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Study of W_2 curvarture tensors on Lorentzian para-Kenmotsu manifolds

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Abstract
In this study we consider a class of Lorentzian Para- Kenmotsu manifolds (briefly l.p Kenmotsu). We study w_2 curvarture tensors in relation to w_2-flatness, $w_2 - Q$, $w_2.\emptyset$, $w_2 - \varepsilon$, $w_2 - n$ and other conditions such as special $n - Einstein$ manifold, Einstein manifold and $n - Einstein$ manifold. Additionally, $R(xy).w_2 = 0, w_2.w_2 = 0$ is also put into account

Keywords: Lorentzian para-Kenmotsu manifolds, w_2-Curvature tensors, Einstein manifolds, para-contact manifold

Introduction

K. Matsumoto in 1989 introduced the notion of Lorentzian para contact particularly L.P sasakiani manifolds [1]. Other geometer studied these manifolds widely such as Mihai and Matsumoto, Mihai and Rosca, Mihai, Shaika and de, Venkatesha and Bagewadi, Pradeel Kumar *et al* [2].

In 1970 Pokhariyal and Mishra introduced a new tensor field called w_2 curvature tensor on Riemannian manifold m on Riemannian correction is given by

$$w_2(x,y,z,u) = R(x,y,z,u) + \frac{1}{n-1}[g(x,z)s(y,u) - g(y,z)s(x,u)] \quad (1)$$

For $R(x,y)$ is the Riemannian curvarture tensor,
$s(x,y)$ the Ricci tensor on m
Equation (1) can be written as

$$w_2(xy)z + \frac{1}{n-1}[g(xy)Qy - g(yz)Qx] \quad (2)$$

Where Q =(n-1)
In the same context, Pokhariyal studied the properties of these curvature tensors on sasakian properties [4]. Matsumoto, Mihai and Rosca, extended these concepts to almost paracontact structures and studied p. s manifolds in relations to these tensors fields and the results were further generalized by De and Sarkar in 2009 Sinha and Sai Prasad described a class of almost paracontact metric manifolds referred to as para-Kenmotsu and special para-Kenmotsu (l. p Kenmotsu) manifolds [3].
In 2015 Sai Prasad studied w_2 curvarture tensor in a special- Kenmotsu manifolds [5].

Preliminaries

An (n)-dimensional differentiable manifolds admitting a (1,1) tensor field $\emptyset$ killing vector ε, 1-form η and Lorentzian metric $g(xy)$ satisfying the following condition

$\emptyset^2 x = x(1 + n(1)\varepsilon$............ (3)

$g(\emptyset x, \emptyset y = g$(xy +η(x) ... (y)).............. (4)

And $\eta(\varepsilon) = 1, \emptyset\varepsilon = 0$

$g(x\varepsilon) = \eta(x)$

$\emptyset = \eta - 1$

Is Lorentzian almost paracontact manifolds.

A Lorentzian almost para contact manifold. We have $\emptyset(xy) = \emptyset(yx)$ where $\emptyset(xy) = g(x, \emptyset y)$

A Lorentzian almost paracontact manifold m is called Lorentzian para -Kenmotsu manifold if

$(\nabla_x \emptyset)y = -g(\emptyset\emptyset x, y)\varepsilon - \eta(y)\emptyset x$

For all xy on m and ∇ is the operator of covariant differentiation with respect to the Lorentzian metric (g) [6]
In the L.P.K. The following relations hold

$\nabla_x \varepsilon = \emptyset^2 x = -x - n(x)\varepsilon$

$(\nabla y n)Y = -g(xy) - \eta(x)\eta(y)$

Additionally, on the l. p Kenmotsu manifold the following condition holds

$(\nabla_x \emptyset)y = -g(\emptyset x, y)\varepsilon - n(y)\emptyset x$

$\nabla_x \varepsilon = x + n(x)\varepsilon$

$(\nabla_x n)y = -g(xy) - n(x)n(y)$

$R(\varepsilon x)y = g(xy)\varepsilon - n(y)x$

$R(\varepsilon x)\varepsilon = -\nabla_x \varepsilon$

$\nabla_x \varepsilon = -x - n(x)\varepsilon$

$R(xy)\varepsilon = n(y)x - n(x)y$

$s(x\varepsilon) = (n-1)n(x)$

$Q\varepsilon = (n-1)\varepsilon$

$g(R(xy)z, \varepsilon) = n(R(xy)z$

$n(R(xy)z = g(y,z)n(x) - g(y,z)n(y)$

$s(\emptyset x, \emptyset y) = s(x,y) + (n-1)n(x)n(y)$

for all vector fields x, y, y on M

- S - Ricci tensor
- Q - Ricci operator
- R - Curvature tensor
- ∇ - Levi-Civita connection

A Lorentzian para-Kenmotsu manifold M is said to be an η-Einstein manifold if its Ricci tensor satisfies the relation $s(xy)$ is of the form [10]

$S(xy) = a\, g(xy) + b\, \eta\,(x)\eta(y)$

Where a and b are scalar function on m.
In particular if $b = 0$ then the manifold is said to be an Einstein manifold.

3. A w_2- flat L.P Kenmotsu manifolds

Definition 3.1

An n dimensional L.p Kenmotsu manifold is termed as w_2 flat if its w_2- curvature tensor satisfies the following condition

$$w_2(xy)z = o$$

Suppose the l. p Kenmstu manifold is w_2 flat then the following condition hold

$$w_2(xy)z = o$$

$$w_2(xy)z = R(xy)z + \frac{1}{n-1}(g(xz)Qy) - g(yz)Qx)$$

$$R(xy)z = -g(xz)y + g(yz)x$$

$$g(yz)x - g(xy)z = -g(xz)y + g(yz)x - g(xy)z = g(xzy)$$

But

$$s(xy)z = (n-1)g(xy)z$$

$$g(xy)z = \frac{s(xy)z}{n-1}$$

Therefore

$$s(xy)z = (1-n)g(xz)y$$

Let $z = \varepsilon$

$$s(xy)\varepsilon = (1-n)g(x\varepsilon)y$$

Contracting w, r, t ε

$$s(xy) = -(n-1)n(x)y$$

Theorem: A w_{2-} flat Lorentzian Para- Kenmotsu manifold is a special type of n −Einstein manifold.

4. A $\varepsilon - w_2$ flat LP Kenmotsu manifold

Definition 4.0 An $n-$ dimensional lotrentzian Para- Kenmotsu manifold is said to be $\varepsilon - w_2$ flat if this condition holds

$$w_2(xy)\varepsilon = 0$$

Let

$$w_2(xy)\varepsilon = 0$$

Then

$$w_2(xy)\varepsilon = R(xy)\varepsilon + \frac{1}{n-1})(g(xz)Qy - g(yz)Qx)$$

$$w_2(xy)\varepsilon = R(xy)\varepsilon + \frac{1}{n-1}(g(x\varepsilon)Qy - g(y\varepsilon)Qx)$$

$$R(xy)\varepsilon = -g(x\varepsilon)y + g\frac{(y\varepsilon)Qx}{n-1}$$

$$g(y\varepsilon)x - g(x\varepsilon)y = -g(x\varepsilon)y + g(y\varepsilon)x$$

$$\frac{s(y\varepsilon)x}{n-1} = n(y)x$$

$$s(y\varepsilon)x = (n-1)n(y)x$$

$g(y\varepsilon)x = n(y)x$

$g(y\varepsilon)g(xu) = n(y)g(xu)$

$$\frac{n(y)s(xu)}{n(y)} = \frac{(n-1)g(xu)}{n(y)}$$

$s(xu) = (n-1)g(xy)$

Theorem: A $\varepsilon - w_2$ flat l.p Kenmotsu manifold is an Einstein manifold.

5. $R.w_2$ curvature tensors on Lorentzian Para Kenmotsu manifolds

Definition 5.1 A Lorentzian Para-Kenmotsu manifolds is said to be semi symmetric if it satisfies their condition [8, 7]

$R(xy).R = 0$

$R(xy)$ is considered as the derivation of the algebra at each point of the manifold.
Definition 5.2
A Lorentzian Para-Kenmotsu manifold satisfies the condition $R(xy)w_2 = 0$ [9]
Consider $R(\varepsilon\, x)w_2(uvy) = 0$
Considering $R(xy)$ as the derivation of the tensor algebra at every point of the manifold
x, y, u, v are vector fields

$$R(\varepsilon, x, w_2(u, v, y) - w_2(R(\varepsilon, x, u), v, y) - w_2(u\, R(\varepsilon, x, v)y - w_2\big(u, v, R(\varepsilon, x, y)\big) = 0$$

$$\eta\big(w_2(u, v, y)\big)x - w_2(u, v, y, x)\varepsilon - n(u)w_2(x, v, y) + g(xu)w_2(\varepsilon vy) - n(v)w_2(u, x, y) + g(x, v)w_2(u, \varepsilon, y) - n(y)w_2(u, v, x) + g(xy)w_2(u, v, \varepsilon)$$

Taking the inner product of above equation with ε and using equations

$$w_2(u, v, y, x) = -\frac{u}{(n-1)}[g(x, u)n(v)n(y) - g(x, v)n(u)n(y)] + \left[\frac{n-1+u}{n-1}\right]g(x, u)n(v)n(y) - g(x, u)n(y)n(v) - \left[\frac{n-1+u}{n-1}\right]g(x, v)n(u)n(y) + g(xv)n(y)n(u)$$

But

$$R(u, v, y, x) = \frac{1}{n-1}[g(yu)s(xv) - g(yv)s(xu)]$$

Set: $(i = 1,2 \dots .)$ be on orthonormal basis with $\nabla e_i = 0$ let $x = u = e_i$ in the above equation and taking summation over i we get

$s(yv) = -ng(yv) + n(y)n(v)$

Hence w_2 curvature tensor on LP manifold is on n −Einstein manifolds
Theorem: w_2 curvature tensor on Lorentzian Para- Kenmotsu manifold satisfying the condition $R.w_2 = 0$ is on n −Einstein manifold.

6. w_2 Lorentzian para-Kenmotsu manifold satisfying the condition $w_2R = 0$

Definition 6.1
A L.P-Kenmotsu manifold is said to satisfy the condition $w_2R = 0$

$\forall$ vector field x, y, z, u, v on m [8]

i.e., $w_2(uv).R(xy)z = 0$

Theorem: A w_2 LP-Kenmotsu manifold satisfies the condition $w_2R = 0$

$$w_2(uv).R(xy)z = w_2(u, v)R(x, y)z - R(w_2(u, v)x, y)z - R(xw_2(uv)y)z - R(xy)w_2(uv)z$$

Let $U = \varepsilon$ in the above equation

$$w_2(\varepsilon v)R(xy)z = w_2(\varepsilon, v)R(x, y)z - R(w_2(\varepsilon v)x, y)z - R(yw_2(\varepsilon v)y)z - R(xy)w_2(\varepsilon v)z$$

By

$R(xy)z = g(yz)x - g(xz)y$
$R(\varepsilon v)w = g(vw)\varepsilon - g(\varepsilon w)v$

$= g(vw)\varepsilon - \eta(w)v$

Compute the four terms separately gives
Term (1)

$w_2(\varepsilon v)R(xy)z$

Let $R(xy)z = w$

Then

$$w_2(\varepsilon v)w = R(\varepsilon v)w = \frac{1}{n-1}[g(xz)Qy - g(yz)Qx]$$

$= g(vw)\varepsilon - g(\varepsilon w)v + g(\varepsilon w)y - g(\forall w)\varepsilon = 0$

Second term

$R(w_2(\varepsilon v)x, y)z$

Let $w_2(\varepsilon v)x = w$

$R(wy)z = g(yz)w - g(wz)y$

$\Rightarrow g(yz)w_2(\varepsilon v)x - g(w_2(\varepsilon v)xz)y$

$\Rightarrow g(yz)w_2(\varepsilon v)x - g(w_2(\varepsilon v)x, z)y$

$$w_2(\varepsilon v)x = R(\varepsilon v)x + \frac{1}{n-1}(g(xz)Qy - g(yz)Qx$$

$\Rightarrow g(vx)\varepsilon - g(\varepsilon x)v + g(\varepsilon x)v - g(vx)\varepsilon$

$\Rightarrow 0$

$R(wy)z = 0$

Third term

$R(X\ W_2(\varepsilon, v)Y)Z$

$R(xw)z = g(wz) - g(xz)w$

$\Rightarrow g(w_2(\varepsilon v)y, z)x - g(xz)w_2(\varepsilon v)y, z)x$

But $w_2(\varepsilon v)y = 0$

Thus

$R9yw)z = 0$

Fourth term

$R(XY)W_2(\varepsilon v)z$

$R(xy)w = g(yw) - g(xw)y$

But $w(\varepsilon v)z = 0$

thus$= 0$

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