic S. Common fixed points under contractive conditions in cone metric spaces. Comput Math Appl. 2009;58(6):1273-8.
17. Rezapour S, Hamlbarani R. Some note on the paper “Cone metric spaces and fixed point theorems of contractive mappings.” J Math Anal Appl. 2008;345:719-24.
18. Rezapour Sh, Haghi RH, Shahzad N. Some notes on fixed points of quasi-contraction maps. Appl Math Lett. 2010;23:498-502.
19. Bhashkar TG, Lakshmikantham V. Fixed point theorems in partially ordered cone metric spaces and applications. Nonlinear Anal Theory Methods Appl. 2006;65(7):825-32.