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Bounded approximate identity on the deformed algebra

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Abstract
The convergence of nets shows that $\{e_n\}$ is a bounded approximate identity for the deformed algebra, (μ_ϵ) as $||e_n\mu_\epsilon u - u||_\varepsilon < \delta$. Using the symbol spaces, $S^{m,q}$ (X, R^n) and the smooth polynomial bounded R^n - actions by endomorphism on the generalised algebra, $G(M)$.

Keywords: Convergence, smooth polynomial, symbol spaces, endomorphism, generalised algebra

Introduction

A bounded approximate identity is a sequence or net of elements in an algebra that approximates the identity element both from the left and right sides and is bounded in the norm. Also, it is a concept in functional analysis and algebra that relates to the structure of an algebra, particularly in the context of Banach algebras, C*-algebras, and other topological algebras. In other words, bounded approximate identity can be applied in functional and harmonic analyses - {Banach and topological algebras}, where it controls the behaviour of multiplication in the absence of an identity and make use of strong convergence tools which localises or smoothing the nets. Bounded approximate identities ensure stability under algebraic operations and appear naturally in harmonic analysis, C*- algebra theory, and partial differential equations. Consequently, in non-commutative geometry and Deformation quantisation, bounded approximate identity can be applied on smooth polynomial bounded R^n-actions, where the smoothness ensures that the integral is well-defined.

Over the years, many researchers have done a good job in this area of mathematics. Ayinde (2022) [3] discusses bounded approximate identities as a key concept and shows that the algebra of compact operators has both right and left locally bounded approximate identities. The author also mentions the algebra of compact operators on Fréchet space, X, is a mathematical structure and it has a relationship to bounded approximate identities. Esmeral, *et al* (2021) [6], defines and discusses norm bounded and operator norm bounded approximate identities, especially on norm algebras and C*-algebras. Fozouni *et al*, (2015) [10] give two notions of approximate identities by changing the concepts of convergence and bounded in the classical notion of bounded approximate identity, where they focus on commutative Banach algebras and their character spaces. Pedersen, (2009) [19] discusses bounded approximate identities in the context of Fréch*et al*gebras, specifically the weighted convolution Fréch*et al*gebras denoted as $A(\omega)$. The author notes that while L_1 (ω_n) has a bounded approximate identity, $A(\omega)$ does not possess a uniformly bounded one under certain conditions. Kochubei (2015) [13], defines and uses the concept a bounded approximate identity in the context of Banach algebras. He discusses the multiplier algebras, which are unital Banach algebras and how they are related to bounded approximate identities. Dales & Ulger (2015) [5], study contractive and point-wise contractive Banach function algebras with contractive or point-wise contractive approximate identities. They also provide examples which include uniform algebras and describe a contractive Banach function algebra is not equivalent to a uniform algebra. Feizi & Soleymani (2014) [7], study the relationship between Fréch*et al*gebras and their ultra-powers, focusing on locally bounded approximate identities. Here, they introduce a new version of bounded approximate identity in Fréch*et al*gebras and examine contractibility in these structures.

Also, they show that a Fréch*et al*gebra has bounded approximate identity if and only if, its ultra-power does. Fozouni (2018) [9], addresses bounded approximate identities by providing a counterexample in Banach algebras. The author also explores the relationship between Ideals, Multiplier algebras and bounded approximate identities within the context of Banach algebras and Fourier algebras. Flores (2024) [8], discusses bounded approximate identities, including their existence and properties in the context of Banach algebras, where the author mentions $L_1\,v(G|C)$ and $L_1\,v(G)$ as mathematical structures where these identities are relevant. Rieffel (1992) [20], considers the bounded approximate identity on Fréch*et al*gebras coming from the R^n- actions on C^*- algebras. Also, Lechner and Waldman (2016) work on locally convex sequentially complete algebra with separable continuous associative product and where α is a smooth polynomial bounded R^n- actions defined by automorphism. In this paper, we consider the bounded approximate identity on smooth polynomial bounded R^n- actions define by endomorphism on deformed generalised algebra, $\mu_\varepsilon \epsilon G(M)$. The paper is structured in this form, section 2 is the Preliminaries and section 3 is result and conclusion.

Preliminaries

The definitions to be used in this work will be given here.

Definition 2.1: (Fréchet space)

A Fréchet space, X is a topological vector space that satisfies two key conditions:

1. **Metrisability:** There exists a metric d on the space X such that X is homeomorphic to a metric space. This means the space can be described with a distance function.
2. **Complete:** The space is complete with respect to the topology induced by the metric. In other words, every Cauchy sequence in the space converges to an element in the space.

In other words, a Fréchet space can be characterised as a locally convex space whose topology is generated by a countable family of seminorms or a locally convex space that is complete with respect to a translation -invariant metric. (See, Allahyari *et al,* (2020) [2], Curiel, (2015) [4], Nwachukwu (2025)) [16].

Definition 2.2: Symbols [12]

Let $S^m_{\rho,\sigma}(X\times\mathbb{R}^N)$ denote the set of all $a\in C^\infty(X\times\mathbb{R}^N)$. Then, for every compact set $K\subset X$ and all multi- orders α,β, where, $m,\rho,\sigma\in\mathbb{R}$ be real numbers, the estimate;

$$\left|D^\alpha_x D^\beta_\zeta a(x,\zeta)\right|\le C_{K,\alpha,\beta,}(1+|\zeta|)^{m-\rho|\alpha|+\sigma|\beta|}x\in K,\zeta\in\mathbb{R}^N \quad (2.1)$$

Is valid for some constant $C_{K,\alpha,\beta}$, where a is the growth condition derivatives of the function $a(x,y,\zeta)$. Consequently, the optimal values of the constants $C_{k,\alpha,\beta}$ provide a set of seminorms on $S^m_{\rho,\sigma}(X\times\mathbb{R}^N)$, which turn $S^m_{\rho,\sigma}(X\times\mathbb{R}^N)$ into a Fréchet space, X.

Definition 2.2: Generalised Functions (Al-Gwaiz, [1]

Let $X\subset R^n$ be an open set. Then, the generalised functions are continuous linear functionals, $D'(X)$, over a space of infinitely differentiable functions $D(X)$ such that all continuous functions have derivatives which are themselves generalised functions.

Definition 2.3 Generalised Space [21, 16]

Let X be a Fréchet space. Then, we can associate a generalised space G_X as follows

Let $I\subset$ R be the interval (0,1). Then, we define the Moderate nets in X to be

$$E(X):=(\alpha_\epsilon)_{\epsilon\in I} \quad (2.2)$$

Such that, for continuous seminorms ρ on X there exists N such that $\rho(\alpha_\epsilon)\sim 0(\epsilon^N)$. The Negligible nets are:

N(X) = (αϵ)ϵ∈I

such that, for continuous seminorms ρ on X and for all m

|ρ (αϵ)ϵ| ~ 0(ϵm).

Then, the generalised space of X is defined as the quotient: (2.3)

$$GX = E(X)/N(X) \quad (2.4)$$

Remark 2.4

The definition of E(X), N(X) suffices to restriction to the defining family of seminorms that generate the locally convex (Fréchet) topology on X.

Definition 2.5: Moderate and Negligible Functions: [11, 18, 16]

We set:

$$\begin{aligned}E(M)\ &:=\left(C^\infty(X)\right)^I\mathcal{E}_u(M):=\{(\alpha_\epsilon)_\epsilon\in\mathcal{E}(M)\mid\forall K\subset\subset X\ \forall\ \beta\in\mathbb{N}^n_o\ \exists\ N\in\mathbb{N}\\ &\quad\text{with}\ \sup_{x\in K}\left|\partial^\beta\alpha_\epsilon(x)\right|=\gamma(\epsilon^{-N})\ \text{as}\ \epsilon\to 0\}.\\ N(M)\ &:=\{(\alpha_\epsilon)_\epsilon\in\mathcal{E}(M)\ \Big|\ \forall K\subset\subset M\ \forall\ \beta\in\mathbb{N}^n_o\ \forall\ u\in\mathbb{N}:\sup_{x\in K}\Big|\partial^\beta\alpha_\epsilon(x)|\\ &\quad=\gamma(\epsilon^u)\ \text{as}\ \epsilon\to 0\}.\end{aligned}$$

Elements of $E(M)$ and $N(M)$ are called moderate and negligible functions respectively. Whereas, elements of moderate functions constitute the differential algebras, the negligible functions constitute a differential ideal. To this end, the Generalised (Colombeau) algebra on M is defined as:

$$G(M)=E(M)/N(M) \quad (2.5)$$

Where, $G(M)$ is an associative, commutative differential algebra.

Definition 2.6: Fréch*et al*gebra [15]

A Fréch*et al*gebra A is a complete topological algebra, whose topology is generated by a sequence $(P_n)_{n\in}$N of seminorms on A satisfying $\forall f,g\in A$ and $\forall\ n\in$ N:

1. $P_n(f.g)\le P_n(f)\cdot P_n(g)$
2. $P_n(e_n)=1$
3. $P_n(f)\le P_{n+1}(f)$,

In other words, a Fréch*et al*gebra is an associative algebra A over the real or complex field that at the same time is also locally convex. Some properties of Fréch*et al*gebra include the following - the continuity of multiplication, group invertible elements and convexity.

Definition 2.7: (Smooth Polynomial Bounded R^n - action) [14]

Let X be a Fréchet space with continuous (countable) defining system of seminorms, P and let m be an order for ρ. Then, the smooth polynomial bounded R^n - action is defined as a map:

- $\alpha: R^n \times X \to X \ni$
- $\alpha(x) \in S^{m,0}(X, R^n)$ for each $x \in X$
- $X \ni x \mapsto \alpha(x) \in S^{m,0}(X, R^n)$ is smooth for any $\rho \in P$, $\sigma \in \mathbb{N}_0^n \; \exists \rho' \in P \ni$

$$\| \alpha(x) \|_{\rho,\sigma}^{m,0} \leq \rho'(h), \forall x \in X. \quad (2.6)$$

As stated earlier, the concept of bounded approximate identity plays a very crucial role in non-commutative geometry and deformation quantisation. Here, the (Definition 2.16, [20], will be modified to suit our situation.

Definition 2.8: Bounded Approximate Identity (Rieffel, [20]**)**

Let $u \in G(M)$ be a generalised algebra and let $\{\rho\} \subseteq P$ be a family of seminorms which determines the topology on $G(M)$. Then, we define a bounded approximate identity for $G(M)$ as a net $\{e_n\}$ of elements of $G(M)$ such that

1. For each $u \in G(M)$, the nets $e_n u$ and ue_n converge to u.
2. For each ε there is a positive constant M_ε such that $||e_n||_\varepsilon \leq M_\varepsilon, \forall n$.

Result

Notations

Here, we set $A = A^\infty$, where A is a Fréch*et al*gebra and A^∞ represent a generalised algebra, $G(M)$. Rieffel, [20] in his work considered A to be a Fréch*et al*gebra with a strongly continuous $\mathbb{R}^n$ -action α by automorphisms α_x which are also isometric for $\rho \in P$. Then, $A = A^\infty$ and A^∞ is also a Fréch*et al*gebra, where α takes A^∞ into itself and these seminorms are isometric and the action of α is strongly continuous and differentiable. Both Lechner *et al* and Rieffel in their works considered the sequentially complete locally convex spaces with filtrating defining systems of seminorms. However, we are to survey the generalised functions $v \in G_X$, alongside the space of test functions, $\phi \in D(R^n)$ as a Fréchet space and the algebra of generalised functions will be covariant to the Fréch*et al*gebra acting by endomorphism and where the multiplication is jointly continuous.

Now, we set X, Y, $Z \in G(M)$ and $A = X = Y = Z$, with jointly continuous product μ and the $\mathbb{R}^n$-action, $\alpha = \alpha^X, \alpha^Y, \alpha^Z$ by endomorphism α_x that are polynomial bounded to $\rho \in P$.

According to Lechner *et al*, [14] and Rieffel, [20], we consider a real $(n \times n)$- matrix ϵ as our deformation parameter defined on the space of test functions and we introduce the functions as $x \in X$, $y \in Y$ such that;

$$\mu_{x,y}^\epsilon: \mathbb{R}^n \times \mathbb{R}^n \to Z,$$

where the actions α^X, α^Y are smooth and polynomially bounded. By Definition (2.2), we say that these functions are symbols in $S^{m,q}(G_X, \mathbb{R}^n)$.

Now, let $A^\infty = G(M)$ be a generalised algebra with jointly continuous product.

Then, $\mu: G(M) \times A \to G(M)$

and we assume that the $\mathbb{R}^n$ - action, α acts by endomorphism. With our deformation parameter, ϵ, we can deform the product μ as; Given $X, Y \in G(M)$, as given in the following definition.

Definition 3.2: (Deformed Algebra)

Let α be a smooth polynomially bounded $\mathbb{R}^n$- action on A and ϵ our deformation parameter, we define the product

$$\mu_\epsilon(x, y) = x \times_\epsilon y = \int \int e^{i\phi(x,\theta)} a(x, \theta) u(x) dx d\theta. \quad (3.1)$$

Lemma 3.3 [14]

Let $y \in Y$ and $z \in Z$. If either y is α^Y- invariant or z is α^Z- invariant, then

$$\mu_\epsilon(y, z) = \mu(y, z) \quad (3.2)$$

Now, we show that the deformed algebra, μ_ϵ, has a bounded approximate identity.

Proposition 3.4: Let $u \in G(M)$ be a generalised algebra, and let α: $R^n \times G(M) \to G(M)$, be a smooth polynomially bounded R^n- action by endomorphisms on $G(M)$. Let $\{e_n\}$ be a bounded approximate identity for $u \in G(M)$. Then, $\{e_n\}$ is also a bounded approximate identity for the deformed algebra, μ_ϵ (3.1)

Proof. First, we check if $u \in G(M)$ has an identity. Since, α acts by endomorphisms, this implies $\alpha_x(1) = 1 \; \forall x \in R^n$. By Lemma 3.*3*, we have

$$u\mu_\epsilon 1 = u1 = u \quad (3.3)$$

and

$$1\mu_\epsilon u = 1u = u. \quad (3.4)$$

for any $u \in G(M)$. From Definition 2.8, since the seminorm is not changed, the bounded condition holds for $\|e_n\mu_\epsilon u - u\|_\varepsilon \leq M_\epsilon$. Now, let $\delta > 0$ and let φ_i and φ_j be defined as test functions such that $\rho \in P$ be the seminorm. Then,

$$e_n\mu_\epsilon u - u = \sum \int \int e^{i(x,\theta)} (\alpha\epsilon_x(e_n)\alpha_\theta(u) - \alpha_\theta(u)\varphi_i(x)\varphi_j dx d\theta. \quad (3.5)$$

Since, $\{e_n\}$ is bounded by definition, we find an m that is independent of n such that the part of the sum above for $|i| \geq m$ and $|j| \geq m$ is lesser for the seminorm $\{\rho\} \subseteq P$ than $\frac{\delta}{2}$ for all n. Now,

$$\int \int e^{i(x,\theta)} \big(\alpha\epsilon_x(e_n)\alpha_\theta(u) - \alpha_\theta(u)\big)\varphi_i(x)\varphi_j dx d\theta$$

$$= \int \int e^{i(x,\theta)} \alpha\epsilon_x \big(e_n\alpha_\theta(u) - \epsilon_x(u) - \alpha_\theta - \epsilon_x(u)\big)\varphi_i(x)\varphi_j(\theta) dx d\theta$$

The compact support of the test functions $\varphi_i(x)$ and $\varphi_j(\theta)$, allows the integral to converge to 0 for the seminorm as $n \to \infty$. This holds for each of the finitely many terms for which $|i| < m$ and $|j| < m$. This implies that, if n is large enough, we have:

$$||e_n\mu_\epsilon u - u||_\varepsilon < \delta. \; (3.6)$$

Thus, the approximate identity for the deformed algebra, μ_ϵ is bounded.

Conclusion

The bounded approximate identity aided the convergence of the net, $\{e_n\}$. With the boundedness of $\{e_n\}$, one can construct and approximate any element $u \in G(M)$. That is, for any $u \in G(M)$, the limit $\lim_{n\to 0} e_n.u = u$.

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