International Journal of Statistics and Applied Mathematics 2025; 10(2): 29-34

ISSN: 2456-1452
Maths 2025; 10(2): 29-34

https://www.mathsjournal.com
Received: 06-12-2024
Accepted: 14-01-2025

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Optimal investment strategies for retirement: A stochastic interest rate approach using Garch type model

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Abstract
This paper presents a stochastic optimal control model to determine optimal investment strategies in a defined contribution pension plan, accounting for pre-retirement phase under a stochastic interest rate environment. Unlike traditional models that assume constant risk-free rates, this study utilizes the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) type models to capture the time-varying and asymmetric volatility of interest rates, a key factor in retirement planning amid uncertain economic conditions. Wealth dynamics are modeled with investments in both a risky asset and a risk-free asset, where the risk-free rate follows a GARCH process to realistically reflect fluctuating interest rates. The resulting stochastic differential equations for wealth evolution incorporate both the volatility of the risky asset and the conditional heteroskedasticity of the interest rate. Simulation results indicate that optimal investment strategies are significantly influenced by the stochastic nature of interest rates, with implications for asset allocation shifts before retirement. By incorporating GARCH-type modeled interest rates, the approach provides a robust framework for managing risk and optimizing returns in retirement planning, offering valuable insights for pension fund managers and financial planners navigating dynamic interest rate environments.

Keywords: Optimal control, investment policy, GARCH, asymmetric volatility and stochastic interest rate

1. Introduction

Effective retirement planning, especially within defined contribution pension schemes, requires tailored investment strategies that evolve through both wealth accumulation before retirement and wealth preservation afterward. The traditional models often assume constant interest rates, but this simplifying assumption limits their applicability in real-world conditions where interest rates are both volatile and asymmetric in response to economic shocks. Merton's foundational work on continuous-time portfolio optimization provided a valuable framework for dynamic investment strategies (Merton,1969, 1971) [7, 8]. However, the assumption of constant or deterministic interest rates is increasingly viewed as inadequate for modeling the complex behaviour of financial markets, particularly for long-term retirement planning (Cairns, 2000; Ricardo and Juan, 2008) [1, 12].

To address these limitations, recent research has incorporated stochastic interest rate models to capture real-world rate fluctuations more accurately. Processes such as Vasicek and Cox-Ingersoll-Ross (CIR) introduced mean-reverting characteristics, which better approximate empirical interest rate behaviour (Vasicek, 1977) [13]. However, these models may not adequately capture asymmetric responses and volatility clustering, which are prevalent in financial data. In response, GARCH-type models, particularly EGARCH and GJR-GARCH, have been applied to model conditional volatility in interest rates. The EGARCH model is widely recognized for its ability to capture asymmetric volatility without requiring positivity constraints (Nelson, 1991) [9], while the GJR-GARCH model provides an alternative that effectively captures leverage effects, making it well-suited for rates that exhibit differing responses to positive and negative shocks (Glosten *et al.*, 1993) [4].

In this study, we compare EGARCH and GJR-GARCH models in representing long-term interest rates for pre-retirement investment periods

By modeling long-term rates for wealth accumulation before retirement, the paper aims to address the objectives of the pre-retirement phase. This approach enables a more responsive allocation strategy, where asymmetric volatility and varying risk exposures can be more precisely managed throughout the retirement horizon**.** Stochastic optimal control techniques, which have been applied in pension fund modeling and annuity contracts (Devolder *et al.*, 2003; Charupat and Milevsky, 2002; Osu and Ijioma, 2012) [3, 2, 11], provide a robust framework for optimizing asset allocation strategies in such dynamic environments. Moreover, applications of stochastic differential equations further enhance the modeling of pension fund dynamics and optimal control solutions (Oksendal, 1998; Mallappa and Talawar, 2019; Lin *et al.*, 2023) [10, 6, 5].

In the present paper, our findings highlight the impact of using a dynamic interest rate model on optimal asset allocation before retirement, highlighting how EGARCH and GJR-GARCH models provide nuanced risk management tools for portfolio adjustments. This study offers actionable insights for pension fund managers and financial planners focused on maximizing retirement security in fluctuating economic conditions, demonstrating the value of distinct models for stochastic interest rate horizons.

2. Materials and Methods

This section formulates the retirement investment strategy, a defined contribution pension plan is considered, where benefits are paid by annuity. The problem is to find the best investment policy for the asset backing the pension liabilities of an investor in the plan before retirement. During the activity period, the contributions can be invested in a riskless asset or in a risky asset and the reserve obtained at retirement age is the amount accumulated without any special guarantee by the insurer. At the age of retirement this reserve is used to purchase a paid-up annuity. After retirement the insurer has to pay this guaranteed annuity. Because of the presence of the liability only after retirement, the problem is split into two periods before and after retirement (Devolder *et al.*, 2003) [3].

A continuous-time stochastic model is considered and used the tools of stochastic optimal control theory (Oksendal, 1998) [10]. In the present paper, the focus is on optimal investment plan before retirement under stochastic interest rate.

The optimal control problem is defined as,

State variable: The asset of pension plan is chosen as a state variable

$F(t)\ (t\ \in [0, T+N]$ (1)

Decision variable: Following the classical model of Merton (1971) [8] the proportion invested in risky asset is chosen as the decision variable.

The financial market is supposed to be described by two assets:

Riskless asset with price dynamics: S_1

$$dS_1(t) = rS_1(t)dt \tag{2}$$

Risky asset with price dynamics: S_2

$$dS_2(t) = \alpha S_2(t)dt + \sigma S_2(t)dW(t) \tag{3}$$

Where, W the standard Brownian motion.

The proportion invested in the risky asset at time t is denoted $u(t)$ and $\big(1-u(t)\big)$ is the proportion in the riskless asset. The problem is to find optimal solution of $u(t)$.

We model the wealth dynamics $F(t)$ over time, for the pre-retirement phase. The wealth $F(t)$ is allocated between a risky asset and a risk-free asset, with the following dynamics:

$$dF(t) = F(t)\big[u(t)\alpha + \big(1-u(t)\big)r(t)\big]dt + F(t)u(t)\sigma dW(t) \tag{4}$$

Where, $r(t)$is the stochastic interest rate on the risk-free asset, α is the expected return on the risky asset, and σ is the volatility of the risky asset.

Since $r(t)$ is the stochastic interest rate, we model the rates for the two cases:

2.1. The EGARCH model

Let $r(t)_l$, denote the observed long-term interest rate and l stands for long term.

The Mean equation is specified as:

$$r(t)_c = \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{j=1}^{q} \theta_j Z_{lt-j} + Z_{lt} \tag{5}$$

Where,Z_{lt} is the innovation process,$\theta_j$is the MA (moving average) coefficients, capturing the influence of past shocks on the current rate,ϕ_i is the AR (autoregressive) coefficients, capturing the influence of past rates on the current rate,

The exponential GARCH model, denoted by EGARCH (p, q), has the variance model as:

$$\log(\sigma(t)_l) = \omega_l + \sum_{i=1}^{p} \alpha_{li} \frac{Z_{lt-i}}{\sqrt{\sigma(t-i)_l}} + \sum_{j=1}^{q} \beta_{lj} \log(\sigma\,(t-j)_l) + \sum_{k=1}^{p} \gamma_{lk} \left|\frac{Z_{lt-k}}{\sqrt{\sigma(t-k)_l}}\right| \tag{6}$$

Where, $\sigma(t)_l$ denotes a volatility process, ω_l is the constant term, controlling the baseline level of volatility, α_{li}'s are the parameters capturing the impact of past standardized returns on volatility (asymmetric shock terms),

β_{lj}'s are the parameters representing the persistence of volatility and γ_{lk}'s terms allowing for asymmetric effects, where negative shocks can impact volatility differently than positive shocks. Where subscript l for long-term interest rate.

2.2. The GJR-GARCH model

The mean Equation of GJR-GARCH is same as specified in equation (5).

The GJR-GARCH model, denoted by GJR-GARCH (p, q), has the variance model as:

$$\sigma(t)_l = \omega_l + \sum_{i=1}^{p} \alpha_{li} Z_{tl-i}^2 + \sum_{j=1}^{q} \beta_{lj} \sigma(t)_{l-j} + \sum_{i=1}^{p} \gamma_{li} Z_{lt-i}^2\, I \tag{7}$$

Where, $\sigma(t)_l$ is the conditional variance of $r(t)_c$, representing volatility at time t, ω_l is the constant, α_{li} denotes the impact of past squared residuals (ARCH effect), capturing short-term volatility shocks.γ_{li}'sare the terms allowing for asymmetric effects, where negative shocks can impact volatility differently than positive shocks**.** I is the indicator function $(Z_{tl} < 0)$and β_{lj}'s are the GARCH term, capturing the persistence of volatility. Where subscript l for long-term rates.

2.3. Optimal Wealth Before Retirement

Before retirement ($t \in [0, N]$), period without liability wherein we optimize the utility of the final wealth at retirement.
Therefore, the final wealth equation for before retirement is as follows:

$$dF(t) = F(t)\big[u(t)\alpha + \big(1 - u(t)\big)r(t)_l\big]dt + F(t)u(t)\sigma dW(t) \quad (8)$$

Where, $r(t)_l$ is stochastic interest rate (long-term) modelled using either EGARCH or GJR-GARCH model.

Objective function: The problem will be to optimize the expected utility of the final wealth at the end of the period.
First period: Maximization of the expected utility of the total fund obtained at retirement age:

$$\max_{u} EU\big(F(N)\big) \quad (9)$$

2.5. Optimal policy before retirement
Using the classical tools of stochastic optimal control, to solve equation (9) with

$$dF(t) = F(t)\big[u(t)\alpha + \big(1 - u(t)\big)r(t)_l\big]dt + F(t)u(t)\sigma dW(t),$$
$$F(0) = P(0 \le t \le N)$$

The value function of the problem is denoted by

$$W(t, F, r(t)_l) = \max_{u} E[U(F(N)|F(t) = F] \quad (10)$$

The maximum principle leads to the following result (Hamilton- Jacobi method):

$$0 = \max_{\{u\}}[\frac{\partial W}{\partial t} + [u(t)(\alpha - r(t)_l + r(t)_l)]F\frac{\partial W}{\partial F} + \frac{1}{2}u^2(t)\sigma^2F^2\frac{\partial^2 W}{\partial F^2} + \mu_{r_l}\frac{\partial W}{\partial r} + \sigma_{r_l}^2\frac{\partial^2 W}{\partial r_l^2}] \quad (11)$$

Or $0 = \max_{u}\{\psi\}$

We can derive from these two equations and second order condition:

$$\text{(i). } \psi(u^*) = 0 \quad (12)$$

$$\text{(ii). } \frac{\partial\psi(u^*)}{\partial u} = 0 \quad (13)$$

$$\text{(iii). } \frac{\partial^2\psi(u^*)}{\partial u^2} < 0$$

Therefore (13) gives

$$0 = (\alpha - r(t)_c)F\frac{\partial W}{\partial F} + u(t)\sigma^2F^2\frac{\partial^2 W}{\partial F^2} \quad (14)$$

Hence the optimal the optimal investment proportion u^* in risky asset is

$$u^*(t) = -\frac{\partial W/_{\partial F}}{F\left(\partial^2 W/_{\partial^2 F}\right)}\frac{(\alpha - r(t)_c)}{\sigma^2} \quad (15)$$

Substituting this in equation (12), the partial differential equation for the value function is obtained as:

$$\frac{\partial W}{\partial t} + r(t)_l F\frac{\partial W}{\partial F} - \frac{1}{2}\frac{(\alpha - r(t)_l)^2}{\sigma^2}\frac{\left(\partial W/_{\partial F}\right)^2}{\left(\partial^2 W/_{\partial^2 F}\right)} + \mu_{r_l}\frac{\partial W}{\partial r} + \sigma_{r_l}^2\frac{\partial^2 W}{\partial r_l^2} = 0 \quad (16)$$

With limit condition $W(t, F, r) = U(F)$.
Solving for the equation (16) for the value function W and replacing it in (15) the optimal policy is obtained.

3. Results and Discussion
The analysis uses daily data from Federal Reserve Economic Data (FRED) database, for the period January 1, 2022, to December 1, 2022. The DGS10 rates were chosen to model long-term interest rates, respectively, due to their relevance in retirement investment strategies.
Since this phase typically spans multiple decades, it needs a rate that reflects long-term expectations for risk-free returns. Longer-term rates tend to be less volatile and better suited for reflecting the long-term nature of retirement saving, where the focus is on accumulating wealth over a significant period. A longer-term interest rate DGS10 (10-Year U.S. Treasury Rate) is considered for the study**.** Long-term interest rates, such as the 10-year Treasury yield (DGS10), exhibit slow mean reversion and are less volatile compared to short-term rates. Since they adjust gradually to macroeconomic conditions, even a one-year period can capture meaningful trends for stochastic modeling.

3.1. Estimation of the Model Parameters
3.1.1. EGARCH Model Estimation
The model used to analyze the data is an EGARCH model for conditional variance dynamics with an Auto Regressive Fractional Integrated Moving Average (ARFIMA) mean structure. The parameters were estimated using the maximum likelihood method under a normal distribution assumption. The results are summarized in Tables below:

Table 1: Parameter Estimates for EGARCH (2, 2) with ARFIMA (3, 0, 4)

Parameters	Estimate	Std. Error	t-value	p-value
Mean Model Parameters				
μ_l	-0.000185	0.000000	-3524.9	0.000000
AR(1)	-0.562195	0.000121	-4647.5	0.000000
AR(2)	-0.099920	0.000027	-3730.4	0.000000
AR(3)	-0.446467	0.000147	-3027.7	0.000000
MA(1)	-0.346307	0.000142	-2435.5	0.000000
MA(2)	-0.598648	0.000182	-3292.8	0.000000
MA(3)	0.398199	0.000155	2570.5	0.000000
MA(4)	-0.449825	0.000238	-1893.5	0.000000
Variance Model Parameters				
ω_l	-0.223474	0.000065	-3464.4	0.000000
α_{l1}	-0.135514	0.000033	-4163.9	0.000000
α_{l2}	0.213894	0.000047	4539.1	0.000000
β_{l1}	0.681179	0.000168	4044.6	0.000000
β_{l2}	0.275256	0.000072	3831.7	0.000000
γ_{l1}	0.692928	0.000179	-3860.9	0.000000
γ_{l2}	0.496132	0.000144	3438.5	0.000000

Table 2: Loglikelihood, AIC and BIC comparison of EGARCH Models

Model	Loglikelihood	AIC	BIC
EGARCH(0, 1)	254.8218	-2.1835	-2.0778
EGARCH(1, 1)	259.1678	-2.2041	-2.0683
EGARCH(1, 2)	266.0853	-2.2386	-2.0576
EGARCH(2, 2)	272.1538	-2.2657	-2.0394

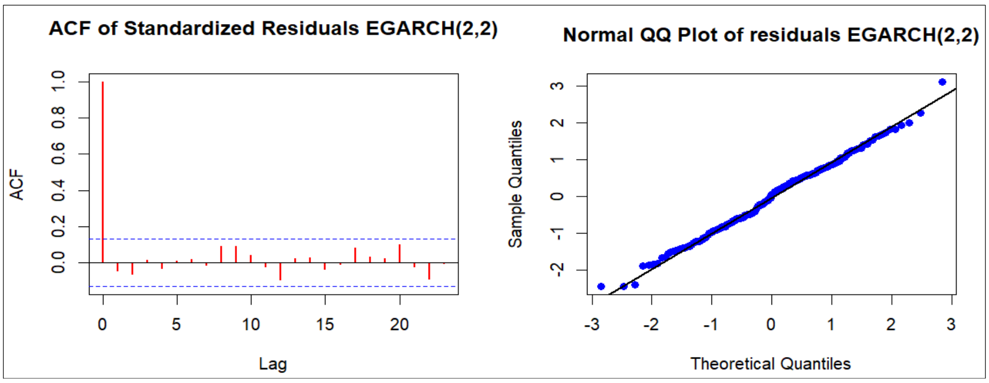

Fig 1: ACF and Normal QQ plot of Standardized Residuals of EGARCH for DGS10.

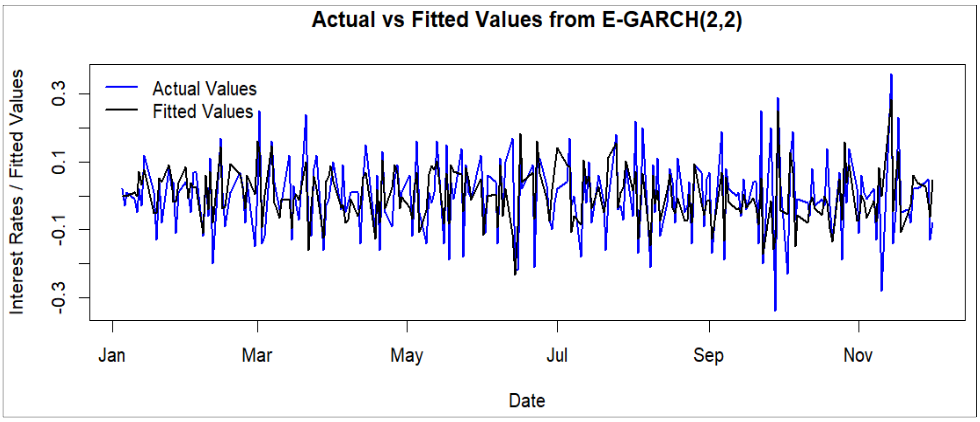

Fig 2: Plot of Actual versus fitted values of EGARCH model for DGS10.

From the above model outputs, it is observed that the parameter estimates (Table 1) of EGARCH (2, 2) are significant, also the log-likelihood (Table 2) is maximum with the lowest AIC and BIC. The ACF and QQ plot (Figure 1) implies that most of the lags fall within the confidence band suggesting no autocorrelation in the residual and are normally distributed. The plot of actual versus fitted value (Figure 2) also shows that the model fits most of the observed values and also indicates that the model has captured the volatility of the long-term interest rate to some extent. Therefore, the EGARCH (2, 2) could be chosen for fitting the long-term interest rate.

3.1.2. GJR-GARCH Model Estimation

The model used to analyze the data is an GJR-GARCH model for conditional variance dynamics with an Auto Regressive Fractional Integrated Moving Average (ARFIMA) mean structure. The parameters were estimated using the maximum likelihood method under a normal distribution assumption. The results are summarized in Tables below.

From the above model outputs, it is observed that the parameter estimates (Table 3) of GJR-GARCH (1, 1) are significant, also the loglikelihood (Table 4) is maximum with the lowest AIC and BIC. The ACF and QQ plot (*Figure 3*) implies that most of the lags fall within the confidence band suggesting no autocorrelation in the residual and they are normally distributed. The plot of actual versus fitted value (Figure 4) also shows that the model fits most of the observed values and also indicates that the model has captured the volatility of the long-term interest rate to some extent. Therefore, the GJR-GARCH (1, 1) could be chosen for fitting the long-term interest rate.

Table 3: Parameter Estimates for GJR-GARCH (1, 1) with ARFIMA (2, 0, 2)

Parameters	Estimate	Std. Error	t-value	p-value
Mean Model Parameters				
μ_l	-0.000214	0.000003	-83.6456	0.000000
AR(1)	-0.884874	0.000076	-11654.4764	0.000000
AR(2)	0.014557	0.000083	175.2033	0.000000
MA(1)	0.040012	0.000002	20991.4096	0.000000
MA(2)	-1.035677	0.000064	-16272.1661	0.000000
Variance Model Parameters				
ω_l	0.000173	0.000001	275.9036	0.000000
α_{l1}	0.000048	0.000030	1.6143	0.10647
β_{l1}	1.000000	0.000015	66061.0546	0.000000
γ_{l1}	-0.056794	0.000292	-194.3459	0.000000

Table 4: Loglikelihood, AIC and BIC comparison of GJR-GARCH Models

Model	Loglikelihood	AIC	BIC
GJR-GARCH(0, 1)	-7168.212	63.218	63.323
GJR-GARCH(1, 1)	266.2895	-2.2669	-2.1311

GJR-GARCH(1, 2)	250.0632	-2.0975	-1.9164
GJR-GARCH(2, 2)	247.1429	-2.0453	-1.8190

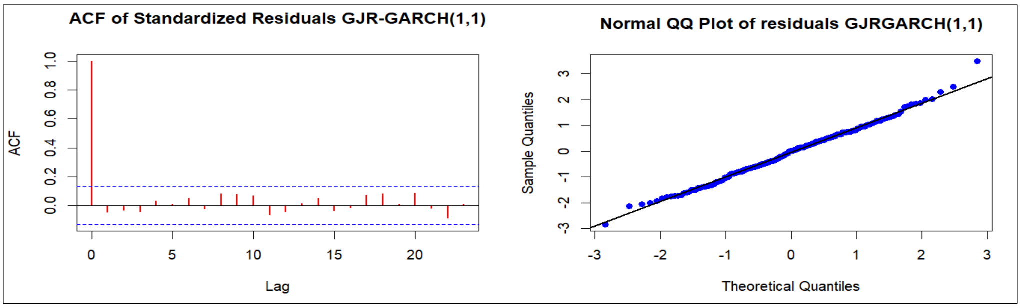

Fig 3: ACF and Normal QQ plot of Standardized Residuals of GJR-GARCH for DGS10.

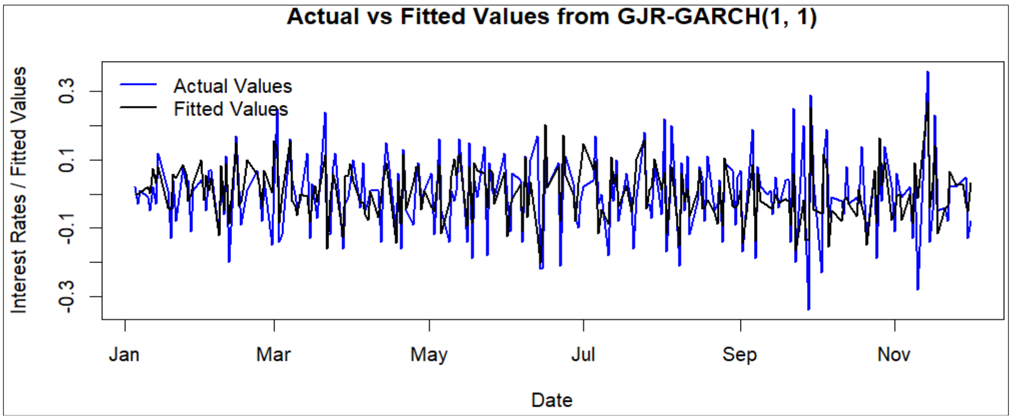

Fig 4: Plot of Actual versus fitted values of GJR-GARCH model for DGS10.

3.2. Optimal Policy and Sensitivity Analysis for long-term interest rates: The below plot illustrates pre-retirement wealth trajectories under different parameter settings: expected return (Alpha), volatility (Sigma) & risk aversion (Gamma). Each subplot corresponds to a unique combination of these parameters, allowing us to observe how different investment strategies and risk preferences affect pre-retirement wealth accumulation over 20 years.

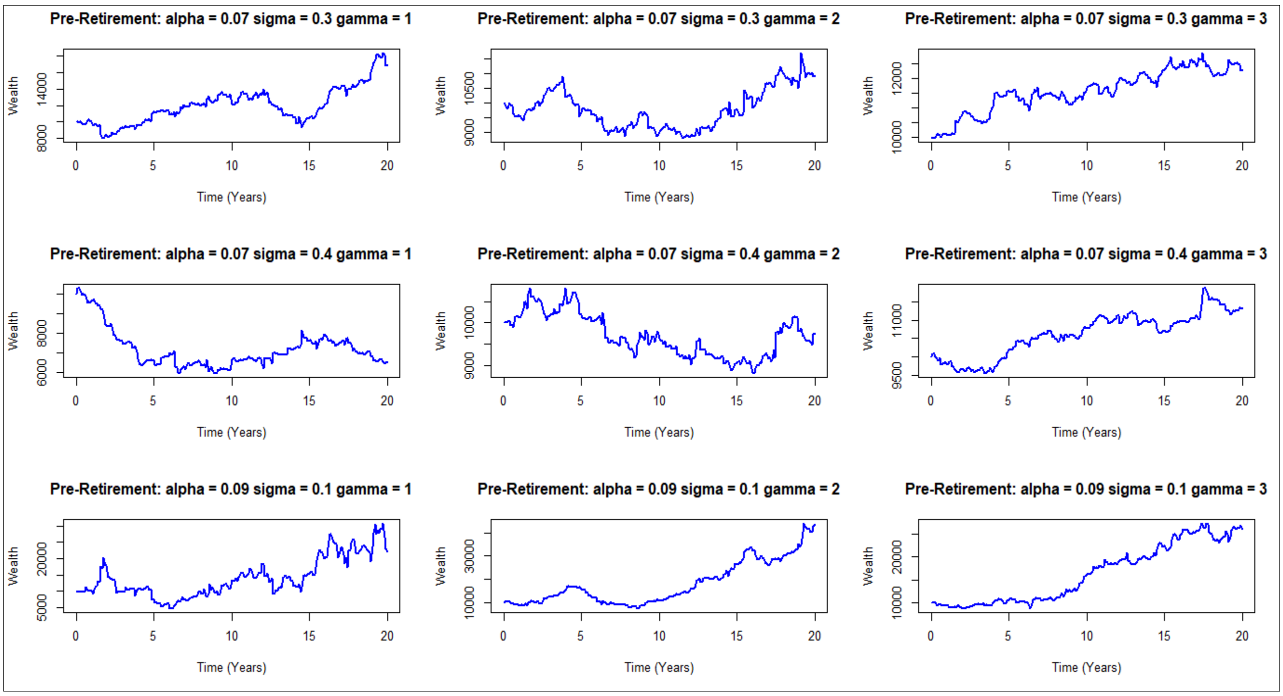

Fig 5: Sensitivity Analysis for Pre-Retirement Wealth

4. Key observations from Figure 5

- **Effect of Alpha (Expected Return):** The last row corresponds to α = 0.09, while the other rows use α = 0.07. Higher α (0.09) results in higher final wealth, indicating that increasing the expected return leads to better long-term wealth accumulation.
- **Effect of Sigma (Volatility):** The middle row (σ = 0.4) shows greater fluctuations compared to the top row (σ = 0.3) and bottom row (σ = 0.1). Higher σ (volatility) leads

to greater wealth variations, with some scenarios even experiencing a decline in wealth over time (middle row, leftmost plot). The bottom row ($\sigma = 0.1$) exhibits steadier growth, showing that reducing volatility smooths out the wealth trajectory.

- **Effect of Gamma (Risk Aversion):** $\gamma = 1$ (leftmost) leads to higher average wealth accumulation but comes with higher fluctuations. $\gamma = 2$ (middle) shows moderate fluctuations and more controlled growth. $\gamma = 3$ (rightmost) has the least fluctuation, with smoother but lower wealth growth. Higher γ leads to more conservative investment strategies, resulting in lower but more stable wealth growth.

5. Conclusion and Recommendation

Both EGARCH and GJR-GARCH models perform well for the long-term stochastic interest rates. When compared selection criterion, of EGARCH and GJR-GARCH, loglikelihood was maximum for EGARCH, the EGARCH model will be chosen to model the optimal investment policy for the long-term stochastic interest rates.

A moderate risk aversion strategy ($\gamma = 2$) with controlled exposure to market volatility (σ between 0.15-0.25) and a return rate of $\alpha = 0.07$ or higher appears to be the most effective approach for balancing growth and sustainability in a defined contribution pension plan. Overly conservative strategies may lead to wealth depletion, while excessive risk-taking could cause major losses. A dynamic investment allocation that transitions from high-risk to low-risk assets over time is recommended to ensure financial security pre-retirement.

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