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The product of two Hypergeometric Functions ${}_1F_1(x)$: Few special cases

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Abstract
The hypergeometric function is a special function denoted as ${}_2F_1$ (a, b; c; z) arising in many areas such as probability, combinatorics, and mathematical physics. It generalizes the geometric series and satisfies a second-order linear differential equation. Its power series converges for $|z|<1$, with extensions possible through analytic continuation. Products of hypergeometric functions appear in advanced mathematical analysis, often in the context of summation identities, integral transforms, or representation theory. These products can reveal deeper symmetries and relationships between special functions and are instrumental in solving complex problems in both pure and applied mathematics. Drawing on the works of Poudel *et al*., this work aims to present special cases of the several product formulas involving two generalized hypergeometric functions.

Keywords: Hypergeometric function, generalized hypergeometric series, Bailey's identity

1. Introduction

The hypergeometric functions are the extension of the geometric series [10]. The generalized hypergeometric function ${}_pF_q$ having p number of numerator and q number of denominator parameters is defined by [13] is given by

$$ {}_pF_q\left[\begin{matrix} h_1,\dots, h_p \\ k_1,\dots, k_q \end{matrix};\; z\right] = \sum_{n=0}^{\infty} \frac{(h_1)_n \dots (h_p)_n}{(k_1)_n \dots (k_q)_n} \frac{z^n}{n!} \qquad (1.1) $$

Where,

$(\lambda)_n$ denotes the Pochhammer symbol commonly represented in terms of Gamma function defined by

$$ (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} $$

$$ = \begin{cases} 1 & (n=0; \lambda \in C \setminus \{0\}) \\ \lambda(\lambda+1)(\lambda+2)\dots\dots(\lambda+n-1) & (n \in N, \lambda \in C) \end{cases} \qquad (1.2) $$

Kummer has obtained the solutions of hypergeometric differential equations by using the theory of differential equations, [4, 7, 8]. Further he obtained the following results [3, 4, 14].

$$ e^{-\frac{x}{2}}\, {}_1F_1\left[\begin{matrix} \alpha \\ 2\alpha \end{matrix};\; x\right] = {}_0F_1\left[\begin{matrix} - \\ \alpha+\frac{1}{2} \end{matrix};\; \frac{x^2}{16}\right] \qquad (1.3) $$

Bailey [1] has derived the product of two ${}_0F_1$ functions as the identity given below;

$$ {}_0F_1\left[\begin{matrix} - \\ \alpha \end{matrix} ; x\right] {}_0F_1\left[\begin{matrix} - \\ \beta \end{matrix} ; x\right] = {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta), \quad \frac{1}{2}(\alpha+\beta-1) \\ \alpha, \quad \beta, \quad \alpha+\beta-1 \end{matrix} ; 4x\right] \tag{1.4} $$

Several studies are done in the product of generalized hypergeometric functions [1, 6, 11, 12]. Very recently Poudel *et al.* [12] has derived a formula given below in the product of two hypergeometric function.

$$ \begin{aligned} &{}_1F_1\left[\begin{matrix} \alpha \\ 2\alpha+2 \end{matrix} ; x\right] {}_1F_1\left[\begin{matrix} \beta \\ 2\beta+2 \end{matrix} ; x\right] \\ &= e^x \Bigg\{ {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+1), \quad \frac{1}{2}(\alpha+\beta) \\ \alpha+\frac{1}{2}, \quad \beta+\frac{1}{2}, \quad \alpha+\beta \end{matrix} ; \frac{x^2}{4}\right] - \frac{x}{2}\Bigg\{ \frac{1}{(\beta+1)} {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+2), \quad \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\frac{1}{2}, \quad \beta+\frac{3}{2}, \quad \alpha+\beta+1 \end{matrix} ; \frac{x^2}{4}\right] \\ &+ \frac{1}{2(\alpha+1)} {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+2), \quad \frac{1}{2}(\alpha+\beta+1) \\ \alpha+\frac{3}{2}, \quad \beta+\frac{1}{2}, \quad \alpha+\beta+1 \end{matrix} ; \frac{x^2}{4}\right]\Bigg\} \\ &+ \frac{x^2}{4}\Bigg\{ \frac{\beta}{(\beta+1)(2\beta+1)(2\beta+3)} {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+3), \quad \frac{1}{2}(\alpha+\beta+2) \\ \alpha+\frac{1}{2}, \quad \beta+\frac{5}{2}, \quad \alpha+\beta+2 \end{matrix} ; \frac{x^2}{4}\right] \\ &+ \frac{\alpha}{(\alpha+1)(2\alpha+1)(2\beta+3)} {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+3), \quad \frac{1}{2}(\alpha+\beta+2) \\ \alpha+\frac{5}{2}, \quad \beta+\frac{1}{2}, \quad \alpha+\beta+2 \end{matrix} ; \frac{x^2}{4}\right] \\ &+ \frac{1}{(\alpha+1)(\beta+1)} {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+3), \quad \frac{1}{2}(\alpha+\beta+2) \\ \alpha+\frac{3}{2}, \quad \beta+\frac{3}{2}, \quad \alpha+\beta+2 \end{matrix} ; \frac{x^2}{4}\right]\Bigg\} \\ &- \frac{x^3}{8(\alpha+1)(\beta+1)}\Bigg\{ \frac{\beta}{(2\beta+1)(2\beta+3)} {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+4), \quad \frac{1}{2}(\alpha+\beta+3) \\ \alpha+\frac{3}{2}, \quad \beta+\frac{5}{2}, \quad \alpha+\beta+3 \end{matrix} ; \frac{x^2}{4}\right] \\ &+ \frac{\alpha}{(2\alpha+1)(2\alpha+3)} {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+4), \quad \frac{1}{2}(\alpha+\beta+3) \\ \alpha+\frac{5}{2}, \quad \beta+\frac{3}{2}, \quad \alpha+\beta+3 \end{matrix} ; \frac{x^2}{4}\right]\Bigg\} \\ &+ \frac{\alpha\beta x^4}{16(\alpha+1)(2\alpha+1)(2\alpha+3)(\beta+1)(2\beta+1)(2\beta+3)} {}_2F_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+5), \quad \frac{1}{2}(\alpha+\beta+4) \\ \alpha+\frac{5}{2}, \quad \beta+\frac{5}{2}, \quad \alpha+\beta+4 \end{matrix} ; \frac{x^2}{4}\right]\Bigg\} \end{aligned} \tag{1.5} $$

2. Main Results

In this section, the special case of the above formula (1.5) will be evaluated. For this, the values of $\alpha = 1,2,3,4$, $\beta=1,2,3,4$ and x = 1 will be taken in (1.5) and the results of the special cases are obtained as follows; The results will be in terms of exponential function (in the form of ‘e’)

$$ \left[\begin{matrix} 1 \\ 4 \end{matrix} ;1\right] \left[\begin{matrix} 1 \\ 4 \end{matrix} ;1\right] = 9(4e^2 - 20e + 25) \tag{2.1} $$

$$\begin{bmatrix}1 & \\ 4 & ;1\end{bmatrix}\begin{bmatrix}2 & \\ 6 & ;1\end{bmatrix} = -60(36e^2 - 188e + 245) \qquad (2.2)$$

$$\begin{bmatrix}1 & \\ 4 & ;1\end{bmatrix}\begin{bmatrix}3 & \\ 8 & ;1\end{bmatrix} = 630(504e^2 - 2630e + 3425) \qquad (2.3)$$

$$\begin{bmatrix}1 & \\ 4 & ;1\end{bmatrix}\begin{bmatrix}4 & \\ 10 & ;1\end{bmatrix} = -9072(9100e^2 - 47432e + 61705) \qquad (2.4)$$

$$\begin{bmatrix}2 & \\ 6 & ;1\end{bmatrix}\begin{bmatrix}2 & \\ 6 & ;1\end{bmatrix} = 400(324e^2 - 1764e + 2401) \qquad (2.5)$$

$$\begin{bmatrix}2 & \\ 6 & ;1\end{bmatrix}\begin{bmatrix}3 & \\ 8 & ;1\end{bmatrix} = -4200(4536e^2 - 24678e + 33565) \qquad (2.6)$$

$$\begin{bmatrix}2 & \\ 6 & ;1\end{bmatrix}\begin{bmatrix}4 & \\ 10 & ;1\end{bmatrix} = 60480(81900e^2 - 445088e + 604709) \qquad (2.7)$$

$$\begin{bmatrix}3 & \\ 8 & ;1\end{bmatrix}\begin{bmatrix}3 & \\ 8 & ;1\end{bmatrix} = 44100(53504e^2 - 345240e + 469225) \qquad (2.8)$$

$$\begin{bmatrix}3 & \\ 8 & ;1\end{bmatrix}\begin{bmatrix}4 & \\ 10 & ;1\end{bmatrix} = 635040(1146600e^2 - 6226682e + 8453585) \qquad (2.9)$$

$$\begin{bmatrix}4 & \\ 10 & ;1\end{bmatrix}\begin{bmatrix}4 & \\ 10 & ;1\end{bmatrix} = 9144576(20702500e^2 - 112303100e + 152300281) \qquad (2.10)$$

3. Conclusion

In this paper, we have established the results on product of two generalized hypergeometric functions as an special cases for ${}_1F_1\left[\begin{matrix}\alpha \\ 2\alpha+n\end{matrix}\,;\; x\right]\cdot{}_1F_1\left[\begin{matrix}\beta \\ 2\beta+m\end{matrix}\,;\; x\right]$ for $n, m = 1,2,3,4$ which are in exponential form. These results may be useful in mathematics, engineering and some branches of sciences.

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