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$\mathbb{F}$-Shadowing property on the field dynamical system

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Abstract
In my work, I have tried to link algebraic field concepts with dynamical system and find mathematical model and study their effect on the properties of actual error in a dynamical system. This research also seeks to find a new way of working that transforms algebraic characteristics into tools that can be employed in complex phenomena occurring in adynamic system. That's why the research is so important because it's between being able to apply algebraic field properties to a dynamical system and presenting a recent study that combines variable behavior of adynamic system with stable algebraic fields structure and offers anew shared concept that can benefit future generations.

Keywords: F-Shadow Property, group action, field action, algebraic dynamics, algebraic field, dynamic system

Introduction

The concept of shadowing is one of the fundamental concepts affecting dynamical systems, in [1] many researchers have studied the continuous functions on space $\mathbb{G}$ and the property of tangent convergence on this metric space were studied. In [2] Shadowing with reparametrics for flows was studied within this framework. Also in [3] Some researchers studied that the property of tangents stabilizes in the $\mathbb{Z}$ depression with specific properties such as structural stability, dilation, expansion, and extension. They also discovered that this property in $\mathbb{Z}$ studies general properties such as sensitivity and multiplicity, which are two important properties in dynamical systems. In this [4] the volume, the researcher studied the fundamentals of ergonomic theory of hyperbolic systems and introduced a field in which he discussed the possibility of combining two seemingly unrelated theories. One is the theory of equilibrium statistics and the determination of states of infinite systems, and the other is smooth hyperbolic dynamics. In [5] a broad introduction to the subject of dynamical systems was presented, and the researchers provided numerous examples of the development of dynamical theory, including topological mechanics, symbolic mechanics, and algebraic theory. Hyperbolic and one-dimensional dynamics, complex algorithms, and theoretical entropy measurement. They also provided examples in number theory, data storage, and search engines. Also [7] Here, the researcher correctly clarified the important and fundamental aspects of algebra. In [8] This research demonstrates that there is a large class of compact metric spaces in which shading can be described as a structural property of the space of dynamical systems. That is, if the dynamical system contains sufficient shading to achieve the shading of continuously generated pseudo-orbits. In [9] It studies the mechanics of iterative homeomorphic mappings from the Riemannian surface, a broad and rapidly growing topic.In [10] This is a very important book for specialists in dynamical systems. It is the first to dedicate a study to the theory of shading and the shading of approximate paths in dynamical systems, and it clarifies the importance of this theory for both qualitative theory and numerical method theory. Also [11] In this book, the researchers studied the stability of dynamical systems, chaos, and symbolic dynamics, and their treatment from a mathematical perspective. In [12] In this research, the symmetry of algebraic actions $\mathbb{Z}d$ was studied, and it was proven that every loop is continuous for the expanding and mixed Hölder $\mathbb{Z}d$ through the self-symmetry of the compact Abelian set.

In [13] Here the research study provides a complete description of the relationship between different properties and topological pluralism. Also [14] The study of periodic misalignment and the stability of $\mathbb{Q}$ is a study of a function that possesses the property of periodic misalignment and the Lobes property, and that it is stable at $\mathbb{Q}$.

Preliminary Definitions

This chapter outlines the essential definitions and fundamental notions that constitute the theoretical basis of the study. These preliminaries provide the logical framework required for the development of the forthcoming theorems and proofs, ensuring coherence and mathematical rigor throughout the research.

Definition 2.1 [12]

A dynamical system is a mathematical framework for describing the evolution of points in a space over an index set, typically representing time or another parameter. Formally, a dynamical system is given by a triple $(\mathcal{V}, T, \varphi)$, where:

- $\mathcal{V}$is a state space (usually a metric or topological space),
- T is the index set (often $\mathbb{R}$ or $\mathbb{Z}$),
- $\varphi: T \times \mathcal{V} \to \mathcal{V}$ is a map describing the evolution of states.

Definition 2.2 [1]

Let $\mathbb{G}$ be a group and $(\mathcal{V}, +, \cdot)$ be an algebraic group. A group action of $\mathbb{G}$ in a dynamic algebraic system is a mapping $\varphi : \mathbb{G} \times \mathcal{V} \to \mathcal{V}$ *satisfying:*

- $\varphi(e, v) = v$ *for all* $v \in \mathcal{V}$, *anywhere* e *is the identity element of* $\mathbb{G}$ *;*
- $\varphi(g_1 g_2, v) = \varphi(g_1, \varphi(g_2, v))$ *for all* $g_1, g_2 \in \mathbb{G}$ *and* $v \in \mathcal{V}$.

Definition 2.3 [1]

Let a function $\varphi: \mathbb{G} \times \mathcal{V} \to \mathcal{V}$ is called a homeomorphism and it have the $\mathbb{G}$-shadowing property if, for any $\epsilon > 0$, $\exists\, \delta > 0 \ni$ each δ-pseudo-orbit can be ϵ-shadowed by an actual orbit. In essence, this property formalizes how closely pseudo-trajectories can approximate true dynamical behavior.

Definition 2.4 [14]

A dynamical system $(\mathcal{V}, f)$, where $\mathcal{V}$ is a metric space with metric d and $f: \mathcal{V} \to \mathcal{V}$ is a continuous function, is called to have the periodic shadowing property if for all $\epsilon > 0$, there exists $\delta > 0 \ni$ every periodic δ-pseudo-orbit $\{x_n\}$ $(i.e.\, \mathbb{d}(f(x_n), x_{n+1}) < \delta \;\forall\; n$, and $v_{n+k} = v_n$ for some period $k)$ is ϵ −shadowed by a true periodic orbit. That is, there exists a periodic point p in $\mathcal{V} \ni \mathbb{d}\,(f^n(p), v_n) < \epsilon$ *for every* n.

Definition 2.5 [13]

A group dynamical system $(\mathcal{V}, \mathbb{G}, \varphi)$ is said to be topologically transitive *if* for every two non-empty open sets $\mathcal{U}, K \subseteq V$, ∃ an element r ∈ $\mathbb{G} \ni \varphi(r, \mathcal{U}) \cap K \neq \emptyset$.

Definition 2.6 [9]

When algebraic group properties are introduced into the dynamical system, a productive point can be viewed as a point $v \in \mathbb{G}$ (where $\mathbb{G}$ is an algebraic group) such that $\exists\, y \in \mathbb{G}$ and $n \in \mathbb{N}$ for which $f^n(v), f^n(y) \in \mathcal{U}_v$, where $\mathcal{U}_v$ is an algebraic neighborhood around v defined by group operations. Thus, the productive point becomes one that contributes to both the stability and recurrence of algebraic and dynamic structures.

Definition 2.7 [2]

A reparameterization is a growing homeomorphism h of the line $\mathbb{R}$; we mean by Rep the set of completely reparameterizations. Aimed at $a > 0$, we mean

$$Rep\,(a) = \left\{ h \in Rep : \left| \frac{h(t) - h\,(s)}{t - s} - 1 \right| < \text{a}, t, s \in \mathbb{R}, t \neq s \right\}$$

Definition 2.8 [12]

A group $\mathbb{G}$ is a set equipped with two operations, addition $\langle + \rangle$ and multiplication$\langle \cdot \rangle$, satisfying the next:-

- $\mathbb{G}$ is an abelian group under addition with identity 0,
- $\mathbb{G} \setminus \{0\}$ is an abelian group under multiplication with identity 1
- Multiplication is distributive over addition.

Examples: Include $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{Q}$ (rational numbers), and finite groups such as $\mathbb{G}_p$.

Definition 2.9

Let $(\mathbb{F}, +, \cdot)$ be an algebraic field gifted thru a metric $\mathbb{d}$ compatible thru its topology. A field action $\varphi : \mathbb{F} \times \mathcal{V} \to \mathcal{V}$ contents $\varphi(0, v) = v$ also $\varphi(t + s, v) = \varphi(t, \varphi(s, v))$. Assume φ interacts smoothly with the field operations:

$\varphi(t, v + y) = \varphi(t, v) + \varphi(t, y)$ also $\varphi\,(t, v \cdot y) = \varphi(t, v) \cdot \varphi(t, y)$

In the dynamic field context, this action is extended to include time-dependent transformations of field action elements, where field operations evolve dynamically with respect to a temporal parameter t.

Thus,
$\varphi_t : \mathbb{F} \times \mathcal{V} \to \mathcal{V}_t$ represents a time-varying field preserving the algebraic field properties, such as associativity, commutativity, and distributivity at each time t.

Definition 2.10

A field Dynamical System be a triple $(\mathcal{V}, \mathbb{F}, \varphi)$ wherever :
$\mathcal{V}$ is a metric $\mathbb{F}$-space,
$\mathbb{F}$ is a field,
$\varphi : \mathbb{F} \times \mathcal{V} \to \mathcal{V}$ is field action map such that:
$\varphi(r + s, v) = \varphi(r, v) \cdot \varphi(s, v)$,
$\varphi(r \cdot s, v) = \varphi([r, \varphi(s, v)])$,
$\forall\, r, s \in \mathbb{F}, v \in \mathcal{V}$,
also $\varphi(0, v) = v$ *and* $\varphi(1, v) = v_0$ (identity condition).

This definition generalizes the concept of a field action or group action to fields, allowing the index parameter $r \in \mathbb{F}$ to represent richer algebraic structure beyond a simple time parameter.
In the following definitions and theories, let $(\mathcal{V}, \mathbb{d})$ be a compact metric $\mathbb{F}$-space on the field dynamic system $(\mathcal{V}, \mathbb{F}, \varphi)$ and $\varphi : \mathbb{F} \times \mathcal{V} \to \mathcal{V}$ be afield action.

Definition 2.11

Given $\delta > 0$, a sequence $\{v_r\}_{\{r \in \mathbb{F}\}}$ is named a δ-$\mathbb{F}$-pseudo-orbit of the field dynamical system $(\mathcal{V}, \mathbb{F}, \varphi)$ if:

$$sup\left[\, \mathbb{d}(\varphi(s, v_r), v_{(\{r+s\})}) < \delta, \mathbb{d}(\varphi(s, v_r), v_{\{r.s\}}) < \delta \right], \text{for all } r, s \in \mathbb{F}.$$

This means that the points approximately satisfy the system's field action under the field.

Also a finite sequence $\{(x_i, t_i)\}_{i=0}^{n-1}$ is a $(\delta, T) - \mathbb{F}$-pseudo-orbit if $t_i \geq T$ and $sup[\, \mathbb{d}(\varphi(s, v_r), v_{(\{r+s\})}) < \delta, \mathbb{d}(\varphi(s, v_r), v_{\{r.s\}}) < \delta], \forall\, r, s \in \mathbb{F}$.

It is a $(\delta, T) - \mathbb{F}$-chain as of v to y *if* $v_0 = v$ *and* $v_n = y$. The field action is $\mathbb{F}$-chain transitive if for each $\delta > 0$ also $T > 0$, $\exists$ a $(\delta, T) - \mathbb{F}$-chain field action connecting any two elements of $\mathbb{F}$.

Definition 2.12
The field dynamical system ($\mathcal{V}$, $\mathbb{F}$, φ) has the $\mathbb{F}$-shadowing property (used for short $\mathbb{F}$-$\mathcal{SP}$) if for each $\in > 0, \exists\, \delta > 0 \ni$ for each δ-$\mathbb{F}$-pseudo-orbit $\{\mathcal{V}_r\}_{\{r \in \mathbb{F}\}}$ of the field, there exists a point $y \in \mathbb{F}$ and a field reparameterization $\varphi : \mathbb{F} \times \mathcal{V} \to \mathcal{V}$ (typically a field automorphism close to the identity) such that: $\mathbb{d}(v_r, \varphi(\theta(r), y)) < \epsilon, \forall\, r \in \mathcal{V}$.
Here, field reparametrization means adjusting the index r to align the $\mathbb{F}$-pseudo-orbit with a true orbit of the system, analogous to time reparametrization in classical shadowing for field action.

Definition 2.13
The field dynamical system ($\mathcal{V}$, $\mathbb{F}$, φ) takes the $\mathbb{F}$-average shadowing property (used for short $\mathbb{F}$-$\mathcal{ASP}$) if for each $\in > 0 \;\exists\, \delta > 0 \ni$ any (δ, T)-$\mathbb{F}$-pseudo-orbit field $\{v_i\}$ tin be $\in -\mathbb{F}$-shadowed field happening $\mathbb{F}$-average by a true orbit $\varphi(t, v_0)$, satisfying $\lim sup\ (1/n)\ \Sigma\ \mathbb{d}(\varphi(t_i, v_0), v_i) < \epsilon$.

Main Results
There are strong links between $\mathbb{F}$-transitivity field and the ($\mathbb{F}$-$\mathcal{SP}$) with field, especially when $\mathbb{F}$-hyperbolicity field or $\mathbb{F}$-expansiveness field conditions are assumed.

Theorem (3.1)
Let ($\mathcal{V}$, $\mathbb{F}$, φ) be situated a field dynamical system, and the field action $\varphi : \mathbb{F} \times \mathcal{V} \to \mathcal{V}$ be expansive and has the $\mathbb{F}$-$\mathcal{SP}$. If the system is $\mathbb{F}$-topologically transitive, then it is $\mathbb{F}$-topologically mixing.

Proof
Let $c > 0$ be an expansivity constant. Fix $0 < \epsilon < c/4$. By $\mathbb{F}$-shadowing field, choose $\delta > 0$ corresponding to $\in$ such that any δ-$\mathbb{F}$-chain is $\epsilon - \mathbb{F}$-shadowed by a true $\mathbb{F}$-orbit.
Using $\mathbb{F}$-transitivity field, for open sets $\mathcal{U}, \mathcal{C} \subseteq \mathcal{V}$, find $r_1 \in \mathbb{F}$ and $v \in \mathcal{U}$ with $\varphi(r_1, v) \in \mathcal{C}$. By repeating and concatenating finite δ-$\mathbb{F}$-chains field, construct long $\mathbb{F}$-pseudo-orbits with cumulative parameter sums arbitrarily large in F.
Use the δ-$\mathbb{F}$-chain shadowing property field to obtain $z \in \mathcal{V} \ni \mathbb{d}(\varphi(T_i, z), v_i) < \epsilon\ \forall$ i, and thus $\varphi(T, z) \in \mathcal{C}$ for some large cumulative parameter T.
Concatenate $\mathbb{F}$-pseudo-orbits starting and ending near $\mathcal{U}$, yielding an $\mathbb{F}$-approximate periodic return with $\mathbb{d}(\varphi(S, w), w) < 2\epsilon$ for some $S \in F$.
Since $\epsilon < c/4$, expansiveness forces $\varphi(S, w) = w$, giving a $\mathbb{F}$-periodic point. Using $\mathbb{F}$-periodic points and continuity, one shows that for sufficiently large additive parameters in $\mathbb{F}$, $\varphi(r, \mathcal{U}) \cap \mathcal{C} \neq \emptyset$, hence φ is $\mathbb{F}$-topologically mixing.
Thus, under expansiveness, $\mathbb{F}$-shadowing, and $\mathbb{F}$-transitivity, a field dynamical system must be $\mathbb{F}$-topologically mixing.

Theorem (3.2)
Let ($\mathcal{V}$, $\mathbb{F}$, φ) be a field dynamical system, and the field action $\varphi : \mathbb{F} \times \mathcal{V} \to \mathcal{V}$ be $\mathbb{F}$-topologically transitive, and it has the $\mathbb{F}$-$\mathcal{SP}$ with field reparameterization, then for every $\mathbb{F}$-pseudo-orbit, there exists a true orbit that not only $\mathbb{F}$-shadows it but also spreads throughout the space (due to $\mathbb{F}$-transitivity).

Proof
Let $(\mathcal{V}, \mathbb{d})$ be situated a compact metric space also assume ($\mathcal{V}$, $\mathbb{F}$, φ) stay a continuous field action dynamical system and by the $\mathbb{F}$-$\mathcal{SP}$ with reparametrization, for each $\epsilon > 0\ \exists\ \delta > 0$ *such that* each δ-$\mathbb{F}$-pseudo-orbit (v_i, t_i) is ε-$\mathbb{F}$-shadowed by around true $\mathbb{F}$-orbit up to (v_i) a reparameterization $\varphi : \mathbb{F} \times \mathcal{V} \to \mathcal{V}$. If the $\mathbb{F}$-pseudo-orbit visits a dense subset of $\mathcal{V}$, then the true orbit $O(y)$ that $\mathbb{F}$-shadows it will also be $\mathbb{F}$-dense.
$\mathbb{F}$-Topological transitivity guarantees the existence of $\mathbb{F}$-dense $\mathbb{F}$-orbits and enables the construction of $\mathbb{F}$-dense pseudo-orbits by concatenation of orbit segments. Applying the $\mathbb{F}$-$\mathcal{SP}$ to such $\mathbb{F}$-pseudo-orbits produces dense true $\mathbb{F}$-orbits. for every $\mathbb{F}$-pseudo-orbit, there exists a true $\mathbb{F}$-orbit that $\mathbb{F}$-shadows it and spreads throughout the $\mathbb{F}$-space.

Theorem (3.3)
Assum that ($\mathcal{V}$, $\mathbb{F}$, φ) be a field dynamical system, also let $\varphi : \mathbb{F} \times \mathcal{V} \to \mathcal{V}$ be a $\mathbb{F}$-chain transitive map. Assume that $\mathbb{F}$ is also endowed with an algebraic field structure $(\mathbb{F}, +, \cdot)$ such that the operations are continuous by respect to $\mathbb{d}$. If φ has the $\mathbb{F}$-periodic shadowing property and preserves the algebraic field procedures $\big(i.e., \varphi(a + b) = \varphi(a) + \varphi(b)$ *and* $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)\big)$, then φ has the $\mathbb{F}$-$\mathcal{SP}$ and $\mathbb{F}$-transitivity in the algebraic dynamical sense.
Proof:
Let $\in > 0$ be given. Since φ has the $\mathbb{F}$-periodic shadowing property field, there exists $\delta > 0 \ni$ each $(\delta, \mathbb{F})$ −pseudo-trajectory field is $\in -\mathbb{F}$-shadowed field in a true $\mathbb{F}$-orbit field under φ.
Because φ preserves the field operations, any $\mathbb{F}$-pseudo-orbit sequence $\{m_i : 0 \leq i \leq t\} \subseteq \mathbb{F}$ that satisfies $\mathbb{d}\big(\varphi(g_i, m_i), m_{\{i+1\}}\big) < \delta$ for some $g_i \in \mathbb{F}$ also maintains algebraic consistency: $\varphi\big(m_i + m_j\big) = \varphi(m_i) + \varphi\big(m_j\big)$ and $\varphi\big(m_i . m_j\big) = \varphi(m_i) . \varphi\big(m_j\big)$.

By $\mathbb{F}$-chain transitivity field, for any two elements $m_t, m_0 \in \mathbb{F}$ there exists a δ-$\mathbb{F}$-chain connecting them, i.e., a finite sequence $\{v_0 = m_t, v_1, \dots, v_n = m_0\} \ni \mathbb{d}\big(\varphi(g_i, v_i), v_{(\{i+1\})}\big) < \delta$ for all $i \in \mathbb{F}$.
Combining the algebraic continuity and $\mathbb{F}$-chain transitivity, the $\mathbb{F}$-shadowing orbit preserves the additive and multiplicative structures of the field, ensuring that the algebraic relationships are stable under φ. Therefore, φ possesses the F-$\mathcal{SP}$ and $\mathbb{F}$-transitivity in the algebraic field context. Hence, φ defines a $\mathbb{F}$-algebraic dynamical system that is both structurally and algebraically stable under perturbations of $\mathbb{F}$-pseudo-trajectories field.

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