International Journal of Statistics and Applied Mathematics 2025; 10(5): 109-115

International Journal of Statistics and Applied Mathematics

**ISSN:** 2456-1452
Maths 2025; 10(5): 109-115

https://www.mathsjournal.com
Received: 16-03-2025
Accepted: 19-04-2025

**Jaufar Mousa Mohammed**
Department of Statistics Sciences, College of Administration & Economic, Kirkuk University, Iraq

**Corresponding Author:**
**Jaufar Mousa Mohammed**
Department of Statistics Sciences, College of Administration & Economic, Kirkuk University, Iraq

# Estimation multivariate fuzzy data in spatial statistics with application

**Jaufar Mousa Mohammed**

**DOI:** https://www.doi.org/10.22271/maths.2025.v10.i5b.2041

**Abstract**
This research addresses how to estimate unmeasured points for fuzzy spatial data when the number of its elements (spatial sample) is small, which is not preferred in the estimation process. As we know, when the data size is large, the estimation results for unmeasured points are better, and thus the estimation variance is lower. Therefore, the idea of this research is how to benefit from other secondary (auxiliary) fuzzy data that has a strong correlation with the primary (main) data for estimating one of its unmeasured points using the Co-kriging technique, after finding the central point of the fuzzy data values. This technique was applied to fuzzy data in the field of wheat cultivation in Iraq, where the production quantity was considered the primary data (primary variable) to estimate one of its unknown points, and the cultivated area (secondary variable) was used. Encouraging results were obtained.

**Keywords:** Co-kriging, multivariate random fields, cross- covariance functions, spatial statistics

## 1. Introduction

This research addresses a study on how to use secondary (auxiliary) variables to estimate primary (main) fuzzy variables that are spatially correlated with each other. One of the techniques used in this field is the Co-kriging technique. The nature of the variables dealt with in this method includes primary variables $\{z_1(x), x \in D\}$ and secondary variables $\{z_2(x), x \in D\}$, and these two variables can either be equal in number or different within the study area $D$. This method is used when the number of primary variables, for which the prediction of one of their unmeasured elements is desired, is small. There are three cases for using the Co-kriging method: simple (known fixed mean), ordinary (unknown fixed mean), and the case when (the means of both variables are equal and unknown). In this study, the second case was used, considering the (unknown fixed mean) in the study area $D$ to build predictions on real data in the field of wheat cultivation in Iraq. The wheat yield was considered the primary variable, and the cultivated area was considered the secondary variable. There was a strong correlation between these two variables with r = 0.879. The results were very encouraging, as they showed a significant similarity between the predicted values and the actual values.

## 2. Research objective

The objective of this research is to estimate the unknown values of data that are characterized by uncertainty and ambiguity in their values, as well as their small size, which is not suitable for estimation. This is achieved by relying on secondary (assisting) data using the Co-kriging technique.

## 3. Fuzzy set theory

In our real world, we encounter many groups, categories, or classifications where elements are classified according to a characteristic or criterion that is not clearly defined, making it difficult to determine the degree or extent of an element's membership whether it belongs or does not belong to that classification or group. This lack of clarity in these groups and classifications, and the phenomena that exhibit this feature, fall under the study of fuzzy set theory, which focuses on uncertainty and vagueness in elements and data. It is assumed that there exists a large set, known as the universal set, denoted by $X$, and that $\tilde{A}$ is a subset of it. If $x$ is an element chosen from the space of the set $X$, we aim to describe its membership in Ã, meaning determining whether $x$ belongs to the subset $\tilde{A}$ within the same space $X$. $\mu_{\tilde{A}}(x)$ is the membership function that measures the degree of membership of the element x in the fuzzy set $\tilde{A}$. Based on this, the fuzzy set is defined according to the following formulas.

$$\tilde{A} = \{x: \mu_{\tilde{A}}(x) > 0 \,, x \in X\}$$
$$\tilde{A} = \{(\mu_{\tilde{A}}(x_1), x_1), (\mu_{\tilde{A}}(x_2), x_2), \dots . (\mu_{\tilde{A}}(x_n), x_n)\} = \{(\mu_{\tilde{A}}(x), x), x \in X\}$$

It is noted that one of the most important topics related to fuzzy sets is the membership function, which allows for the visualization of the fuzzy set. The membership function is called so because it expresses the degree of membership of each element in the fuzzy set based on a certain characteristic or criterion that is not clearly defined (i.e., a fuzzy characteristic or criterion). The domain of this function is the set of real numbers, representing the values of the fuzzy set on the horizontal axis (x-axis), while its corresponding range is the closed interval [0,1] on the vertical axis (y-axis). If the degree of membership of an element equals one, this indicates a certain membership, while a degree of membership equal to zero means that the element does not belong to the set at all. When the membership function takes binary values, i.e., only providing the values zero and one to express whether an element belongs or does not belong to the set, the membership function indicates a crisp set. Membership functions provide a way to visualize the data of the fuzzy set and understand the significance of each value based on its degree of membership in the phenomenon being studied. There are many types of membership functions, and each is applied to a phenomenon that is suitable for it, where the data of the phenomenon consist of fuzzy sets that are appropriate for the chosen membership function. One of the membership functions is the triangular membership function, which is widely used and has three basic parameters $a_1, a_2, a_3$ It can be expressed in the following form:

$$\mu_{\tilde{A}}(x) = \begin{bmatrix} \frac{x-a_1}{a_2-a_1} \; if \; a_1 \le x \le a_2 \\ \\ \frac{a_3-x}{a_3-a_2} \; if \; a_2 \le x \le a_3 \\ \\ 0 \; other \; wise \end{bmatrix} \qquad \dots (1)$$

Where $a_1, a_2, a_3$ represent the parameters of the left limit, the middle limit, and the right limit, respectively. The triangular membership function can be graphically illustrated as shown in the figure [7].

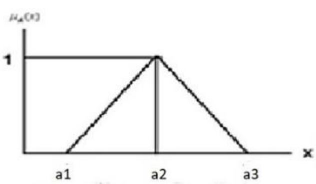


**Fig 1:** Triangular fuzzy number

In most cases, a researcher or decision-maker needs to determine a reasonable level of fuzziness for the data or fuzzy sets at hand. Therefore, they resort to the concepts of the cut level and the cut level $\alpha$ set. The cut level $\alpha$ is a fixed value that lies within the interval $\alpha \in [0,1]$, and it is a chosen value from the vertical axis of the membership function. The value of α is represented by a constant function (i.e., a horizontal straight line) that divides the membership function into two parts (upper and lower). It controls the boundaries of the possible fuzzy numbers that the sample points fall into, i.e., it controls the boundaries of the fuzzy number (the upper and lower limits). This is illustrated in the figure.

The α-cut set is a crisp set whose membership function is the upper part of the membership function that is cut by the cut level α. Thus, the fuzzy set after the α-cut can be expressed as follows:

$$\tilde{A}_{\alpha} = \{\, \mu_A \ge \alpha \,, x \in X : \} \,, \alpha \in [0,1]$$

Formula (7) is considered the general form for any fuzzy set, since α lies within the interval [0,1]. Formula (5) is a special case of formula (7) above when substituting $\alpha = 0$ into it.

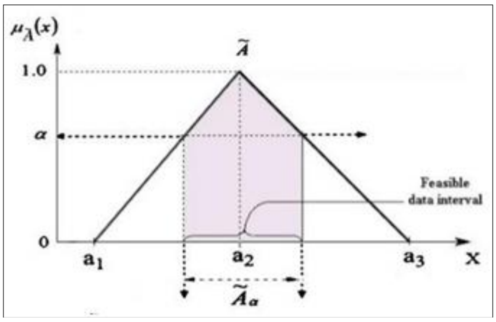


**Fig 2:** $\alpha - cut$

It is worth mentioning that most fuzzy field applications use fuzzy numbers. A fuzzy number is a fuzzy subset of the real number line (the set of real numbers), and this fuzzy set is convex and normalized. Fuzzy numbers are the core of fuzzy computations, as they allow us to describe and deal with them more easily, in addition to enabling us to perform arithmetic operations on these numbers.

In figure (7), there are the two limits $a_{al}{}^{a}$ and $a_{au}$, which represent the upper and lower limits, respectively, of the fuzzy number $\tilde{A}$ after the $\alpha - cut$. These limits can be expressed by the following formulas [8].

$$a_{\alpha}^{l} = (1-\alpha)a_1 + \alpha a_2 \,, a_{\alpha}^{u} = (1-\alpha)a_3 + \alpha a_2$$

It is important to mention that arithmetic operations can be performed on triangular fuzzy numbers. There is no doubt that the four arithmetic operations (addition, subtraction, multiplication, division) on fuzzy numbers differ from their implementation on crisp numbers (ordinary real numbers) due to the nature of the fuzzy number in its composition. The basic arithmetic operations are performed on the fuzzy numbers after they have been $\alpha - cut$, where the four basic operations are carried out on the upper and lower limits of the fuzzy numbers for the cut level $\alpha$. The calculations for these four basic operations are carried out as follows:

Assume that:

$$\tilde{A} = (a^l, a^m, a^u), \tilde{B} = (b^l, b^m, b^u)$$

are two fuzzy numbers, then the limits of the fuzzy numbers after the $\alpha - cut$ will be:

| $Operation$ | $\alpha - cat\ Arthmetic\ results$ |
| --- | --- |
| $\tilde{A}\oplus\tilde{B}$ | $[a_\alpha^l + b_\alpha^l\ , a_\alpha^u + b_\alpha^u]$ |
| $\tilde{A}\ominus\tilde{B}$ | $[a_\alpha^l - b_\alpha^u\ , a_\alpha^u - b_\alpha^l]$ |
| $\tilde{A}\otimes\tilde{B}$ | $\{min[a_\alpha^l b_\alpha^l\ , a_\alpha^l\ b_\alpha^u, a_\alpha^u b_\alpha^l\ , a_\alpha^u b_\alpha^u]\ , max\ [a_\alpha^l b_\alpha^l\ , a_\alpha^l\ b_\alpha^u, a_\alpha^u b_\alpha^l\ , a_\alpha^u b_\alpha^u]\}$ |
| $\tilde{A}\oslash\tilde{B}$ | $\{min[a_\alpha^l/b_\alpha^l\ , a_\alpha^l\ /b_\alpha^u, a_\alpha^u/b_\alpha^l\ , a_\alpha^u/b_\alpha^u]\ , max\ [a_\alpha^l/b_\alpha^l\ , a_\alpha^l\ /b_\alpha^u, a_\alpha^u/b_\alpha^l\ , a_\alpha^u/b_\alpha^u]\}$ |

Additionally, $\tilde{A}$ can be converted into a crisp number using the centroid method. This process is called defuzzification. The centroid method is expressed by the following formula:

$$\tilde{A}_C = \frac{\int x\mu_A(x)dx}{\int \mu_A(x)dx} = \frac{1}{3}(a^l + a^m + a^u) \qquad \dots (2)$$

This method, designed by M. Sugeno in 1985, is the most commonly used and is a very accurate technique. With this method, arithmetic operations can be performed with ease [7].

**4. Co-regionalization**

The variables dealt with in spatial statistics differ from those in conventional statistics, as each spatial variable $z(x)$ has coordinates $x = (u, v)$ if in the plane, or $x = (u, v, w)$ if in three-dimensional space. For all $\forall\ x \in D \subset R^p$ in the study domain, and $p = 1$ or $p = 2$, or $p = 3$, then a sample of size $n$ from the variables in $D$ will be $z(x_1), z(x_2), z(x_3), \dots\dots..z(x_n)$ at locations $x_1, x_2, x_3, \dots\dots\dots x_n$, separated by displacements. These displacements can be calculated using the Euclidean distance formula. These variables are called the sample function of the stochastic process $\{z(x), x \in D\}$ [3].

**5. The order-2 stationary multivariate model**

The random variable $z(x)$ is called a second-order realization if the expectation exists and is independent of the location $x$ , meaning that:

$$E[z(x)] = \mu\ , \forall x \in D$$

A pair of spatial random variables $[z(x), z(x + h)]$ has covariance that is defined and depends only on the displacement $h$ , meaning that [2, 10]

$$C(h) = E[z(x) - z(x+h) - \mu^2 = Cov[z(x), z(x+h)]\ , \forall x\ , x+h \in D$$

**6. Cross-variogram**

The variogram function has been developed into an equation known as the cross-variogram function, where this function deals with the values of spatial random variables with different distributions, meaning that [1]

$$\gamma(\mathrm{h})_{z_1z_2} = \frac{1}{2\mathrm{n(h)}}\sum_{\mathrm{i=1}}^{\mathrm{n(h)}}[z_1(x_i) - z_1(x_i+h)][z_2(x_i) - z_1(x_i+h)] \qquad \dots (3)$$

Where:

- $n(h)$ is the number of observation pairs at locations with displacement $h$,
- $z_1(x_i)$are the values of the primary variables at locations $x_i$, where $i = 1,2,3 \dots\dots n$,
- $z_2(x_i)$are the values of the secondary variables at locations $x_i$, where $i = 1,2,3 \dots\dots n$.

There is a relationship that links the variogram, covariance, and variance, which can be obtained under the assumption of stationarity, and it is:

$$\gamma_{z_1z_2}(h) = C_{z_1z_2} - \frac{1}{2}[C_{z_1z_2}(-h) + C_{z_1z_2}(h)] = C_{z_1z_2}(0) - C_{z_1z_2}(h) \qquad \dots (4)$$

Where $C_{z_1z_2}(0) = \sigma^2$ , $C_{z_1z_2}(-h) = C_{z_1z_2}(h)$

The equation (8) can be used to derive the mathematical form of the cross-covariance function, which is in the form:

$$C_{z_1z_2}(h) = \begin{bmatrix} \psi_0 + \psi\ h = 0 \\ \psi\left[1 - \frac{3h}{2a} + \frac{1}{2}\left(\frac{h}{a}\right)^3\right]\ 0 < h \le a \\ 0\ h > a \end{bmatrix} \qquad \dots (5)$$

Where

$$\gamma(h) = \begin{bmatrix} \psi_0\ h = 0 \\ \psi_0 + \psi\left[\frac{3h}{2a} - \frac{1}{2}\left(\frac{h}{a}\right)^3\right]\ 0 < h \le a \\ \psi_0 + \psi\ h > a \end{bmatrix} \qquad \dots (6)$$

The reason for studying the variogram function is that its mathematical form represents the variance of the differences between spatial observations that are separated by a displacement $h$. As the displacement $h$ between the observations increases, the covariance becomes larger until it stabilizes at a certain distance, such as $h = a$. This distance $a$ is called the range. After this point, the covariance in the variogram function diminishes and stabilizes at a constant value equal to the variance $\sigma^2$of the observations. This variance is called the Sill, as shown in Figure (1).

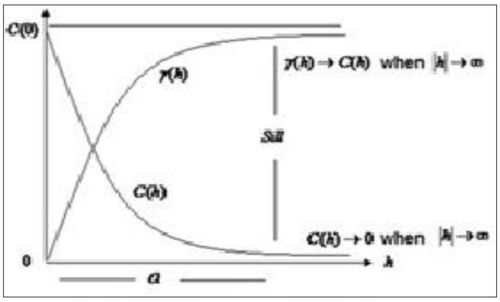


**Fig 3:** Variogram and covariance

When $h$ approaches zero from the right, the semi-variogram function does not equal zero but instead has a value equal to $\psi_0$. This phenomenon represents a discontinuity or a break in the variogram function at $h = 0$ , and it is referred to in spatial statistics as the Nugget Effect. It represents random measurement errors that occur when the displacement $h$ suddenly changes from millimeter units to meter or kilometer units [9].

**7. Co-kriging**

The ordinary Co-kriging estimator (OCK) is a linear combination of both primary and secondary variables of the data values. In this case, we also assume (stationarity) where the mean is constant but unknown in the study area $D$ . Therefore, we need to estimate the mean. The ordinary Co-kriging estimator for the point $z(x_0)$ at location $x_0$ is given by the following expression:

$$\hat{z}(x_0) = \sum_{i=1}^{n} a_i z_1(x_i) + \sum_{j=1}^{m} b_j z_2(x_j) \qquad \dots (7)$$

Conditions for unbiasedness of this estimator:

$$E[R] = 0 = E[\hat{z}_0 - z_0] = E[\sum_{i=1}^{n} a_i z_1(x_i) + \sum_{j=1}^{m} b_j z_2(x_j) - z_0]$$
$$= \sum_{i=1}^{n} a_i\, E[z_{1i}] + \sum_{j=1}^{m} b_j\, E[z_{2j}] - E[z_0] = m_1 \sum_{i=1}^{n} a_i + m_2 \sum_{j=1}^{m} b_j - m_1$$

Therefore, the condition of unbiasedness is satisfied under the following constraints:

$$\sum_{i=1}^{n} a_i = 1\ , \sum_{j=1}^{m} b_j = 0$$

To reduce the variance of the estimation error in (11) under the above constraints of unbiasedness, the expression is as follows:

$$w = var[R] + 2\mu_1(\sum_{i=1}^{n} a_i - 1) + 2\mu_2(\sum_{j=1}^{m} b_j) \qquad \dots (8)$$

We take the partial derivative of the above expression (8) with respect to $a_i$ ،$b_j$ ، $\mu_1$ ، $\mu_2$ and set them equal to zero. This results in $(n + m + 2)$equations of Co-kriging, which are as follows:

$$\begin{cases} \sum_{j=1}^{m} a_j C(z_{1i} z_{1j}) + \sum_{j=1}^{m} b_j\, C(z_{2i} z_{1j}) + \mu_1 = C(z_{1i} z_0)\ i = 1,2, \dots n \\ \sum_{i=1}^{n} b_i C(z_{2i} z_{2j}) + \sum_{1=1}^{n} a_i\, C(z_{1i} z_{2j}) + \mu_2 = C(z_{2j} z_0)\ j = 1,2, \dots m \\ \sum_{i=1}^{n} a_i = 1 \\ \sum_{j=1}^{m} b_j = 0 \end{cases}$$

Where $\mu_1$ ، $\mu_2$ Lagrange multipliers , and In brief [4, 11]

$$\begin{bmatrix} C_{z_1z_1} & C_{z_1z_2} & . & 1 & 0 \\ C_{z_2z_1} & C_{z_2z_2} & . & 0 & 1 \\ . & . & . & . & . \\ 1 & 0 & . & 0 & 0 \\ 0 & 1 & . & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ . \\ \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} C_{z_1z_0} \\ C_{z_2z_0} \\ . \\ 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} C_{z_1z_1} & C_{z_1z_2} & . & 1 & 0 \\ C_{z_2z_1} & C_{z_2z_2} & . & 0 & 1 \\ . & . & . & . & . \\ 1 & 0 & . & 0 & 0 \\ 0 & 1 & . & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} C_{z_1z_0} \\ C_{z_2z_0} \\ . \\ 1 \\ 0 \end{bmatrix}, \lambda = \begin{bmatrix} a \\ b \\ . \\ \mu_1 \\ \mu_2 \end{bmatrix}$$

then, we find the inverse of $C$ in order to obtain the weights $\lambda$ from the following relation:

$$\lambda = C^{-1}B \qquad \dots (9)$$

And the variance of ordinary Co-kriging is given by the following expression:

$$\sigma^2_{OCK} = C(z_0 z_0) - \sum_{i=1}^{n} a_i C(z_{1i} z_0) - \sum_{j=1}^{m} b_j C(z_{1j} z_0) - \mu_1 \qquad \dots (10)$$

## 8. Fuzzy spatial variable

If we assume that the available primary and secondary spatial data are fuzzy numbers of the triangular type with a membership function, as in equation (1), dealing with such numbers is somewhat complex. Therefore, there are fuzzy logic methods that can be used to simplify the process of handling them, such as finding the centroid of the fuzzy numbers for the primary data. This would transform the data into the form $z_{1C}(x_1), z_{1C}(x_2), \dots \dots . . z_{1C}(x_n)$ at fixed locations $x_1, x_2, \dots \dots \dots \dots . x_n$ and the secondary data into $z_{2C}(x_1)\ , z_{2C}(x_2), \dots \dots . . z_{2C}(x_m)$ at locations $x_1, x_2, \dots \dots \dots \dots . x_m$. Such data is called fuzzy data with fragile values, meaning the fuzziness has been removed using the formula [6].

## 9. Ordinary fuzzy co-kriging method

After converting the data from fuzzy to crisp data using the midpoint method, the linear combination of Co-kriging can be written as:

$$\hat{z}_C(x_0) = \sum_{i=1}^{n} a_i z_{1C}(x_i) + \sum_{j=1}^{m} b_j z_{2C}(x_j) \qquad \dots (11)$$

$$\sum_{i=1}^{n} a_i = 1\ and\ \sum_{j=1}^{m} b_j = 0$$

Where we follow the same method as in item (7) in order to obtain the Co-kriging matrix, but in the crisp form.

$$\begin{cases} \sum_{j=1}^{m} a_j C(z_{1Ci} z_{1Cj}) + \sum_{j=1}^{m} b_j\, C(z_{2Ci} z_{1Cj}) + \mu_1 = C(z_{1Ci} z_0)\ i = 1,2, \dots n \\ \sum_{i=1}^{n} b_i C(z_{2Ci} z_{2Cj}) + \sum_{1=1}^{n} a_i\, C(z_{1Ci} z_{2Cj}) + \mu_2 = C(z_{2Cj} z_0)\ j = 1,2, \dots m \\ \sum_{i=1}^{n} a_i = 1 \\ \sum_{j=1}^{m} b_j = 0 \end{cases}$$

Then, we obtain the weights as before from...

$$\lambda = C^{-1}B \qquad \dots (12)$$

And the Ordinary Co-kriging variance is given by the following form:

$$\sigma^2_{OCK} = C(z_0 z_0) - \sum_{i=1}^{n} a_i C(z_{1Ci} z_0) - \sum_{j=1}^{m} b_j C(z_{1Cj} z_0) - \mu_1 \qquad \dots (13)$$

## 10. Application

### a) Data description

This research was applied to real data in the field of wheat cultivation in Iraq. The data were obtained from the Annual Statistical Bulletin issued by the Ministry of Planning / Central Statistical Organization. As we can see from the data shown in Table (1), it consists of (15) governorates (locations), each with coordinates $u(x)$ representing East-West and $v(x)$ representing North-South. In addition, there are two variables for each location. The first variable is the wheat yield $z_1(x)$, which represents the primary (core) variables, and we aim to estimate one of its points. The second variable, $z_2(x)$,, represents the area of cultivated land, which is considered a secondary (auxiliary) variable. This secondary variable had a strong correlation with the first, with a correlation coefficient $r = 0.879$.

**Table 1:** Iraqi governorates with their coordinates and primary and secondary variables

| Governorates | $z_1(x)$ | $z_2(x)$ | $u(x)$ | $v(x)$ |
|---|---|---|---|---|
| Nineveh | 1417208 | 2700326 | 30 | 62 |
| Kirkuk | 627324 | 653295 | 38 | 54 |
| Diyala | 571748 | 680606 | 40 | 40 |
| Anbar | 285088 | 493942 | 30 | 38 |
| Baghdad | 133831 | 155683 | 39 | 39 |
| Babylon | 295465 | 300720 | 37 | 28 |
| Karbala | 98040 | 103879 | 35 | 30 |
| Wasit | 811384 | 947374 | 48 | 30 |
| Salahuddin | 633101 | 748189 | 33 | 47 |
| Najaf | 186730 | 254948 | 36 | 25 |
| AL-Qadisiyah | 495224 | 551045 | 42 | 27 |
| Muthanna | 154975 | 253898 | 43 | 20 |
| Dhi Qar | 257243 | 348710 | 50 | 20 |
| Maysan | 248296 | 350000 | 55 | 25 |
| Basra | 22735 | 31068 | 60 | 15 |

### b) Fuzzification of data

There was uncertainty or doubt regarding the values of $z_1(x)$and $z_2(x)$ in Table (1). Therefore, these data converted into triangular fuzzy data as shown in the following figure:

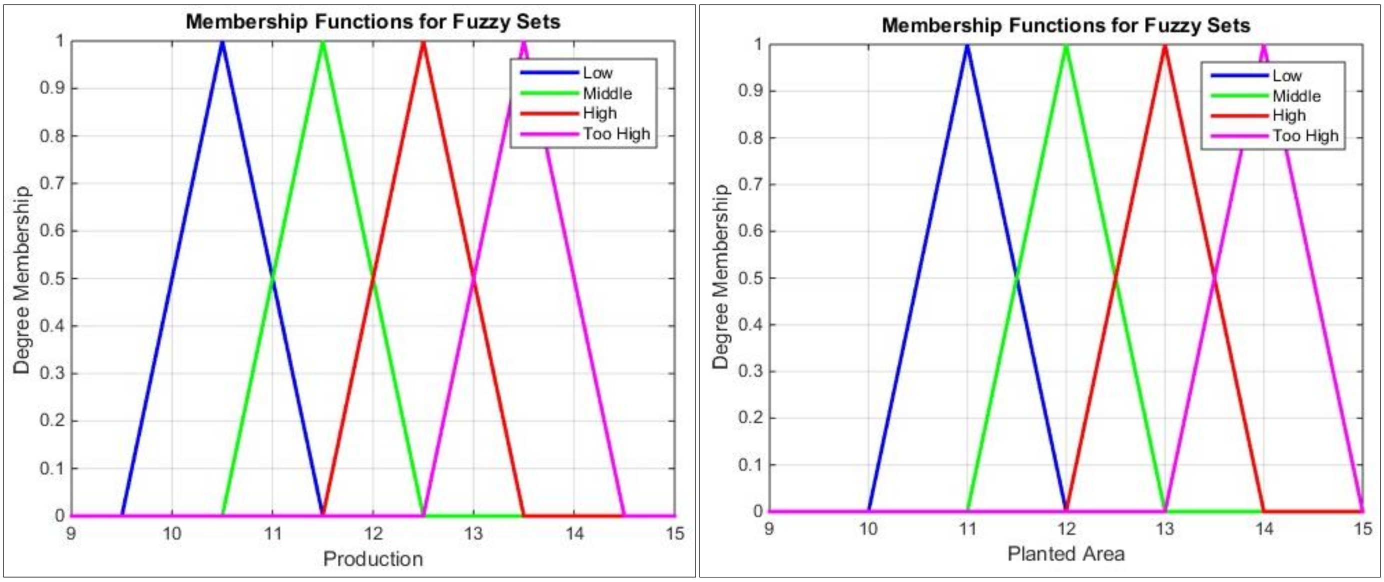


**Fig 4:** Triangular fuzzy data

After finding the degree of membership for each location with respect to the two variables, the data in Table (1) becomes fuzzy. Then, by finding the midpoint of each location, the result is as shown in the following Table (2):

**Table 2:** Fuzzy membership values and midpoints for governorates based on variables u(x) and v(x)

| Governorates | $z_1(x)$ | $z_2(x)$ | $u(x)$ | $v(x)$ |
|---|---|---|---|---|
| Nineveh | 13.5 | 14 | 30 | 62 |
| Kirkuk | 13.5 | 13 | 38 | 54 |
| Diyala | 13.5 | 13 | 40 | 40 |
| Anbar | 12.5 | 13 | 30 | 38 |
| Baghdad | 11.5 | 12 | 39 | 39 |
| Babylon | 12.5 | 13 | 37 | 28 |
| Karbala | 11.5 | 12 | 35 | 30 |
| Wasit | 13.5 | 14 | 48 | 30 |
| Salahuddin | 13.5 | 14 | 33 | 47 |
| Najaf | 12.5 | 12 | 36 | 25 |
| AL-Qadisiyah | 13.5 | 13 | 42 | 27 |
| Muthanna | 11.5 | 12 | 43 | 20 |
| Dhi Qar | 12.5 | 13 | 50 | 20 |
| Maysan | 12.5 | 13 | 55 | 25 |
| Basra | 10.5 | 11 | 60 | 15 |

Let us assume that the location (Nineveh) is unknown (unmeasured). Ordinary Co-kriging can be applied to estimate this location, assuming that the mean is unknown but constant in the study area. Thus, the mean $E(x)$ is estimated. From this data, the variogram for each variable was calculated, in addition to the cross-variogram between the two variables, as per equation (3). The results are shown in Table (3) in the appendix. Then, the relationship between the variogram function and the distance h is plotted, as shown in Figures (5), (6), (7) ,and This shape is closer to the spherical mathematical model, from which the covariance function for each of the variables and the cross-covariance function are formulated, after estimating the parameters $a$, $\psi$, and $\psi_0$ from the plot, where $\sigma^2 = sill = \psi_0 + \psi$ .

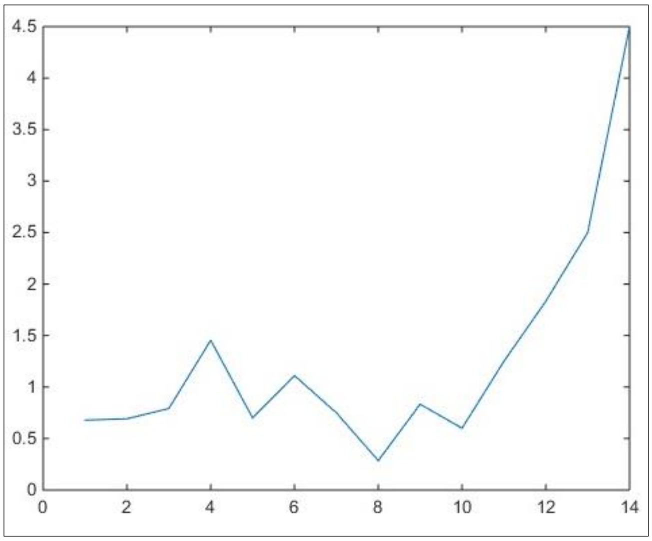

**Fig 5:** Variogram primary data

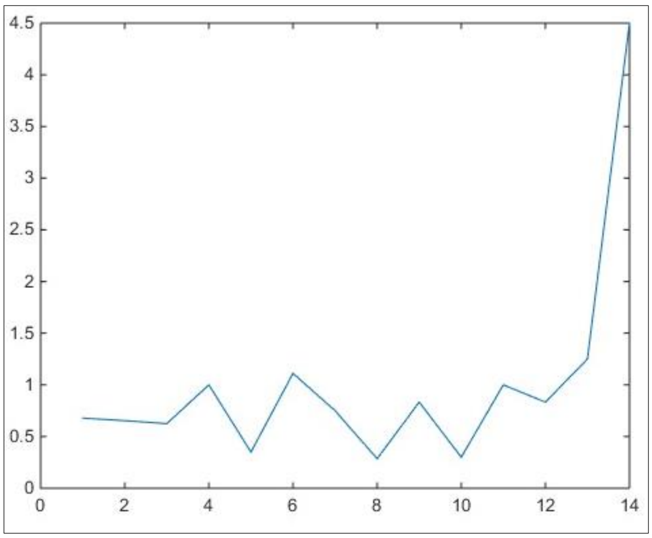

**Fig 5:** Variogram secondary data

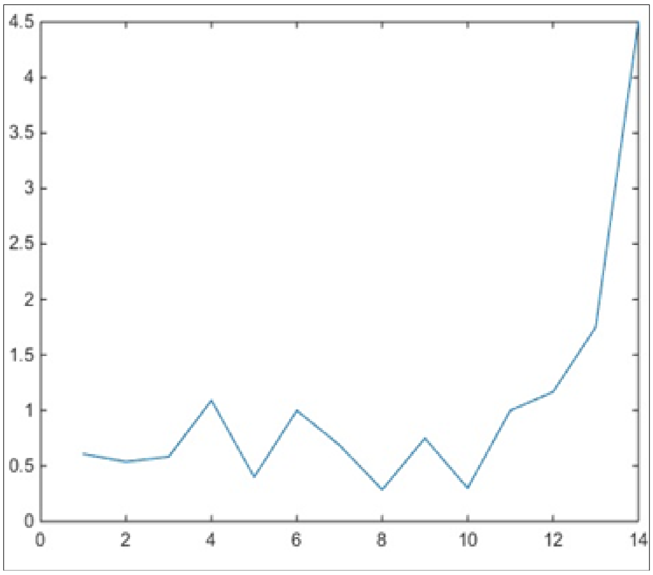


**Fig 7:** Cross variogram

$$C_{z_1}(h) = \begin{bmatrix} 1.5\ h = 0 \\ 0.9\left[1 - \frac{3h}{14} + \frac{1}{2}\left(\frac{h}{7}\right)^3\right]\ 0 < h \le 7 \\ 0\ h > 7 \end{bmatrix} \quad \ldots (14)$$

$$C_{z_2}(h) = \begin{bmatrix} 1\ h = 0 \\ 0.3\left[1 - \frac{3h}{14} + \frac{1}{2}\left(\frac{h}{7}\right)^3\right]\ 0 < h \le 7 \\ 0\ h > 7 \end{bmatrix} \quad \ldots (15)$$

$$C_{z_1 z_2}(h) = \begin{bmatrix} 0.7\ h = 0 \\ 0.1\left[1 - \frac{3h}{14} + \frac{1}{2}\left(\frac{h}{7}\right)^3\right]\ 0 < h \le 7 \\ 0\ h > 7 \end{bmatrix} \quad \ldots (16)$$

The ordinary Co-kriging technique for estimating $\hat{z}_1(x_0)$satisfies the constraints $\sum_{i=1}^{n} a_i = 1 , \sum_{i=1}^{m} b_i = 0$. The primary and secondary variables are treated as independent variables in the estimation of $\hat{z}_1(x_0)$, and this is applied to equations (14), (15), and (16). We obtain the cross-covariance matrix $C$ and then its inverse $C^{-1}$. The same process is repeated between the measured locations and the location to be estimated in order to obtain the vector $B$. After that, the weights $(n \times 1)$ for the primary and secondary variables are obtained from equation (12), and by multiplying the final result by the values of the primary and secondary variables for the data, we get $\hat{z}_1(x_0) = 12.5073$ and the estimation variance for ordinary Co-kriging $\sigma^2_{Fcok} = 0.8189$, which is calculated from equation (13). The entire computational process has been implemented in Matlab program, and the results are shown in Table (3).

**Table 3:** Results of the arithmetic processes

| $\lambda$ | $z_1$ | $z_2$ | $\gamma_{z_1}$ | $\gamma_{z_2}$ | $\gamma_{z_1 z_2}$ |
|---|---|---|---|---|---|
| $\lambda_1$ | 0.081161 | -0.00406 | 0.678571 | 0.678571 | 0.607143 |
| $\lambda_2$ | 0.052055 | 0.0103 | 0.692308 | 0.653846 | 0.153846 |
| $\lambda_3$ | 0.081161 | -0.00406 | 0.791667 | 0.625 | 0.375 |
| $\lambda_4$ | 0.052055 | 0.0103 | 1.454545 | 1 | 0.545455 |
| $\lambda_5$ | 0.032511 | 0.022028 | 0.7 | 0.35 | 0 |
| $\lambda_6$ | 0.065844 | 0.000648 | 1.111111 | 1.111111 | 0.722222 |
| $\lambda_7$ | 0.082113 | -0.00617 | 0.75 | 0.75 | 0.0625 |
| $\lambda_8$ | 0.081161 | -0.00406 | 0.285714 | 0.285714 | 0 |
| $\lambda_9$ | 0.067546 | -0.00145 | 0.833333 | 0.833333 | 0.333333 |
| $\lambda_{10}$ | 0.079747 | -0.00723 | 0.6 | 0.3 | -0.1 |
| $\lambda_{11}$ | 0.081161 | -0.00406 | 1.25 | 1 | 0.5 |
| $\lambda_{12}$ | 0.081161 | -0.00406 | 1.833333 | 0.833333 | 0.166667 |
| $\lambda_{13}$ | 0.081161 | -0.00406 | 2.5 | 1.25 | 0.25 |
| $\lambda_{14}$ | 0.081161 | -0.00406 | 4.5 | 4.5 | 1.5 |
| $\sum \lambda$ | 1 | 0 | | | |
| $\mu_1$ | $-0.1189$ | | | | |
| $\mu_2$ | $-0.0528$ | | | | |
| $\hat{z}_{(x_0)}$ | 12.5073 | | | | |
| $\sigma^2_{Fcok}$ | 0.8189 | | | | |

## 11. Conclusion

- Secondary variable data can be used to estimate its unknown points using a method called Co-kriging.
- Primary and secondary data can be Fuzzification in case of uncertainty, turning them into fuzzy triangular data, then the mean value can be found and used for estimation.
- The Co-kriging method can be applied in various fields, such as estimating agricultural crop yields.
- Multivariate estimation for fuzzy data can be performed using the ordinary Co-kriging method.

## 12. Recommendations

- Fuzzification the data into a trapezoidal or Gaussian form and compare the results.
- Study the estimation in cases where the mean is constant and known or constant and unknown, as well as when the mean between primary and secondary variables is equal and known.
- It is preferred to have a large dataset, as this provides more accurate results.
- Consult with a survey engineer to determine the coordinates of the measured and unmeasured points.
- Choose the appropriate model based on the graph of the cross-variogram function and the displacement h; otherwise, inaccurate results will be obtained.

## References

1. Bohling G. Introduction to geostatistics and variogram analysis. Kansas Geological Survey. 2005;1:1-20.

2. Guo D, Guo R, Thiart C. Integrating GIS with fuzzy logic and geostatistics: predicting air pollutant PM10 for California, using fuzzy kriging, Doctoral Dissertation, 2003.
3. Hesamian G, Akbari MG. A kriging method for fuzzy spatial data. International Journal of Systems Science. 2020;51(11):1945-1958.
4. Jaufar MM. Kriging estimation technique in geological statistics with application. Administrative and Economic Journal, University of Kirkuk. 2015;5(2):220-236.
5. Jaufar MM. A proposed method for selecting a spatial sample for the purpose of estimating an unknown point. Administrative and Economic Journal, University of Kirkuk. 2013;3(2):170-190.
6. Lee ES. Neuro-fuzzy estimation in spatial statistics. Journal of Mathematical Analysis and Applications. 2000;249(1):221-231.
7. Mohammed JM, Tawfeeq AF, Aziz MM. Maximum likelihood for fuzzy pure spatial autoregressive model. International Journal of Nonlinear Analysis and Applications. 2022;13(1):2267-2277.
8. Ross TJ. Fuzzy logic with engineering applications. Hoboken: John Wiley & Sons, 2005.
9. Studzinski J. Application of Kriging Algorithms for Solving Some Water Nets Management Tasks. In: EnviroInfo, c2011; p. 477-488.
10. Smith TE. Spatial data analysis. University of Pennsylvania, School of Engineering and Applied Science, 2015.
11. Zhang A. Statistical analysis of fuzzy linear regression model based on centroid method. Applied Mathematics. 2016;7(7):579-586.