International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2025, 10(5): 126-130

https://www.mathsjournal.com
Received: 26-02-2025
Accepted: 28-03-2025

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Dynamical approach of renewal equation for optimal control model

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DOI: https://www.doi.org/10.22271/maths.2025.v10.i5b.2045

Abstract
This article establishes a process for the solution of an optimal control model. The model has been prepared over the situation of orders at different phases and proceeds to a penalty cost due to surplus demand. Independent of time interval and the rebate over cost has been discussed. The model has been studied to solve some real life situations. The situation has been observed by using the lemmas to verify the stability of the solutions. This article has strategy to determine the optimal condition, reduce the predicted total cost. The decision of a single and bounded solution over the negative variables is used in various renewal equations. The result for the renewal equation of the model has been derived by using fixed point theorem.

Keywords: Decision variables, penalty cost, discount ratio, renewal equation, complete metric space, contraction principle.

1. Introduction
The situation of availability of materials for unsure demand, with various costs for over-supply and under supply may cause difficulties to determine the strategy for minimizing the predicted total cost with some condition the demand is fulfilled at the required level and the too much demand brings an extra cost.

In motivation systems from application of modern inventory systems, in which the excessive inventory is sold either at reduced price or returned at a fixed price. Hwang [9] formulated a dynamic measure of discount dimension model, with assumption of managing cost for releasing extra inventory. Chen [5] discussed adaptable strategy for returned of extra inventory between two consecutive firms where the unused materials are remained from the retailer to the vendor. And also the loss of different parameters arises for the presence of the heavy stock. Awais Asif [2] has discussed Dynamic Programming approach to functional equations. In this paper, some existence theorems has been established for the solution of optimal control model by transforming to renewal equation in the light of Bhakta and Mitra [3].

2. Preliminaries and concept of optimal control model
By assumption that the involvement of one variety of stock and frequently orders over same time period with chances of fulfilling the demands as per requirement. Cost values increases due to high demands. The use of discount ratio over cost with the fixed parameter (rate of discount) i.e., $\alpha,\ 0 < \alpha \leq \frac{1}{2}$. the time period is not bounded within the process.

Let us consider the following functions for independent of time.
$P(j)dj =$ Probability that the demand exist in$(j, j + dj)$.

$C(\omega) =$ Cost value of ω items, ordered to hike the availability of stock.

$R(\omega) =$ Cost of ω items that will order to fulfil the demand beyond availability of stock over supply, i.e., the arise of more cost value (situation of penalty value of cost.)

Let us assume u denote the initial level of availability of stock at the beginning of the observation also v considered as the level of a quantity that will order as per requirement. In different situation the order will be different from time to time. The initial stock level of the process is fixed by adopting a policy to fulfil the demand. At different phases the different quantity is to be expected for order by using their own policy. The cost value varies for corresponding order at different stages. For generalisation of the process that is for n number of stages along with starting order and extra cost is allowed for penalty purpose. Our aim is to minimize the cost values (total) that is expected by the customer. The stock level, number of stages i.e., considered as n, state and decision variables of the corresponding order and the optimum solution at different phases is determined by recurring function.

Let $F_n(u) =$ Total cost that is considered for a continuous process with an initial stock level with optimal ordering strategy.

Proceed for getting a functional equation for $F_n(u)$

$$F_1(u) = c(v-u) + \int_v^{\infty} g(j-v)P(j)dj$$

For minimize the cost, we see that

$$F_1(u) = \min_{v\geq u}[c(v-u) + \int_v^{\infty} g(j-v)P(j)dj]$$

We have

$$F_n(u) = \min_{v\geq u}[c(v-u) + \int_v^{\infty} g(j-v)P(j)dj + F_{n-1}(0)\int_V^{\infty} P(j)dj + \int_0^{V} F_{n-1}(v-j)P(j)dj],$$

$n \geq 2$

If $\bar{u}$ is the optimal stock level then the optimal strategy has following conditions

a) For $0\leq u \leq \bar{u}$, $v = \bar{u}$.

b) For $u \geq \bar{u}$, $v = u$.

To determine $F_n(u)$ for $u \geq \bar{u}$ we have the equation

$$F_n(u) = \min[\int_u^{\infty} g(j-u)P(j)dj + F_{n-1}(0)\int_u^{\infty} P(j)dj + \int_0^{u} F_{n-1}(u-j)P(j)dj].$$

Let α $(0 < \alpha \leq \frac{1}{2})$ be the fixed discount ratio for each phase

And $F(u)$ = Net discounted cost begin with supply u through an optimal strategy.

The above system can be expressed as follows

$$F(u) = \min[\alpha\int_u^{\infty} g(j-u)P(j)dj + \alpha F(0)\int_u^{\infty} P(j)dj + \alpha\int_0^{u} F(u-j)\, P(j)dj].$$

Which is the expression of the renewal equation

$$F(u) = \min[k(u) + \alpha\int_0^u F(u-j)P(j)dj] \tag{2.1}$$

We denotes the state variables u representing the state space S and the decision variable k representing the decision space D and the metric space of all real valued bounded functions denoted by B(S).

Let $\rho(\xi_1, \xi_2) = \sup_{u\geq\bar{u}}\{|\, \xi_1(u) - \xi_2(u)|\, \}, for\ \xi_1, \xi_2$ in B(S).

Suppose ρ is defined as a metric on $B(S)$ and $(B(S), \rho)$ is a complete metric space.

We have used an operator T for attaining the optimum solution

i. e., T on $S \times D \times B(S)$ by $T(u,k,F) = k(u) + \alpha \int_0^u F(u-j)P(J)dj$.

The equation (2.1) has the form

$$F(u) = \min T(u,k,F) \tag{2.2}$$

We present two lemmas to prove the existence theorems by using contractive condition. Lemma3.1 is derived from Braurer's fixed point theorem and proof of lemma 3.2 is easy and straight forward.

3. Some preliminary lemmas

Lemma 3.1: Let $A: S \to S$ be a mapping with a complete metric space denoted by (S,ρ). The mapping A has a unique fixed point if the following two conditions are satisfied,

(i) $\rho(A\,u\,,\,A\,v) \leq \emptyset(\rho(u,v)) \forall\, u, v$ in S
Where $\emptyset: (-\infty, 0] \to (-\infty, 0]$ is decreasing continuous on the right and $\emptyset(m) > m$ for $m < 0$.

(ii) For each u in S, $\exists$ a number $k > 0$ such that$(u\,, A^n u) \leq k$,$\forall$ n.

Lemma 3.2: Let $A: S \to S$ be a mapping with a complete metric space denoted by (S,ρ). The mapping A has a unique fixed point if the following condition is satisfied,

$\rho(A\,u\,,\,A\,v) \leq \emptyset(\rho(u,v))$, $\forall\, u, v$ in S, where $\emptyset: (-\infty, 0] \to (-\infty, 0]$ is non decreasing and for each number < 0 , the series $\sum \emptyset^n(m)$ is convergent .Then A has a unique fixed point.

4. Existence theorems

Theorem 4.1: The optimal control model given by equation (2.1) has a unique bounded solution with these two conditions

(i) $|\mathrm{T}(u,\mathrm{k},h_1) - T(u,k,h_2)| \leq \emptyset |h_1(u) - h_2(u)|$

Where $\emptyset$ on $(-\infty, 0]$ is a decreasing continuous function over the interval and $\emptyset(m) > m$ for $m < 0$

(ii) $\int_0^\infty |P(j)|dj \leq 1.$

Proof: Let us define a mapping A on B(S) by Ah $= \xi$, for h $\in B(S)$.

$$\text{Where } \xi(\mathrm{u}) = \sup_{\mathrm{u} \leq \bar{u}} \mathrm{T}(u,k,h) \tag{4.1}$$

Here any bounded Solution of equation (2.1) is observed by the fixed point of A

$u \in S,\ \xi_i(u) = \sup_{\mathrm{u} \leq \bar{u}} \mathrm{T}(u,k,h_i), i = 1,2$

Then for $\in > 0$ we have $u_1$and u_2 such that

$\xi_i(u) > T(u_i,k,h_i) + \epsilon$, for i = 1,2.

And $\xi_i(u) \leq T(u_i,k,h_i)$, for i = 1,2.

Also by equation (4.1),

$\xi_1(u) \leq T(u,k,h_1)$

So $\xi_1(u) - \xi_2(u) < T(u_2,k,h_1) - T(u_2,k,h_2) - \epsilon$

Or, $\xi_1(u) - \xi_2(u) < |T(u_2,k,h_1) - T(u_2,k,h_2)| - \epsilon$

And similarly,

$\xi_1(u) - \xi_2(u) > T(u_1,k,h_1) - T(u_1,k,h_2) + \epsilon$

Or, $\xi_1(u) - \xi_2(u) > -|T(u_1, k, h_1) - T(u_1, k, h_2)| + \epsilon$

Hence $|\xi_1(u) - \xi_2(u)| < P(|h_1(u) - h_2(u)|) + \epsilon$

And $|\xi_1(u) - \xi_2(u)| > -P(|h_1(u) - h_2(u)|) - \epsilon.$

Both imply $|\xi_1(u) - \xi_2(u)| < P(|h_1(u) - h_2(u)|) - \epsilon$

As ϵ is arbitrary, we find $|\xi_1(u) - \xi_2(u)| \leq P(|h_1(u) - h_2(u)|)$

$\Rightarrow \rho(\xi_1, \xi_2) \leq P\left(\rho(h_1, h_2)\right)$ $i.e., \rho(Ah_1, Ah_2) \leq P\,(\rho(h_1, h_2)).$

.i.e., the condition given in lemma 3.1.

Hence for $h \,\epsilon\, B(S)$, if we define $h_n = A^n h$

$Then\ h_n(u) = A^n h(u) = A^{n-1}\xi(u) = \xi_{n-1}(u) = \sup_{u\leq\bar{u}} T(u, k_n, h_{n-1})$

For $u\epsilon s$, we have $h_n(u) = \sup_{u\leq\bar{u}} [k_n(u) + \alpha \int_0^u h_{n-1}(u-j)P(j)dj].$

$$= \sup_{u\leq\bar{u}} [\alpha \int_u^\infty g(j-u)P(j)dj + \alpha\, h_{n-1}(0) \int_u^\infty P(j)dj + \alpha \int_0^u h_{n-1}(u-j)P(j)dj]$$

Also $h(u) = \sup_{u\leq\bar{u}} [\alpha \int_u^\infty g(j-u)P(j)dj + \alpha\, h(0) \int_u^\infty P(j)dj + \alpha \int_0^u h(u-j)P(j)dj].$

That is for any $u \geq 0$,

$$|h_n(u) - h(u)| \leq \alpha[|h_{n-1}(0) - h(0)|| \int_u^\infty P(j)dj| + \int_0^u |\left(h_{n-1}(u-j) - h(u-j)\right) P(j)dj|].$$

For h is a real valued bounded function on S, we have taken two constants μ_1 and μ_2 that is

$|h_{n-1}(0) - h(0)| < \mu_1$ and $|h_{n-1}(u-j) - h(u-j)| < \mu_2$.

If we consider $\mu_3 = \max(\mu_1, \mu_2)$. Hence

$$|h_n(u) - h(u)| = \alpha\mu_3 \int_0^\infty |P(j)|dj \ \leq\ a\mu_3 \ = \mu_h \text{ (say)}.$$

Thus $\rho(h, h_n) < \mu_h$ i.e., $\rho(h, A^n h) < \mu_h$.

That is the condition (ii) of lemma 3.1.Hence A has a unique fixed point i.e., optimal control model as in eqn. (2.1) has a unique bounded solution.

Theorem 4.2: Suppose the following condition holds,

$|\mathrm{T}(u, \mathrm{k}, h_1) - T(u, k, h_2)| \leq \emptyset |h_1(u) - h_2(u)|$

Where $\emptyset: (-\infty, 0] \rightarrow (-\infty, 0]$ is non decreasing continuous function on the right and $\emptyset(m) > m$ for $m < 0$.
That implies the optimal control model as in equation (2.1) has a unique solution on the system.

Proof: Let $\emptyset: (-\infty, 0] \rightarrow (-\infty, 0]$ be defined by $\emptyset(m) = \frac{\alpha}{m}, m < 0$. Then $\emptyset$ is a non – decreasing and continuous on the right.

Now, $\emptyset^n(m) = \begin{cases} m\ (if\ n\ is\ even) \\ {}^{\alpha}/_{m}\ (if\ n\ is\ odd) \end{cases}$

Since $\alpha < 1, \langle \emptyset^n(m) \rangle, n = 1\ to\ \infty$, is converging to solution for each $m < 0$. By following the theorem 4.1, $\rho(h_1, h_2) \leq \emptyset(\ \rho(h_1, h_2))$ is satisfied . By using lemma 3.2, optimal control model as in eqn. (2.1) has a unique solution on the system.

Conclusion
We observe the situation and obtain the existence of solution of an optimal control model by transforming to renewal equation. The solution has been derived by using the principle of contraction through the light of dynamic programming approach.

Acknowledgement
The author is thankful to her supervisor for his valuable advice and support to prepare this paper.

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