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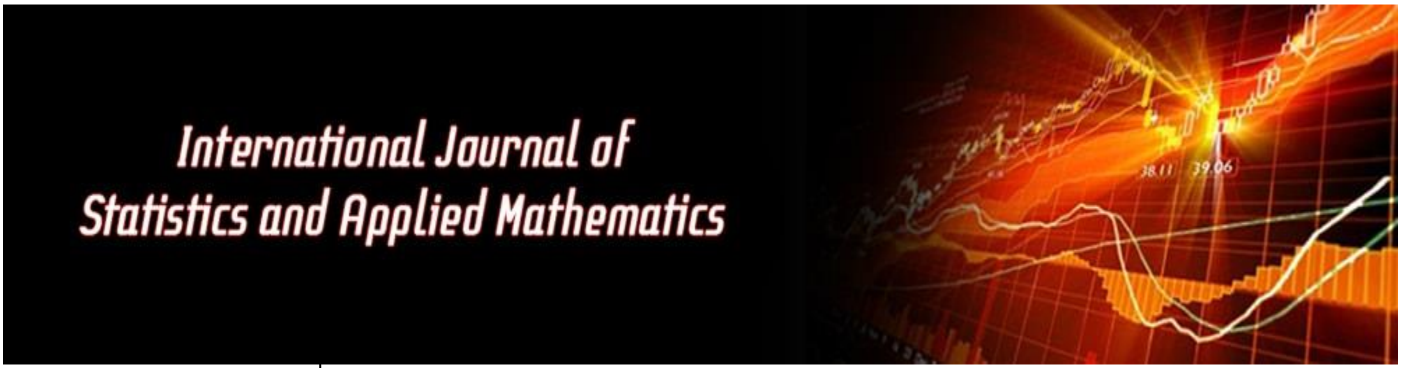

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Sanjeev Kumar Sinha
Department of Mathematics and Statistics, Guru Nanak Khalsa college of Arts Science and Commerce, Mumbai, Maharashtra, India

Corresponding Author:
Sanjeev Kumar Sinha
Department of Mathematics and Statistics, Guru Nanak Khalsa college of Arts Science and Commerce, Mumbai, Maharashtra, India

Bayesian estimation for the scale parameter of the Pareto distribution under asymmetric LINEX loss function

Sanjeev Kumar Sinha

Abstract
This paper explores Bayesian estimation of the scale parameter of Pareto distribution under the asymmetric LINEX loss function, assuming a known shape parameter. Suitable priors are employed to derive the Bayes estimators, which are then compared with the maximum likelihood estimator (MLE) in terms of risk performance. A Monte Carlo simulation study with 1000 samples demonstrates the effectiveness of the proposed approach.

Keywords: Bayesian estimation, Pareto distribution, Scale parameter, Asymmetric LINEX loss, Bayes estimator, Maximum likelihood estimator (MLE), Risk, Monte Carlo simulation, Conjugate priors

1. Introduction

The Pareto distribution was introduced by Vilfredo Pareto in 1897. The Pareto distribution is a widely used model in economics, finance, reliability engineering, and actuarial science due to its ability to describe heavy-tailed phenomena (Arnold, 2015) [1]. The distribution and their close relatives provide very flexibly family of heavy-tailed distributions which may be used as a model for income distribution of higher income group. Later this distribution has been used in connection with studies of wealth by Wold & Whittle (1957) [2] and service time queuing system by Harris (1968) [3]. Davis and Feldstein (1979) [4] have viewed the Pareto distribution as a potential model for life testing problems.

The probability density function of the classical Pareto distribution is given as

$$f(x;\, \alpha, \theta) = \frac{\alpha\theta^{\alpha}}{x^{\alpha+1}} I(x-\theta);\; x > \theta, \theta > 0, \alpha > 0 \qquad (1)$$

In (1), $I(t) = 1\,;\; t \neq 0$ Otherwise $I(t) = 0$, α and θ are referred to as shape and scale parameters respectively. The scale parameter of density given in (1), determines the the minimum possible value of the distribution and shape parameter represents the decay rate of extreme data of the distribution. Estimating scale parameter of Pareto distribution is crucial for accurate statistical inference and decision-making. Classical methods such as maximum likelihood estimation (MLE) are commonly employed, but they may lack robustness when the underlying loss structure is asymmetric (Zellner, 1986) [5].

It has always been recognizing that the most used Squared Error Loss function is inappropriate in many situations when the loss occurred in asymmetric form i.e. in some estimation problems, overestimation may be more serious than underestimation, or vice-versa i.e. in many real-world scenarios, underestimation and overestimation carry different consequences, see Parsain and Kirmani (2002) [6]. In such cases, the usual method of measuring loss may be inappropriate. To deal with such cases, a useful and flexible class of asymmetric LINEX (Linear-Exponential) loss function was introduced by Varian (1975) [7] as

$$L(\Delta, c\,) = \{e^{c\Delta} - c\Delta - 1\}\,;\; \Delta = \left(\hat{\theta} - \theta\right), c \neq 0 \qquad (2)$$

In (2), c is shape parameter of LINEX loss function, $\hat{\theta}$ is an estimate of the parameter θ, $\triangle$ is estimation error, The sign and magnitude of 'c' represents respectively direction and degree of asymmetry. Positive value of c is used when overestimation is riskier than underestimation while negative value of 'c' used when underestimation is more costly than overestimation. The LINEX Loss function was popularized in Bayesian econometrics by Zellner (1986) [5], provides a flexible framework for such cases by allowing different weights for over and under-estimation errors. Bayesian methods, with their ability to incorporate prior knowledge, are particularly well-suited for estimation under asymmetric loss, offering improved risk performance compared to classical estimators (Parsian & Kirmani, 2002) [6]. Several studies have explored Bayesian estimation under LINEX loss for various distributions. Calabria & Pulcini (1996) [8] examined Bayesian inference for the inverse Gaussian distribution, while Asgharzadeh *et al.* (2013) [9] studied estimation for the Pareto model under different loss functions. It has been widely used by several authors, Basu and Ebrahimi (1991) [10], Soliman (2002) [11], Singh *et al.* (2005) [12]. In this paper, we have obtained the Bayes estimator of scale parameter when shape is known under LINEX loss function considering a natural family of conjugate prior for the parameter. The estimator has been derived and its risk has been calculated under LINEX loss function. Results of a Monte Carlo simulation study have been conducted to evaluate the performances of proposed estimators. This paper bridges this gap by:-Deriving Bayes estimators for the Pareto scale parameter using both informative (conjugate) and non-informative priors, comparing their risk efficiency against the MLE under LINEX loss via Monte Carlo simulations and demonstrating the practical advantages of the Bayesian approach, especially when prior information is available.

Results contribute to the growing literature on asymmetric loss-based estimation, offering practitioners a more nuanced tool for Pareto model applications in fields like income distribution modelling and reliability analysis.

2. Bayes Estimator under LINEX loss

Let $\underline{X} = X_1, X_2 \ldots X_n$ be a random sample from Pareto distribution with density function given in (1). The maximum likelihood estimator of the scale parameter σ of the classical Pareto distribution is

$$\hat{\theta}_{MLE} = X_{(1)} = min(X_{(1)}, X_{(2)}, \ldots X_{(n)}) \quad (3)$$

Conjugate prior for scale parameter σ is considered Pareto density given as follows

$$f(\theta) = \frac{\alpha_0 {\theta_0}^{\alpha_0}}{\theta^{\alpha_0+1}} I(\theta-\theta_0); \theta > \theta_0, \alpha_0 > 0 \quad (4)$$

The posterior density of σ is obtained as

$$f(\theta/\underline{X}) = \frac{m}{X_{(1)}^m - \theta_0^m}\theta^{m-1}; \theta_0 \le \theta \le X_{(1)}, m > 0 \quad (5)$$

In (5), $m = (n\alpha - \alpha_0)$ and θ_0 are the posterior hyper parameters. In this study the shape parameter is considered known α_0. The Bayes estimator of θ under LINEX Loss function when θ prior is considered Pareto distribution, is obtained as

$$\hat{\theta}_{B_1LL} = -\frac{1}{c}\ln\left[E\left(e^{-c\theta}\right)\right]$$

$$\hat{\theta}_{B_1LL} = -\frac{1}{c}ln\left[\frac{m}{X_{(1)}^m - \theta_0^m}\left\{\int_0^{X_{(1)}} e^{-c\theta}\,\theta^{m-1}d\theta - \int_0^{\theta_0} e^{-c\theta}\,\theta^{m-1}d\theta\right\}\right]^{-1} \quad (6)$$

Another natural family of conjugate prior for the scale parameter $\boldsymbol{\theta}$ is considered exponential density as

$$g(\theta) = \lambda e^{-\lambda\theta}; \theta > 0, \lambda > 0 \quad (7)$$

The Posterior density of σ is obtained as

$$f(\theta/\underline{X}) = \frac{e^{-\lambda\theta}\theta^{n\alpha}}{\lambda^{n\alpha+1} I(cX_{(1)}, n\alpha+1)}; \theta > 0, \lambda > 0 \quad (8)$$

Where $I(cX_{(1)}, n\alpha+1) = \int_0^{cX_{(1)}} e^{-u} u^{(n\alpha+1)-1} du.$ Bayes estimator of θ under LINEX Loss when prior is exponential, is obtained as

$$\hat{\theta}_{B_2LL} = -\frac{1}{c}ln\left[\frac{I((c+\lambda)X_{(1)}, n\alpha+1)}{\lambda^{n\alpha+1}(c+\lambda)^{n\alpha+1} I(cX_{(1)}, n\alpha+1)}\right]^{-1} \quad (9)$$

3. Estimation of Risks under LINEX loss function

To study the performance of the proposed estimators, a Monte Carlo study has been conducted to compare the simulated risks of various estimators. This is because most of the concerned expressions are not solvable analytically. Therefore, risks of the estimators are empirically evaluated based on a Monte-Carlo simulation study of 1000 samples. The value of shape parameter α is considered known as 1.5, we use numerical technique to compare the proposed Bayes estimates with maximum likelihood estimate of the scale parameter θ and made comparison between the Bayes estimates according to the following steps:

1. Assuming $\theta = 1$ and=1.5, Pareto sample of size n (=5, 10, 20, 50) are generated using transformation $X_i = \theta(1-U_i)^{-\frac{1}{\alpha}}$, where U_i is the uniformly distributed random variable.
2. From each generated sample, the Maximum Likelihood Estimates of θ, $\hat{\theta}_{MLE} = X_{(1)} = \min\left(X_{(1)}, X_{(2)}, \ldots . X_{(n)}\right)$ has computed.
3. Considering known values of LINEX loss function parameter, c=-1.0, -0.5, 0.5, 1.0 and prior as Pareto distribution with known hyper parameters $\theta_0 = 0.5$ α_0=1.5, Bayes estimates have been computed from each generated sample. Similarly other Bayes estimates when prior is considered as exponential distribution, are computed for known values of hyper parameters as λ=0.5, 1.0, 2.0; α_0=1.5.
4. Steps 1 to 3 are repeated N=1000 times and the risks under LINEX loss for $\hat{\theta}_{MLE}, \hat{\theta}_{B_1LL}$ and $\hat{\theta}_{B_2LL}$ i.e $R\left(\hat{\theta}_{MLE}\right)$, $R\left(\hat{\theta}_{B_1LL}\right)$ and $R\left(\hat{\theta}_{B_2LL}\right)$ respectively have been computed using following risk function

$$R\left(\hat{\theta}\right) = \frac{1}{N}\sum_{i=1}^{N}\left\{e^{c\left(\hat{\theta}_i-\theta\right)} - c\left(\hat{\theta}_i - \theta\right) - 1\right\}; \; c \neq 0 \quad (10)$$

4. Results and Discussion

The estimators $\hat{\theta}_{B_1LL}$ and $\hat{\theta}_{B_2LL}$ are Bayes estimators of the parameter θ, assuming Pareto and Exponential priors respectively, while $\hat{\theta}_{MLE}$ represents the maximum likelihood estimator. As previously mentioned, these estimators and their associated risks have been empirically evaluated through a Monte Carlo simulation study based on 1000 generated samples. It is easy to notice that the risk of the estimators are functions the function sample size, population parameters,

parameters of the prior distribution and loss function parameters. The shape parameter of the Pareto distribution is assumed to be known with a value of 1.5, and the sample size is varied to examine the effect of both small and large samples on the performance of the estimators. Variations in the estimators and their associated risks are also assessed by altering the value of the LINEX loss function parameter c=-1.0, -0.5, 0.5, 1.0, while keeping the sample size constant. Additionally, the Pareto prior parameter $\theta = 0.5$ and the Exponential prior parameter $\lambda = 0.5$ are arbitrary considered known throughout the evaluation of the estimators and their risks. The findings from the study are presented and summarized in Figures 1-10.

Based on an extensive analysis of the obtained results, conclusions have been drawn regarding the behavior of the estimators, which are illustrated in the graphs below. Additionally, risk values were computed for other arbitrary known values of the prior parameters Pareto $\theta = 1$ & 2 and Exponential $\lambda = 1$ &2 though, due to space constraints, these results are not displayed in the graphs. A selected set of findings is presented to highlight the observed effects and support the conclusions. Overall, the proposed Bayes estimators demonstrate superior performance compared to the maximum likelihood estimator. From the figures depicting the computed risks of the estimators, it is observed that the risk decreases as the sample size increases. In examining the effect of varying the LINEX loss parameter c on the risks of the estimators $R(\hat{\theta}_{MLE})$, $R(\hat{\theta}_{B_1LL})$ and $R(\hat{\theta}_{B_2LL})$), it is evident that for both positive and negative values of c, the risks associated with the proposed Bayes estimators remain consistently lower than that of the maximum likelihood estimator. This difference is notably larger for smaller sample sizes (N<20), as seen in Figures 1-8. Furthermore, it is observed that the risks corresponding to negative values of c are lower than those for positive values seen in Figures 1-10. This suggests that underestimation should be regarded as more serious than overestimation, and negative values of c are preferable in such decision-making contexts.

It can be observed from Figures 11 and 12 that the risks associated with the proposed estimators $\hat{\theta}_{B_1LL}$ and $\hat{\theta}_{B_2LL}$ are nearly identical across all considered values. Therefore, either of these estimators can be suitably used for estimating θ.

Estimated risks of Bayes estimator θ_{B_1LL} and ML Estimator of θ

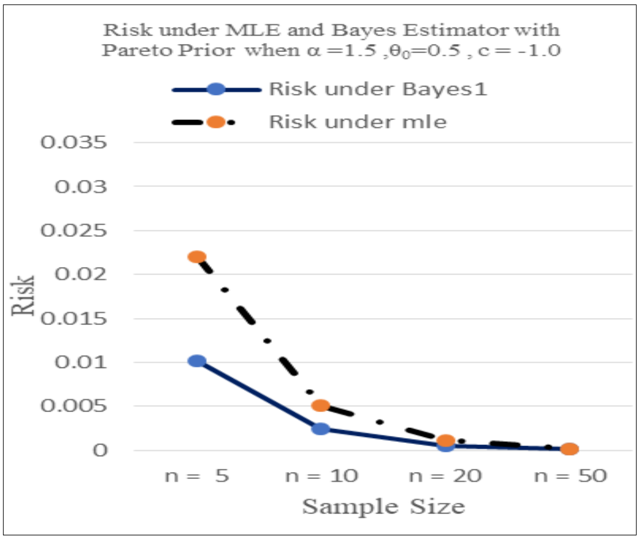

Fig 1: $R(\hat{\theta}_{MLE})$ and $R(\hat{\theta}_{B_1LL})$, under the case of underestimation (c negative)

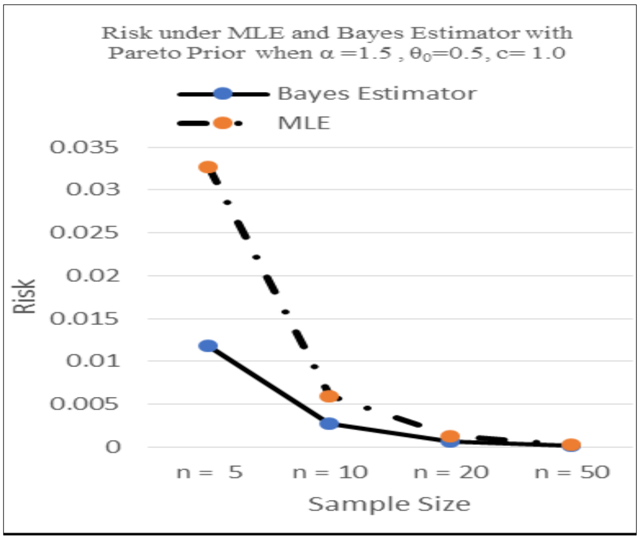

Fig 2: $R(\hat{\theta}_{MLE})$ and $R(\hat{\theta}_{B_1LL})$, under the case of overestimation (c positive)

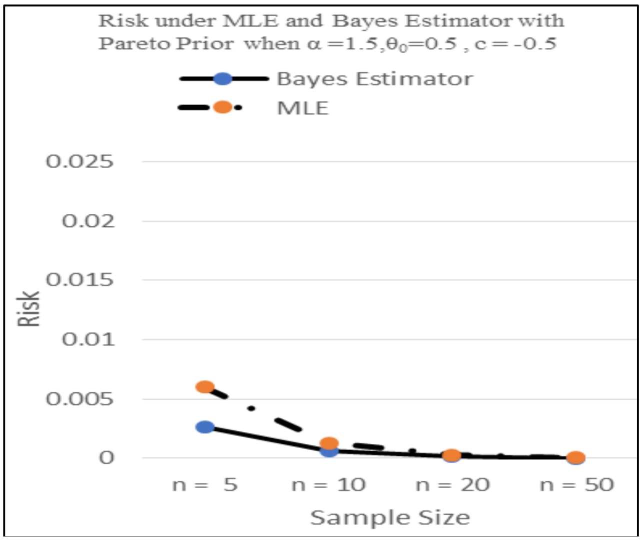

Fig 3: $R(\hat{\theta}_{MLE})$ and $R(\hat{\theta}_{B_1LL})$, under the case of underestimation (c negative)

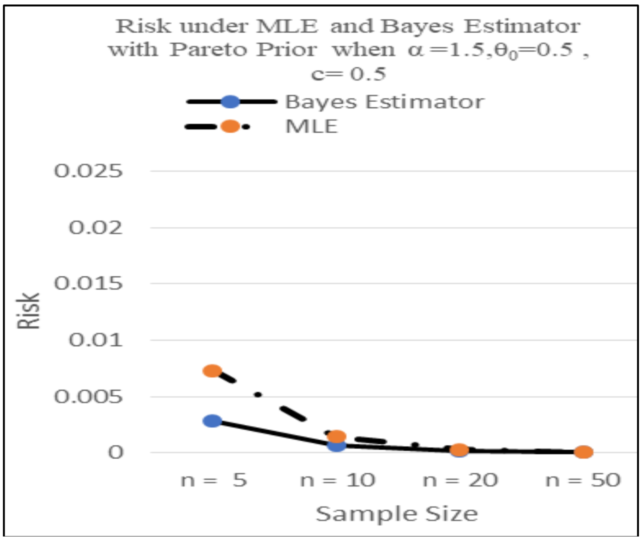

Fig 4: $R(\hat{\theta}_{MLE})$ and $R(\hat{\theta}_{B_1LL})$, under the case of overestimation (c positive)

Estimated Risks of Bayes Estimator θ_{B_2LL} and ML Estimator of $\boldsymbol{\theta}$

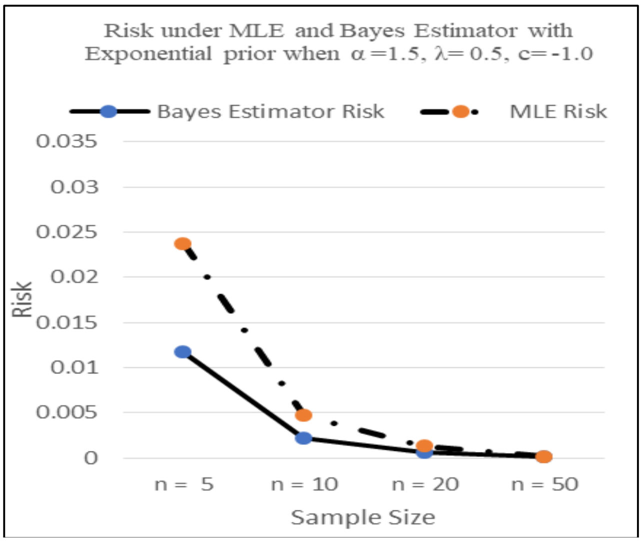

Fig 5: $R(\hat{\theta}_{MLE})$ and $R(\hat{\theta}_{B_2LL})$, under the case of underestimation (c negative)

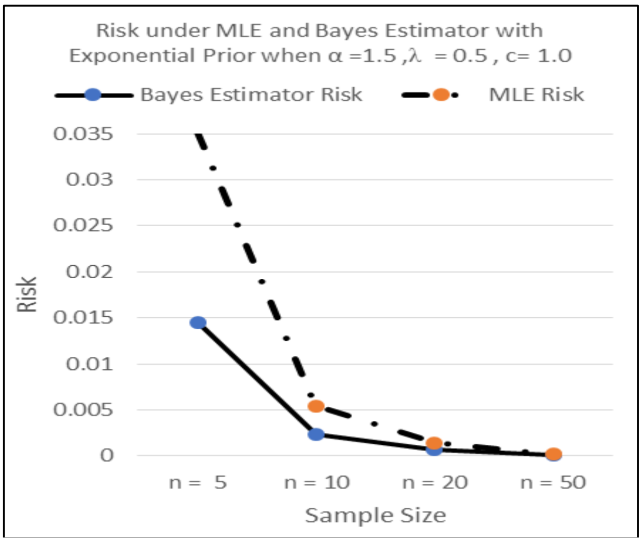

Fig 6: $R(\hat{\theta}_{MLE})$ and $R(\hat{\theta}_{B_2LL})$, under the case of overestimation (c positive)

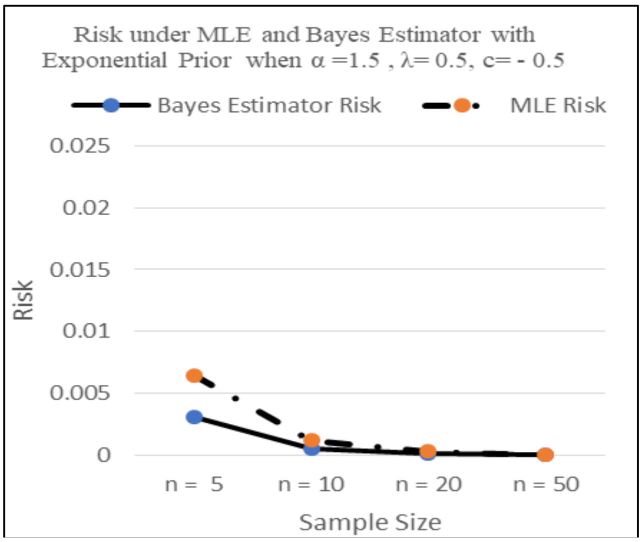

Fig 7: $R(\hat{\theta}_{MLE})$ and $R(\hat{\theta}_{B_2LL})$, under the case of underestimation (c negative)

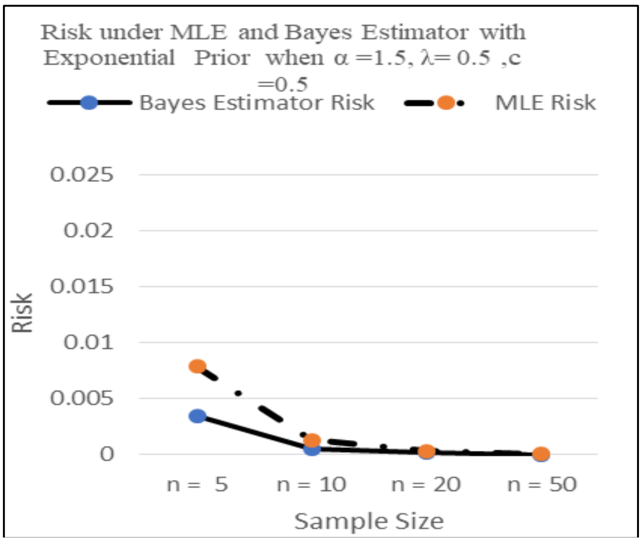

Fig 8: $R(\hat{\theta}_{MLE})$ and $R(\hat{\theta}_{B_2LL})$, under the case of overestimation (c positive)

Estimated risks of Bayes estimator θ_{B_1LL} and Bayes Estimator θ_{B_2LL} of parameter $\boldsymbol{\theta}$

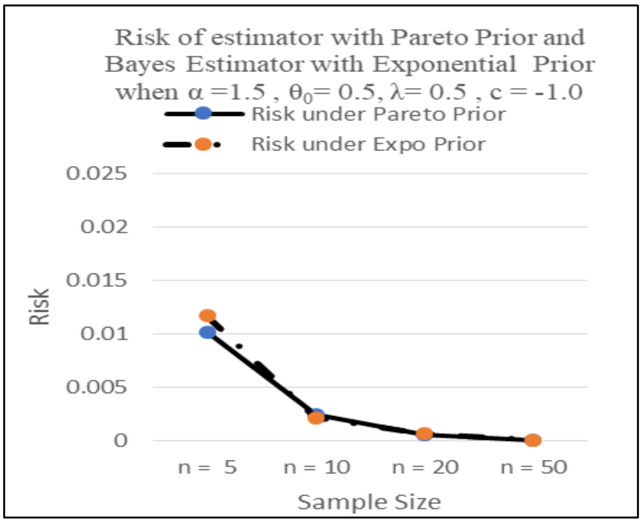

Fig 9: $R(\hat{\theta}_{MLE})$ and $R(\hat{\theta}_{B_2LL})$, under the case of underestimation (c negative)

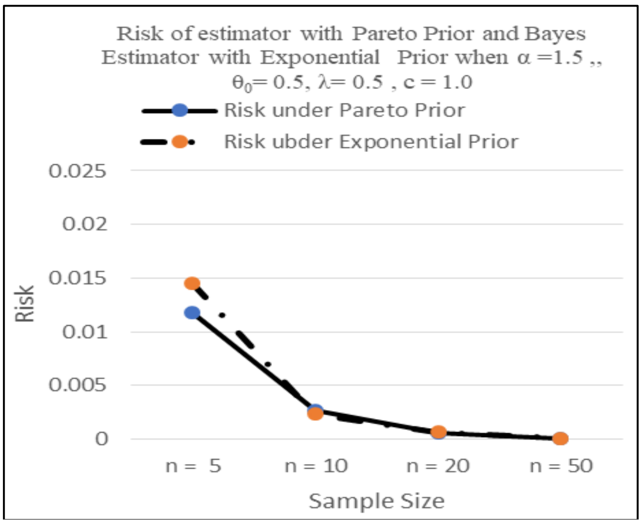

Fig 10: $R(\hat{\theta}_{MLE})$ and $R(\hat{\theta}_{B_2LL})$, under the case of overestimation (c positive)

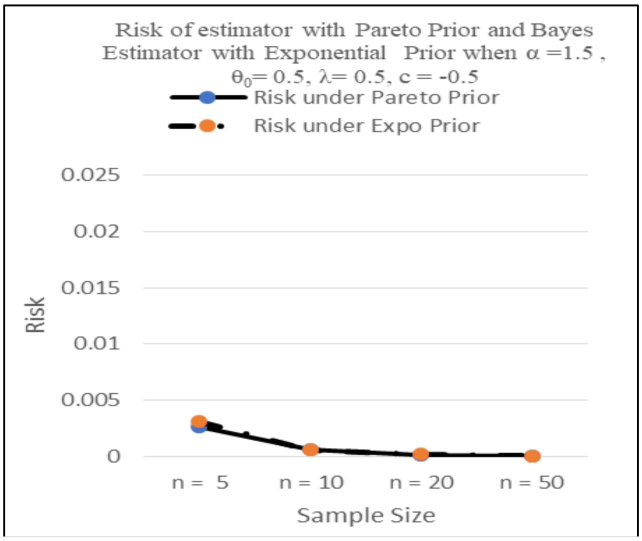

Fig 11: $R(\hat{\theta}_{B_1LL})$ and $R(\hat{\theta}_{B_2LL})$, under the case of underestimation (c negative)

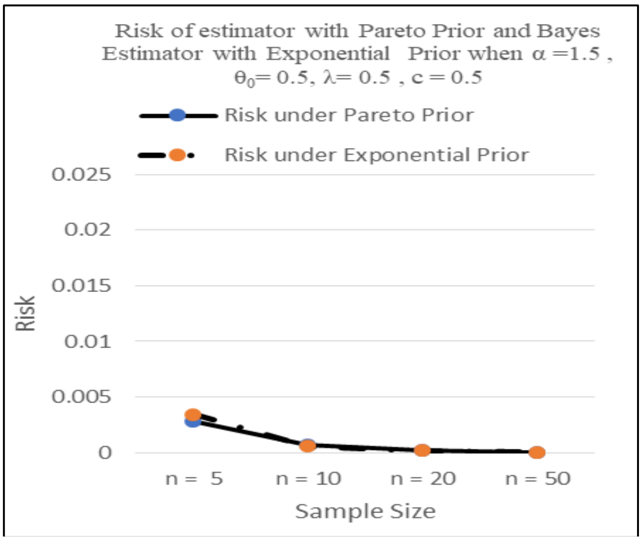

Fig 12: $R(\hat{\theta}_{B_1LL})$ and $R(\hat{\theta}_{B_2LL})$, under the case of overestimation (c positive)

5. Conclusions

The proposed Bayes estimators under the LINEX loss function have been compared with the maximum likelihood estimator through graphs. The findings suggest that for both negative and positive values of the LINEX loss parameter c, the Bayes estimators perform better than the maximum likelihood estimator, especially for small sample sizes. Moreover, risks are consistently lower for negative values of c. Hence, when underestimation is more critical than overestimation (i.e. for negative c values), the proposed Bayes estimators are preferable.

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